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Stability in the Presence of Persistent Plant Changes for a Compact Set of Plant Models

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***Abstract*— In this paper we consider the use of periodic controllers for simultaneous stabilization and performance, with a focus on the case when there is an occasional, though persistent, plant change. We consider the case of a compact set of admissible models; we provide a design procedure which yields a controller which stabilizes each such model and provides near optimal LQR performance. We also demonstrate that this control law has the facility to tolerate occasional (but persistent) switches between these models. The controller is periodic with a slight nonlinearity.**

I. INTRODUCTION

A classical control problem is that of providing good performance in the face of plant uncertainty, especially when time-variations are involved. There are two standard approaches to the problem: robust control and adaptive control; an implicit goal of the latter is to deal with (possibly rapidly) changing parameters, while in the case of robust control, the typical approach yields a linear time-invariant (LTI) controller which deals with fixed (but unknown) parameters, with changes restricted to the occasional, but infrequent, jump, so that LTI theory can be applied. Here we consider an open problem in which the set of uncertainty is very large - larger than LTI robust control theory can tolerate - and where we demand good transient performance - better than what traditional adaptive control will provide. Furthermore, we wish to do this in the context of an occasional, though persistent, plant change.

Robust control approaches e.g. [1], [2], and [3] have been used to prove that every finite set of models can always be simultaneously stabilized using an LTV controller, but the papers indicate that performance will be quite poor. For larger classes of uncertainty such as compact sets of plant parameters, previous results appear to be mostly restricted to adaptive approaches including logic based switching e.g. [6], [7], and [8]; however these approaches typically provide poor transient behaviour and possibly large control signals. Recent research by the second author provides some additional related work, although the work on time-varying plants is limited. Specifically, in [4] it is proven that good performance can be provided for rapidly varying minimum-phase systems; in [5] it is proven that near optimal LQR performance can be obtained for a compact set of LTI models, but with no time variations allowed; in [10] it is proven that the problem considered here can be solved in the context of a finite set of models, with the measure of performance being the classical LQR setup. Here the goal is to extend the approach of [10] to handle a compact set of models; we combine the ideas of [10] with the techniques of [5]. Recent work by Vu and Liberzon [11] uses completely different techniques (it is based on supervisory control) to prove similar types of results, although there the focus is more on stability and disturbance rejection than it is on performance.

We use the Holder 2-norm for vectors and the corresponding induced norm for matrices, and denote the norm of a vector or matrix by $\|\cdot\|$. We measure the size of a piecewise continuous signal x by $\|x\|_2 := [\int_0^\infty \|x(\tau)\|^2 d\tau]^{1/2}$.

II. PROBLEM FORMULATION

The first class of plant models are of the form

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu, \quad x(t_0) = x_0 \\ y & = & Cx, \end{array} \tag{1}$$

with $x(t) \in \mathbf{R}^n$ representing the state, $u(t) \in \mathbf{R}$ the control signal, and $y(t) \in \mathbf{R}$ the measured output. We associate the plant with the triple (A, B, C). We let Γ denote the subset of $\mathbf{R}^{n\times n} \times \mathbf{R}^{n\times 1} \times \mathbf{R}^{1\times n}$ which corresponds to triples (A, B, C) for which (A, B) is controllable and (C, A) is observable. In this paper an initial goal is to control the plant when the model is uncertain: we assume that it lies in a compact subset of Γ, which we label $\mathcal{P}$.

Here we allow for persistent plant changes as well. To this end, we consider a time-varying plant obtained by switching between elements of $\mathcal{P}$:

$$\begin{array}{rcl} \dot{x}(t) & = & A(t)\ x(t) + B(t)\ u(t), \quad x(0) = x_0, \\ y(t) & = & C(t)\ x(t). \end{array} \tag{2}$$

With $T_s > 0$, define the time varying uncertainty set by

$\mathcal{P}_{T_s} := \{(A, B, C)(t)$:

1)$(A, B, C)(t) \in \mathcal{P}, \quad t \geq 0,$
2)$(A, B, C)(t)$ is a piecewise constant function of t, and
3) There is at least T_s time units between discontinuities$\}$.

Notice that $\mathcal{P}_\infty = \mathcal{P}$.

