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Parameter continuity of the ergodic cost for a growth optimal portfolio with proportional transaction costs

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***Abstract*— Some results are given for a continuous time long run growth optimal portfolio that has proportional costs consisting of the sum of a fixed proportional cost and a cost that is proportional to the volume of each transaction. An obligatory portfolio diversification is given that requires at least a small portion of the wealth be invested in each asset. It is assumed that the price of each asset is obtained from a Lévy noise stochastic equation whose coefficients depend on an unknown parameter from a compact set. It is shown that the optimal cost is a continuous function of the unknown parameter.**

I. Introduction

To investigate the continuity of the optimal cost, it is first necessary to formulate the portfolio optimization problem and to verify some properties of the solution. Assume that on a given complete probability space $(\Omega,\mathscr{F},\mathscr{P})$, there are given three independent processes: a d-dimensional standard Brownian motion $(B(t))$, a d-dimensional compensated Poisson random measure $\tilde{N}(dt,du)$ and a time homogeneous Markov process $(z(t))$ with values in a finite space D and transition matrix P_t at time t. There are also d assets with the ith asset price $S_i(t)$ at time t. It is assumed that the evolution of $S_i(t)$ is of the form

$$\delta_i(t) = \delta_i(0)e^{X_i^{\theta^0}}(t) \tag{I.1}$$

where $X_i(0)=0$ and $X^{\theta^0}(t) = (X_1(t),\ldots,X_d(t))$ is a solution to the following Lévy stochastic differential equation:

$$dX^{\theta^0}(t) = \alpha(z(t),\theta^0)\,dt + \sigma(z(t),\theta^0)\,dB(t) + \int_{\mathbb{R}^n} \gamma(z(t),\theta^0,u)\,\tilde{N}(dt,du)\,. \tag{I.2}$$

θ^0 is a parameter that belongs to a compact space Θ. We assume that α, σ, and γ are continuous bounded functions of θ, and

$$\sup_{\theta}\int_{\mathbb{R}} \gamma_{i_k}^2(z,\theta,u)\,\nu_k(du) < \infty$$

for $i,k=1,2,\ldots,d$ with ν_k being the Lévy measure corresponding to $\tilde{N}(dt,du)$. While the solution of (I.2) depends on θ^0, sometimes for notational simplicity this dependence is suppressed.

In what follows, let $\pi_i(t)$ be the portion of our wealth process invested in the ith asset. Let $e^{X(t)} = \left(e^{X_1(t)},\ldots,e^{X_d(t)}\right)$ and for $\pi,\zeta\in[0,\infty)^d$,

$$\pi\diamond\zeta = (\pi_1\zeta_1,\pi_2\zeta_2,\ldots,\pi_d\zeta_d)$$

and

$$g(\zeta) = \left(\frac{\zeta_1}{\sum\zeta_i},\frac{\zeta_2}{\sum\zeta_i},\ldots,\frac{\zeta_d}{\sum\zeta_i}\right)\,.$$

If the portfolio strategy is not changed in the time interval $[0,T]$, the portions of the wealth invested in the assets at time T are of the form

$$\pi(T) = g\left(\pi(0)\diamond e^{X(T)}\right)\,. \tag{I.3}$$

By the form of (I.2), it is clear that the pair $(\pi(t),z(t))$ is a Markov process with transition operator Π_t.

It is assumed that

(A1) The solution to (I.2) with the initial condition $X(0)=x$ has a continuous density for each fixed $z(t)=z$ with respect to the Lebesgue measure l^d at time $t>0$ – i.e., for a Borel set $A\subset\mathbb{R}^d$,

$$P_{X_z}\{X^z(t)\in A\} = \int_A P^z(x,x')\,l^d(dx') \tag{I.4}$$

where $X^z(t)$ is a solution to (I.2) with $z(t)\equiv z$, and $P^z(x,x')$ is a continuous function of x and x'.