Here our first goal is to design a controller which not only provides closed loop stability but also provides near optimal LQR performance for each possible model in $\mathcal{P}$. We would also like to maintain stability in the presence of persistent, but sufficiently infrequent, plant changes; specifically, we

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would like to stabilize $\mathcal{P}_{T_s}$ for sufficiently large T_s. To this end, with $r > 0$, consider the classical performance index

$$\int_0^\infty [y^2(t) + ru^2(t)]\, dt.$$

As is well-known, the optimal controller for $(A, B, C) \in \mathcal{P}$ is state-feedback and of the form $u = F(A, B, C)x$. The associated closed loop system is

$$\dot{x} = (A + BF(A, B, C))x =: A_{cl}(A, B, C)x.$$

It is known how to solve our first objective using a sampled-data linear periodic controller [5]. However, to handle the time-switching we will use a mild nonlinearity; we will also use a more complicated time-variation. To this end, we consider sampled-data controllers of the form

$$\begin{aligned} z[k+1] &= G[k]z[k] + J[k]y(kh), \quad z[0] = z_0 \in \mathbf{R}^l, \\ u(kh+\tau) &= K(k, z[k], y(kh), \tau), \qquad \tau \in [0, h). \end{aligned} \tag{3}$$

It turns out that G, J, and K are periodic functions of k, of period ℓ; the period of the controller is therefore $T := \ell h$. Furthermore, we impose a natural boundedness condition on K: there exists a $c > 0$ such that

$$\|K(k, z, y, \tau)\| \le c(\|z\| + \|y\|), \quad k \in \mathbf{Z}^+, \ \tau \in (0, h).$$

We associate this system with the 5-tuple (G, J, K, h, ℓ).

Here our notion of closed loop stability is the usual one:

Definition 1: The sampled-data controller (3) stabilizes (1) if, for every $x_0 \in \mathbf{R}^n$ and $z_0 \in \mathbf{R}^l$, we have

$$\lim_{t\to\infty} x(t) = 0 \text{ and } \lim_{k\to\infty} z[k] = 0.$$

Notice that stability ensures that $\lim_{t\to\infty} u(t) = 0$.

The first goal of this paper is to design (3) so that it stabilizes every plant in $\mathcal{P}$ and so that it is near optimal in the LQR sense. The second goal of this paper is to ensure that the controller stabilizes $\mathcal{P}_{T_s}$ if T_s is large enough.

At this point we provide a high level motivation of our approach. It combines ideas from [5] and [10], with several new twists. We divide the period $[kT, (k+1)T)$ into two phases: the Estimation Phase and the Control Phase. In [5], in the Estimation Phase we estimate $F(A, B, C)x[kT]$, and in the Control Phase we apply a suitably scaled estimate of this quantity; for this to work we required the probing signals used during the Estimation Phase to be modest in size, and for T and T'/T to be small. Here we proceed in a similar but slightly modified fashion. In the Estimation Phase, we estimate $F(A, B, C)e^{A_{cl}(A,B,C)(t-kT)}x(kT)$ by applying a sequence of test signals, which are constructed on the fly. In the Control Phase we apply a suitably weighted estimate of $F(A, B, C)e^{A_{cl}(A,B,C)(t-kT)}x(kT)$. Hence, if T'/T is small (we no longer require T to be small!), then we would expect that this controller should be close to the optimal one. Figure 1 illustrates the differences between these two approaches. As in [5], we would like to carry out each phase in (almost) a linear fashion in order to end up with (almost) a linear controller.

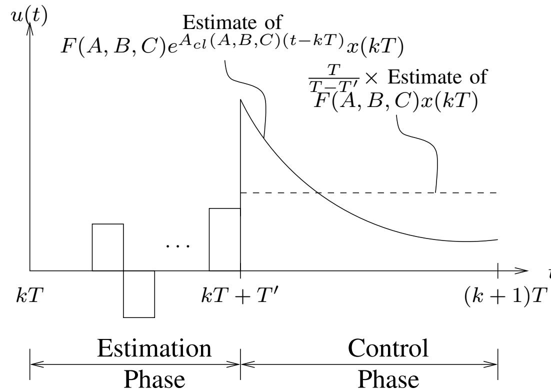

Fig. 1. Comparison of old and new methods.

III. The Approach

Here (versus in [5]) we will let T be large, so that $u(t)$ is closer to the optimal signal, which is aesthetically pleasing. The other major advantage over [5] is that here we will be able to tolerate time variations.

As discussed above, the goal is to periodically estimate the desired control signal and then to apply this estimate. The first step is to choose a parameterization of the plant model which is amenable to estimation.

A. A Special Canonical Form

Let $(A, B, C) \in \mathcal{P}$ and define

$$O_i(C, A) := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^i \end{bmatrix}, \quad i \in \mathbf{Z}^+$$

as well as

$$w := O_{n-1}(C, A)x.$$

It is easy to see that our transformed system is

$$\begin{aligned} \dot{w} &= \underbrace{\begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_0 & \cdots & \cdots & -a_{n-1} \end{bmatrix}}_{=:\bar{A}} w + \underbrace{\begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{n-1}B \end{bmatrix}}_{=:\bar{B}} u \\ y &= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}}_{=\bar{C}} w; \end{aligned} \tag{4}$$

the optimal control law is now written as

$$u = \bar{F}(\bar{A}, \bar{B}, \bar{C})w.$$

Observe, in particular, that $\bar{B}$ is composed of Markov parameters while $\bar{C}$ is constant; we can prove that $\bar{A}$ and $\bar{F}$ are also nice functions of the plant Markov parameters.

Lemma 1: (Parametrization Lemma) [5] $\bar{A}(A, B, C)$ and $\bar{F}(\bar{A}, \bar{B}, \bar{C})$ are analytic functions of the first $2n$ Markov parameters $\{CB, CAB, ..., CA^{2n-1}B\}$ for all $(A, B, C) \in \Gamma$.

Hence, at this point we adopt a more compact notation: we define

$$M := \left\{ \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}}_{=:p} := \begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{m-1}B \end{bmatrix} : (A,B,C) \in \mathcal{P} \right\}; \tag{5}$$

since $\mathcal{P}$ is compact, M is as well. The goal is to have the plant be parameterized by p, so by Lemma 1 we know that setting $m = 2n$ in the definition of p (and M) will always work; of course, if there is a lot of structure in $\mathcal{P}$ (e.g. it is a gain margin problem) then a smaller choice of m may do. In any event, at this point we choose $m \leq 2n$ so that p uniquely identifies the plant. Hence, in this new parameterization, the optimal state feedback gain $\bar{F}(\bar{A}, \bar{B}, \bar{C})$ is an analytic function of p, which we label $f(p)$, so the optimal control law is

$$u = f(p)w = f(p)O_{n-1}(C,A)x. \tag{6}$$

Now we turn to the estimation issue.

Define several matrices:

$$S_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^m \\ & & \vdots & & \\ 1 & m & m^2 & \cdots & m^m \end{bmatrix},$$

$$H_m(h) = \text{diag}\{1,\, h,\, h^2/(2!), ..., h^m/(m!)\}.$$

We will be using a sequence of samples of y: we define

$$\mathcal{Y}_m(t) := \begin{bmatrix} y(t) & y(t+h) & \cdots & y(t+mh) \end{bmatrix}^T.$$

Lemma 2: (Key Estimation Lemma) [5] Let $\bar{h} \in (0,1)$ and $m \in \mathbf{N}$. There exists a constant $\gamma > 0$ so that for every $t_0 \in \mathbf{R}$, $x_0 \in \mathbf{R}^n$, $h \in (0, \bar{h})$, $\bar{u} \in \mathbf{R}$, and $(A,B,C) \in \mathcal{P}$, the solution of (1) with

$$u(t) = \bar{u}, \;\; t \in [t_0, t_0 + mh)$$

satisfies the following:

$$\left\| H_m(h)^{-1} S_m^{-1} \mathcal{Y}_m(t_0) - O_m(C,A)x(t_0) - \begin{bmatrix} 0 \\ p \end{bmatrix} \bar{u} \right\| \leq \gamma h (\|x(t_0)\| + \|\bar{u}\|),$$

$$\|x(t) - x(t_0)\| \leq \gamma h (\|x(t_0)\| + \|\bar{u}\|), \;\; t \in [t_0, t_0 + mh].$$

To see how the Key Estimation Lemma (KEL) can be applied, suppose that we first set

$$u(t) = 0, \;\; t \in [t_0, t_0 + mh),$$

so a good estimate of $O_m(C,A)x(t_0)$ is $H_m(h)^{-1} S_m^{-1} \mathcal{Y}_m(t_0)$. Since we may very well have a plant switch in $[t_0, t_0 + mh)$, we must proceed with caution. Hence, we adopt the trick of [10]: we set

$$u(t) = 0, \;\; t \in [t_0 + mh, t_0 + 2mh)$$

as well, so another good estimate of $O_m(C,A)x(t_0)$ is $H_m(h)^{-1} S_m^{-1} \mathcal{Y}_m(t_0 + mh)$. If we insist that $2mh < T_s$ (the minimum time between switches), then at least one interval will **not** contain a plant switch; so we will set

$$\text{Est}[O_m(C,A)x(t_0)] := \text{argmin}\{\|H_m(h)^{-1} S_m^{-1} \mathcal{Y}_m(t_0)\|, \|H_m(h)^{-1} S_m^{-1} \mathcal{Y}_m(t_0 + mh)\|\};$$

if h is also small, then we will be guaranteed that the above estimate will be accurate when there is no plant switch and will be modest in size when there is one.