Sufficient conditions for (A1) can be found in [1], [8], [10], [13].

Since $(z(t))$ is a finite state continuous time, time homogeneous Markov process, its evolution can be described in the following form:

$$\tau_1 = \inf\{s\geq 0\colon z(s)\neq z(0)\}$$
$$\tau_{n+1} = \inf\{s\geq 0\colon z(s+\tau_n)\neq z(\tau_n)\}$$

for $z(0)=z$

$$P_z\{\tau_1\leq t\} = \int_0^t n(z,s)\,ds$$
$$P_z\{P_{z(\tau_n)}\{t_{n+1}^0\leq t\}\} = E_z\left\{\int_0^t n(z(\tau_n),s)\,ds\right\}$$
$$P_z\{z(\tau_1)=z'\} = P(z,z')\,. \tag{I.5}$$

The following continuity property will be crucial in further investigations:

Proposition 1: Under (A1), the operator Π_t is continuous in variation norm for $(\pi,z)\in S_\delta\times D$ – i.e., for $(\pi_{(n)},z)\to$

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$(\pi,z)\in S_\delta\times D$, and $(\pi_{(n)},n\geq 1)$ is a sequence in S_δ, it follows that

$$\sup_{A\in B(S\times D)}\left|\Pi_t(\pi_{(n)},z,A)-\Pi_t(\pi,z,A)\right|\to 0 \qquad \text{(I.6)}$$

as $n\to\infty$, with

$$S=\{v=(v_1,\ldots,v_d),v_i\geq 0,\sum v_i=1\}$$

and

$$S_\delta=\{v\in S,v_i\geq\delta,i=1,2,\ldots,d\}$$

for $0<\delta<1/d$.
This result is verified in [7].

Using the same arguments as in the proof of Proposition 1, there is

Corollary 1: If the transition density $p_t^\theta(x,x',z,z')$ for $(X^\theta(t)),(z(t))$ depends in a continuous way on θ, x, and x', then for $(\theta_n,\pi_{(n)})\to(\theta,\pi)\in\Theta\times S_\delta$ and $z\in D$,

$$\sup_{A\in B(S\times D)}\left|\Pi_t^{\theta_n}\left(\pi_{(n)},z,A)-\Pi_t^\theta(\pi,z,A)\right)\right|\to 0\,. \qquad \text{(I.7)}$$

Having shown the properties of the uncontrolled process $\pi(t)$, consider the control problem. Assume that proportional transaction costs are a sum of a fixed proportional managing cost and a cost proportional to the volume of transactions. Let ζ_i^-, $i=1,2,\ldots,d$ denote the amount of wealth process invested in the ith asset. Clearly, $W^-=\sum_{i=1}^d\zeta_i^-$ is the wealth before a possible transaction. To change the portfolio to $(\zeta_1,\zeta_2,\ldots,\zeta_d)$ requires paying immediately transaction costs of the form

$$kW^-+\sum_{i=1}^d c_i^1(\zeta_i-\zeta_i^-)^+ + c_i^2(\zeta_i-\zeta_i^-)^- \qquad \text{(I.8)}$$

with $k>0$ corresponding to a fixed managing cost. Short selling or short borrowing are not allowed and assume that the portfolio is self-financing. Therefore, the wealth W after the transaction is equal to

$$W^- - ksW^- - \sum_{i=1}^d c_i^1(\zeta_i-\zeta_i^-)^+ + c_i^2(\zeta_i-\zeta_i^-)^-\,. \qquad \text{(I.9)}$$

Let $\pi_i^-=\zeta_i^-/W^-$ and $\pi_i=\zeta_i/W$ be respectively the portion of wealth invested in the ith asset before and after transaction. From (I.9), it follows that

$$1-k-\sum_{i=1}^d c_i^1\left(\pi_i\frac{W}{W^-}-\pi_i^-\right)^+ + c_i^2\left(\pi_i\frac{W}{W^-}-\pi_i^-\right)=\frac{W}{W^-}$$

or

$$c\left(\pi\frac{W}{W^-}-\pi^-\right)+\frac{W}{W^-}=1\,,$$

with

$$c(v)=k+\sum_{i=1}^d\left(c_i^1v_i^+ + c_i^2v_i^-\right)\,.$$

In what follows, assume that $0<c_i^1,c_i^2<1-k$.