With $g \in \mathbf{R}^{1 \times (m+1)}$, suppose we define a test signal to be a linear functional of our above estimate:

$$\bar{u} = g\text{Est}[O_m(C,A)x(t_0)].$$

If we now set

$$u(t) = \bar{u}, \;\; t \in [t_0 + 2mh, t_0 + 4mh),$$

then by the KEL we should define

$$\begin{aligned} \text{Est}[p\bar{u}] := \begin{bmatrix} 0 & I_m \end{bmatrix} \times \\ \text{argmin}\{\|H_m(h)^{-1} S_m^{-1} [\mathcal{Y}_m(t_0 + 2mh) - \mathcal{Y}_m(t_0)]\|, \\ \|H_m(h)^{-1} S_m^{-1} [\mathcal{Y}_m(t_0 + 3mh) - \mathcal{Y}_m(t_0 + mh)]\|\}. \end{aligned}$$

Of course, this can be repeated a number of times, for different choices of g, so it should be possible to estimate terms of the form $\phi(p)O_m(C,A)x(t_0)$ with $\phi : \mathbf{R} \longmapsto \mathbf{R}^{m+1}$ a polynomial in its arguments and m a positive integer. Since f is an analytic function of its arguments, as long as $m \geq n$, we can always estimate (6) at a given point in time as close as we wish by a sequence of experiments (assuming, of course, that there is no plant change during the experiments).

Henceforth, we assume that

$$m \in \{n, n+1, ..., 2n\}$$

and we define

$$W := \begin{bmatrix} I_n & 0 \end{bmatrix} \in \mathbf{R}^{n \times (m+1)};$$

the optimal control law has the form

$$u = f(p)w = f(p)WO_m(C,A)x. \tag{7}$$

IV. Approximation by a Sampled-Data Controller

In closed loop, (7) yields a control signal of the form

$$u(t) = f(p)w(t) = \underbrace{f(p)e^{(\bar{A}+\bar{B}f(p))t}}_{=:H(p,t)} w(0).$$

Of course, since the proposed controller is periodic of period T, this is equivalent to the sampled-data controller

$$u(t) = H(p, t - kT)w(kT), \;\; t \in [kT, (k+1)T);$$

this could be implemented via a generalized sampler (to generate $w(kT)$) and a generalized hold (to generate $u(t)$). The problem is that p is unknown, so what we'd like to do is estimate $H(p, t-kT)w(kT)$. To proceed, first observe that H is an analytical function of its two arguments. Let's fix an upperbound on the period, say T_{max}. By the Stone-Weierstrass Approximation Theorem, for every $\varepsilon > 0$ there exists a polynomial H_ε satisfying

$$\|H(p,t) - H_\varepsilon(p,t)\| \le \varepsilon, \;\; p \in M, \;\; t \in [0, T_{max}].$$

When the optimal control law is applied to the plant (1), we label the corresponding state response by x^0, output by y^0, and control signal by u^0. Similarly, when the sampled-data controller

$$u(t) = H_\varepsilon(p, t-kT)w(kT), \;\; t \in [kT, (k+1)T),$$

for $k \in \mathbf{Z}^+$ is applied to the plant, we label the corresponding responses by x^ε, y^ε, and u^ε. In both cases, we omit the dependence of w, x, and u on the parameter $p \in M$. One last piece of notation: when we apply the above sampled-data controller to the plant we obtain

$$\begin{aligned} x(t) \;\; &= \;\; \Big[e^{A(t-kT)} + \int_0^{t-kT} e^{A(t-kT-\tau)} B H_\varepsilon(p,\tau) \times \\ & \quad\;\; WO_m(C,A) d\tau \Big] \, x(kT) \\ &=: \;\; \Phi_\varepsilon^p(t-kT, 0)x(kT), \;\; t \in [kT, (k+1)T). \end{aligned}$$

Proposition 1: There exists a $\bar{\varepsilon} > 0$, a constant $\gamma_0 > 0$, and a constant $\lambda_0 < 0$ so that for every $\varepsilon \in (0, \bar{\varepsilon})$, $x_0 \in \mathbf{R}^n$, $T \in (0, T_{max})$ and $p \in M$, we have that

$$\|x^0(t) - x^\varepsilon(t)\| \;\; \le \;\; \varepsilon \gamma_0 e^{\lambda_0 t}\|x_0\|, \qquad (8)$$

$$\|u^0(t) - u^\varepsilon(t)\| \;\; \le \;\; \varepsilon \gamma_0 e^{\lambda_0 t}\|x_0\|, \qquad (9)$$

$$\|\Phi_\varepsilon^p(T,0)^k\| \;\; \le \;\; \gamma_0 e^{\lambda_0 kT}, \;\; k \ge 0, \qquad (10)$$

and

$$\left\| \begin{bmatrix} y^\varepsilon \\ r^{1/2}u^\varepsilon \end{bmatrix} - \begin{bmatrix} y^0 \\ r^{1/2}u^0 \end{bmatrix} \right\|_2^2 \le \varepsilon^2 \gamma_0^2 \|x_0\|^2. \qquad (11)$$

At this point we apply Proposition 1 and choose $\bar{\varepsilon} > 0$, a constant $\gamma_0 > 0$, and a constant $\lambda_0 < 0$ which have the required properties; we freeze $\varepsilon \in (0, \bar{\varepsilon})$ and proceed.

A. Polynomial Notation

Here we adopt the notation of [5] and modify it to our needs, which we now quickly summarize. The goal is to parametrize our polynomial approximation in such a way that we can estimate the various terms in a straight-forward and systematic fashion. Following Rudin [9], we introduce the notion of a multi-index, which is an ordered $m+1$-tuple

$$\alpha = (\alpha_1, ..., \alpha_{m+1}), \;\; \alpha_i \in \mathbf{Z}^+.$$

For such a multi-index, we can define

$$|\alpha| := \alpha_1 + \cdots + \alpha_{m+1} \text{ and } (p,t)^\alpha := p_1^{\alpha_1} \cdots p_m^{\alpha_m} t^{\alpha_{m+1}};$$

since we are dealing with integer exponents, we define $0^0 := \lim_{x \to 0} x^0 = 1$. Hence, given that H_ε is a polynomial which maps $\mathbf{R}^{m+1} \longmapsto \mathbf{R}^{1\times n}$, it follows that there exists a finite index set $I \subset (\mathbf{Z}^+)^{m+1}$ and constant matrices $c_\alpha \in \mathbf{R}^{1\times n}$, $\alpha \in I$, so that we can write H_ε in the form

$$H^\varepsilon(p,t) = \sum_{\alpha \in I} (p,t)^\alpha c_\alpha, \;\; (p,t) \in M \times [0, T_{max}].$$

Now we turn to the realization problem. From Section III,

$$w(t) = WO_m(C,A)x(t) \in \mathbf{R}^n$$

can be easily estimated using the KEL as motivation. Now consider the problem of estimating

$$\sum_{\alpha \in I} (p,t)^\alpha c_\alpha w(t). \qquad (12)$$

We define q to be the largest multi-index of the first m elements and $\bar{q}$ to be the largest index of the $m+1^{th}$ element:

$$q := \max_{\alpha \in I} \sum_{i=1}^m |\alpha_i| \text{ and } \bar{q} := \max_{\alpha \in I} |\alpha_{m+1}|.$$

From the KEL we know that for each $j \in \{1, ..., n\}$, it is possible to estimate $pw_j(t)$ by carrying out a simple experiment. Indeed, by doing a succession of n experiments we can estimate $w(t) \otimes p \in \mathbf{R}^{nm}$. Using the same logic, we can estimate $(w(t) \otimes p) \otimes p \in \mathbf{R}^{nm^2}$ using a succession of nm experiments, and so on. To this end, we now define

$$w(t) \otimes^0 p := w(t) \text{ and } w(t) \otimes^{i+1} p := (w(t) \otimes^i p) \otimes p, \;\; i \in \mathbf{N};$$

notice that $w(t) \otimes^i p \in \mathbf{R}^{nm^i}$. It is easy to see that the vector $w(t) \otimes^i p$ contains all possible terms of the form $\{(p,t)^\alpha w_j(t) : |\alpha| = i, \alpha_{m+1} = 0, \;\; j = 1, ..., n\}$. Hence, (12) can be rewritten: we can choose row vectors $d_{i,j}$ of length nm^i so that

$$\sum_{\alpha \in I} (p,t)^\alpha c_\alpha w(t) = \sum_{j=0}^{\bar{q}} t^j \sum_{i=0}^{q} d_{i,j} (w(t) \otimes^i p).$$

In the next section we use the KEL to iteratively estimate terms in the second summation on the RHS.