It appears that starting from portfolio $(\pi_1^-,\pi_2^-,\ldots,\pi_d^-)$ the portfolio $(\pi_1,\pi_2,\ldots,\pi_d)$ is available. Naturally it follows that (see Lemma 1 of [14], or [15])

Lemma 1: There is a unique continuous function $e\colon S\times S\to(0,1-k]$ such that for $\pi^-,\pi\in S$,

$$c(\pi e(\pi^-,\pi)-\pi^-)+e(\pi^-,\pi)=1\,. \qquad \text{(I.10)}$$

The function e is bounded away from zero and

$$e(\pi,\pi')e(\pi',\pi'')<e(\pi,\pi'')\,,$$

which means that it is not profitable to make two instantaneous portfolio changes. The wealth process W^- after the change of portfolio from π^- to π is diminished to $e(\pi^-,\pi)W^-=W$.

Denote by $W^-(t)$, $W(t)$, $\pi^-(t)$, $\pi(t)$, the wealth process before and after transaction or the portfolio before and after transaction at time t respectively. The purpose is to maximize the following long run wealth growth rate:

$$y^{\theta^0}((\pi(t))=\liminf_{T\to\infty}\frac{1}{T}E_{\pi z}\{\ln W(T)\}\,. \qquad \text{(I.11)}$$

Since $k>0$, the strategy is of impulsive form – i.e. it is a sequence $V=(\tau_n,\pi^n)$ consisting of transaction times (stopping times τ_n for $n=1,2,\ldots$) and portfolios π^n which are chosen at time τ_n. Thus

$$W(t)=W(\tau_n)\sum_{i=1}^d\pi_i(\tau_n)e^{X_i(t)-X_i(\tau_n)} \qquad \text{(I.12)}$$

$$\pi(t)=g\left(\pi(\tau_n)\diamond e^{X(t)-X(\tau_n)}\right) \qquad \text{(I.13)}$$

for $\tau_n<t<\tau_{n+1}$, and

$$W(\tau_n)=e(\pi(\tau_n^-),\pi^n)W^-(\tau_n)\,. \qquad \text{(I.14)}$$

Additionally the portfolio $\pi(t)$ is not allowed to be too close to the boundary of the simplex S. An obligatory diversification of the portfolio is introduced. Let $0<\delta<\delta'<1/d$, and

$$S_\delta^0=\{v\in S\colon v_i>\delta\ \text{for}\ i=1,2,\ldots,d\}\,.$$

As soon as the portfolio $(\pi(t))$ enters the set $S\setminus S_\delta^0$, it is changed by choosing a new portfolio from the set $S_{\delta'}$. Both parameters δ and δ' are assumed to be fixed in the paper. The following remark justifies the use of obligatory portfolio diversification:

Remark 1: Assume that there is a unique invariant measure μ^{θ^0} for Markov process $(z(t))$. Under the assumptions, the law of large numbers for the martingale

$$\int_0^t\sigma(z(s),\theta)\,dB(s)+\int_0^t\int_{\mathbb{R}^d}\gamma(z(s),\theta,u)\tilde{N}(ds,du)$$

is applicable and therefore

$$\lim_{t\to\infty}\frac{1}{t}X_i(t)=\lim_{t\to\infty}\frac{1}{t}\int_0^t\alpha_i(z(s),\theta^0)\,ds = \sum_{z'\in D}\alpha_i(z',\theta^0)\,\mu^{\theta^0}(z')=r_i^{\theta^0}\quad \mathbb{P}\ \text{a.e.}$$