Remark 1: Perhaps the most problematic feature of this approach is that of obtaining a closed form description of $H(p,t)$ and constructing the approximation $H_\varepsilon(p,t)$. As discussed in [5], unless special structure is available, the best approach is a numerical one: grid the M parameter space, compute the optimal gain at each point on the grid, then fit a good polynomial approximation to it; unfortunately this will be difficult to do if m or the set of parameter uncertainty is large.

V. The Controller

Here we adopt the notation from Section IV and combine it with the KEL to design an algorithm to implement the proposed control law, the general operation of which was briefly discussed at the end of Section II. This proposed control law is periodic of period T; we begin by describing its open loop behaviour on a period of the form $[kT,(k+1)T)$. With p defined in (5) we have

$$\begin{aligned} u^{\varepsilon}(t) &= H_{\varepsilon}(p,t-kT)WO_m(C,A)x(t) \quad (13)\\ &= H_{\varepsilon}(p,t-kT)w(t)\\ &= \sum_{j=0}^{\bar{q}}(t-kT)^j\sum_{i=0}^{q}d_{i,j}(w(t)\otimes^i p). \quad (14)\end{aligned}$$

Following Section III.A, envision setting

$$u(t)=0,\ \ t\in[kT,kT+2mh),$$

so it follows from the KEL that a good estimate of $w(kT)$ is given by

$$\begin{aligned}\hat{w}(kT) &:= \text{Est}[w(kT)]=\text{Est}[w(kT)\otimes^0 p]\\ &:= \text{argmin}\{\|WH_m(h)^{-1}S_m^{-1}\mathcal{Y}_m(kT)\|,\\ &\qquad \|WH_m(h)^{-1}S_m^{-1}\mathcal{Y}_m(kT+mh)\|\}\\ &= w(kT)+\mathcal{O}(h)x(kT),\end{aligned}$$

with the last equality holding if there is no plant switch on $[kT,kT+2mh)$. To estimate terms of the form $w(kT)\otimes^i p,\ \ i=1,...,q$, recall that $w(kT)\otimes^i p$ are column vectors of height $nm^i=:n_i$. With $\rho>0$ a scaling factor, set

$$u(t)=\begin{cases}\rho\hat{w}_1(kT) & t\in[kT+2mh,kT+4mh),\\ \vdots & \vdots\\ \rho\hat{w}_n(kT) & t\in[kT+2nmh,kT+2(n+1)mh).\end{cases}$$

It follows from the KEL and the discussion of Section III.A that we should define

$$\begin{aligned}&\text{Est}[pw_i(kT)]=\tfrac{1}{\rho}\left[\begin{array}{cc}0 & I_m\end{array}\right]\times\\ &\text{argmin}\{\|H_m(h)^{-1}S_m^{-1}[\mathcal{Y}_m(kT+2mh)-\mathcal{Y}_m(kT)]\|,\\ &\|H_m(h)^{-1}S_m^{-1}[\mathcal{Y}_m(kT+3mh)-\mathcal{Y}_m(kT+mh)]\|\}\\ &\qquad\qquad =pw_i(kT)+\mathcal{O}(h)x(kT)\end{aligned}$$

for $i=1,...,n$, with the last equality holding if there is no plant switch on $[kT,kT+4mh)$. By stacking these estimates we can obtain an estimate of $w(kT)\otimes p$, which we label Est$[w(kT)\otimes p]$. Of course, now we can estimate $w(kT)\otimes^2 p$ in an analogous way, probing with successive elements of Est$[w(kT)\otimes p]$; since $w(kT)\otimes p$ is of dimension $n_1=nm$, this will take n_1 experiments, each of length $2mh$, yielding a total of $nm(2mh)=2n_2h$ units of time. This can be repeated in the same fashion to yield estimates of $w(kT)\otimes^i p$, $i=3,...,q$, with the i^{th} term taking $2n_ih$ units of time. We can now construct a good estimate of

$$\sum_{j=0}^{\bar{q}}(t-kT)^j\sum_{i=0}^{q}d_{i,j}(w(t)\otimes^i p)$$

to be applied during the Control Phase.

To this end, we define certain important points in time:

$$T_1:=2mh=\text{ the time to estimate } w(kT),$$

$$T_{i+1}=T_i+2n_ih,\ \ i=1,...,q.$$

The idea is that on the interval $[kT,kT+T_1)$ we estimate $w(kT)$, while on the interval $[kT+T_i,kT+T_{i+1})$ we estimate $w(kT)\otimes^i p$. Last of all, with $T>T_{q+1}$ an integer multiple of h, on the interval $[kT+T_{q+1},(k+1)T)$ we implement the Control Phase. With this in mind, we can now write down our proposed controller, presented in open loop form. To make this more transparent, we partition each interval $[T_i,T_{i+1})$, $i=1,...,q$, into n_{i-1} consecutive sub-intervals of length $2mh$ on which probing takes place:

$$[T_i,T_{i+1})=[T_{i,1},T_{i,2})\cup\cdots\cup[T_{i,n_{i-1}},T_{i,n_{i-1}+1}).$$

Now introduce the $n_i\times(n_i+n_{i-1})$ matrix $V_i(h)$, $i\in\mathbf{N}$, which consists of n_{i-1} copies of $\left[\begin{array}{cc}O & I_m\end{array}\right]H_m(h)^{-1}S_m^{-1}$ arranged in a block diagonal form. The proposed sampled-data controller is given in three phases - with $\rho>0$ a scaling factor and for each $k\in\mathbf{Z}^+$:

State Estimation Phase: $[kT,kT+T_1)$

Set

$$u(t)=0,\ \ t\in[kT,kT+T_1)=[kT,kT+2mh), \quad (15)$$

and define

$$\begin{aligned}\text{Est}[w(kT)\otimes^0 p]:=\text{argmin}\{&\|WH_m(h)^{-1}S_m^{-1}\mathcal{Y}_m(kT)\|,\\ &\|WH_m(h)^{-1}S_m^{-1}\mathcal{Y}_m(kT+mh)\|\}.\end{aligned}$$

Control Estimation Phase: $[kT+T_1,kT+T_{q+1})$

For $i=1,...,q$ and $j=1,...,n_{i-1}$, set

$$u(t)=\rho\text{Est}[w(kT)\otimes^{i-1}p]_j,$$

$$t\in[kT+T_{i,j},kT+T_{i,j}+2mh), \quad (16)$$

and define

$$\text{Est}[w(kT)\otimes^i p]:=\frac{1}{\rho}\text{argmin}\{$$

$$\left\|V_i(h)\left[\begin{array}{c}\mathcal{Y}_m(kT+T_{i,1})-\mathcal{Y}_m(kT)\\ \vdots\\ \mathcal{Y}_m(kT+T_{i,n_{i-1}})-\mathcal{Y}_m(kT)\end{array}\right]\right\|,$$

$$\left.\left\|V_i(h)\left[\begin{array}{c}\mathcal{Y}_m(kT+T_{i,1}+mh)-\mathcal{Y}_m(kT+mh)\\ \vdots\\ \mathcal{Y}_m(kT+T_{i,n_{i-1}}+mh)-\mathcal{Y}_m(kT+mh)\end{array}\right]\right\|\right\}. \quad (17)$$

Control Phase: $[kT+T_{q+1},(k+1)T)$

$$u(t)=\sum_{j=0}^{\bar{q}}(t-kT)^j\sum_{i=0}^{q}d_{i,j}\text{Est}[w(kT)\otimes^i p(kT)],$$

$$t\in[kT+T_{q+1},(k+1)T). \quad (18)$$

At this point we examine the behaviour of the closed loop system over a single period $[0,T)$ with $T\in(0,T_{max}]$. To proceed, we let $\hat{x}^{\varepsilon}$, $\hat{y}^{\varepsilon}$ and $\hat{u}^{\varepsilon}$ denote the closed loop state response, output response, and control signal, respectively, when the proposed controller is applied.

Lemma 3: (One Period Lemma) There exist constants $\gamma > 0$ and $\bar{T}_{q+1} \in (0, T_{max})$ so that for every $T \in (0, T_{max})$, $T_{q+1} \in (0, \min\{T, \bar{T}_{q+1}\})$, $P \in \mathcal{P}_{T_{max}}$, and $k \in \mathbf{Z}^+$:
(i) If $P(t)$ is constant on $[kT, (k+1)T)$ then

$$\|\hat{x}^\varepsilon(t) - \Phi_\varepsilon^p(t-kT, 0)\hat{x}^\varepsilon[kT]\| \le \gamma T_{q+1}\|\hat{x}^\varepsilon[kT]\|, \quad t \in [kT, (k+1)T],$$

$$|\hat{u}^\varepsilon(t) - H^\varepsilon(p, t-kT)WO_m(C, A)\hat{x}^\varepsilon[kT]| \le \gamma T_{q+1}\|\hat{x}^\varepsilon[kT]\|, \quad t \in [kT + T_{q+1}, (k+1)T).$$

(ii) In all cases

$$\|\hat{x}^\varepsilon(t) - \hat{x}^\varepsilon(kT)\| \le \gamma T\|x^\varepsilon[kT]\|,$$

$$|\hat{u}^\varepsilon(t)| \le \gamma\|\hat{x}^\varepsilon(kT)\|, \quad t \in [kT, (k+1)T).$$

VI. Performance and Stability for $\mathcal{P}_\infty$

Here we analyze the closed loop behaviour when there are no plant changes. We first look at stability.

Theorem 1: For every $T \in (0, T_{max})$ there exists a $\bar{T}_{q+1} \in (0, T)$ so that for all $T_{q+1} \in (0, \bar{T}_{q+1})$, the controller (15)-(18) stabilizes every plant in $\mathcal{P}$.

For performance we also need to have to have $\varepsilon > 0$ small.

Theorem 2: For every $\delta > 0$ and $T \in [0, T_{max})$, there exists a controller of the form (3) which stabilizes every $p \in M$ and which, for every $x_0 \in \mathbf{R}^n$ and $p \in M$, yields a closed loop system response which satisfies

$$\left\| \begin{bmatrix} y \\ r^{1/2}u \end{bmatrix} - \begin{bmatrix} y^0 \\ r^{1/2}u^0 \end{bmatrix} \right\|_2^2 \le \delta\|x_0\|^2.$$

VII. Stability in the Face of Plant Switches

Here we allow for persistent plant changes and show that our controller is stabilising under the condition that switches occur slowly enough. Let's start with the special case in which $\mathcal{P} = \{P_1, P_2\}$ and the time-varying plant simply switches back and forth between P_1 and P_2, spending τ_1 time units at P_1 and τ_2 time units at P_2, and then repeating - for each $k \in \mathbf{Z}^+$:

$$P(t) = \begin{cases} P_1 & t \in [k(\tau_1+\tau_2), k(\tau_1+\tau_2)+\tau_1) \\ P_2 & t \in [k(\tau_1+\tau_2)+\tau_1, (k+1)(\tau_1+\tau_2)). \end{cases}$$

We choose the LQR-optimal feedback law which corresponds to the plant at those times:

$$u(t) = \begin{cases} F_1 x(t) & t \in [k(\tau_1+\tau_2), k(\tau_1+\tau_2)+\tau_1) \\ F_2 x(t) & t \in [k(\tau_1+\tau_2)+\tau_1, (k+1)(\tau_1+\tau_2)). \end{cases}$$

In this case, stability is dictated by the eigenvalues of the matrix

$$e^{(A_2+B_2F_2)\tau_2} e^{(A_1+B_1F_1)\tau_1} \qquad (19)$$

with a sufficient condition being that

$$\|e^{(A_2+B_2F_2)\tau_2} e^{(A_1+B_1F_1)\tau_1}\| < 1. \qquad (20)$$

Of course, if one knows in advance that the time-varying plant is as indicated, then one can always stabilize the system with a more cleverly designed controller even if (19) has eigenvalues outside the open unit disk; unfortunately, in our case no such a priori information is available.

This brings us to the general case. Using (20) as motivation, a sufficient condition for stability should be

$$\sup_{(A_i,B_i,C_i)\in\mathcal{P}} \sup_{\tau_i > T_s} \|e^{A_{cl}(A_1,B_1,C_1)\tau_1} e^{A_{cl}(A_2,B_2,C_2)\tau_2}\| < 1.$$

To simplify this condition, observe that because of compactness there exists a $\gamma > 0$ and $\lambda < 0$ so that

$$\|e^{A_{cl}(A,B,C)t}\| \le \gamma e^{\lambda t}, \;\; t \ge 0, \;\; (A, B, C) \in \mathcal{P}. \qquad (21)$$

Theorem 3: If $\gamma > 0$ and $\lambda < 0$ satisfy (21) and

$$T_s > \frac{\ln(\gamma)}{-\lambda},$$

then there exists a $\bar{T} \in (0, T_s/2)$ so that for every $T \in (0, \bar{T})$, there exists a $\bar{T}_{q+1} \in (0, T)$ so that for every $T_{q+1} \in (0, \bar{T}_{q+1})$ the controller (15)-(18) stabilizes $\mathcal{P}_{T_s}$.

VIII. Summary and Concluding Remarks

In this paper we consider the problem of designing a controller for a compact set of models which tolerates occasional, but persistent, switches between these models. Here we show how to construct a mildly nonlinear periodic controller which stabilizes every admissible model, provides near optimal LQR performance for every admissible model, and provides stability in the presence of occasional, persistent, switches between these models; we provide an easily computable bound on how often a switch is allowed. Although we have not discussed this here, it can be proven that near optimality (between plant switches) can be maintained if the proposed controller is designed properly.

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