Consequently, $X_i(t)$ is of order $tr_i^{\theta^0}$. If $r_i^{\theta^0}>r_j^{\theta^0}$, for $j\neq i$, then provided that $\pi_i(0)>0$ and the portfolio is not changed,

$$\pi_i(t)=\frac{\pi_i(0)e^{X_i(t)}}{\sum_j\pi_j(0)e^{X_j(t)}}\to 1\quad \mathbb{P}\ \text{a.e.}$$

as $t\to\infty$, while $\pi_j(t)\to 0$ for $j\neq i$ as $t\to\infty$ $\mathbb{P}$ a.e.

In other words, assuming that the $r_i^{\theta^0}$ are not the same for $i=1,2,\ldots,d$, the process $\pi(t)$ in the limit converges to the boundary of S, provided that $\pi(0)>0$. As a consequence,

$$\liminf_{T\to\infty}\frac{1}{T}E\left[\ln W(t)\right]=\max_{i=1,2,\ldots,d} r_i^{\theta^0}\ .$$

This is the value of the wealth process that can be guaranteed over the long run. It may happen however that it is more profitable to change the portfolio regularly than just wait for the guaranteed value.

The portfolios from the boundary of S are unacceptable from a risk sensitivity point of view. To eliminate risk, a portfolio is diversified. Therefore, in the paper an obligatory diversification is considered.

A model with fixed proportional transaction costs ($k>0, c\equiv 0$) was studied by Morton and Pliska in [12]. A simple one asset Black Scholes model with fixed proportional plus proportional transaction costs was considered in [9], and the control was restricted to a diversification boundary and the choice of a new portfolio when this boundary was reached. In the paper [14], general discrete and continuous time models with an obligatory diversification were studied. For a continuous time model, a certain transaction delay was introduced, which played an important role in the proofs. This paper generalizes [14] in various directions. A more specific asset growth model based on Lévy noise is considered. As in [14], the vanishing discount approach is used. To obtain continuity results time discretization is used. The main result, existence of the smooth solutions to the ergodic Bellman equation is obtained by the continuity properties of the transaction operator and finiteness of the space D. This result is complemented by continuity properties of the optimal value, which allows to use adaptive control algorithms.

II. Discounted growth optimal portfolio

The transition operator Π_t of the Markov process $(\pi(t),z(t))$ has a nice continuity property (see Proposition 1). To use this property for a continuous time model, time discretization is considered. Let

$$\sigma=\inf\{s\geq 0\colon\, \pi(s)\in S\setminus S_\delta^0\}$$

and

$$\sigma_n=\inf\{2^{-n}s, s=0,1,2,\ldots,\pi(2^{-n}s)\in S\setminus S_\delta^0\}\ ,$$

i.e. σ and σ_n are first exit times of S_δ^0 for a continuous time or discrete time $2^{-n}s$ Markov processes $(\pi(t)),(\pi(2^{-n}s))$ respectively. Clearly, $\sigma_n\geq\sigma$. Assume furthermore that

(A2) $\sup_{z\in D}\sup_{\pi\in S_\delta^0}E_{\pi_2}\{\sigma\}<\infty$

(A3) $\sup_{z\in D}\sup_{\pi\in S_\delta^0}E_{\pi_2}\{\sigma_n-\sigma\}\to 0$ as $n\to\infty$.

(A4) there is $T>0$ and $\Delta<1$ such that

$$\sup_{z,z'\in D}\sup_{A\subset D}\left|P_T(z,A)-P_T(z',A)\right|=\Delta<1\ .$$

Remark 2: It is clear in view of Remark 1 that if $r_i^{\theta^0}$ are not the same for all i, then $\sigma<\infty$ $\mathbb{P}$ a.e. The fact that $E_{\pi z}\{\sigma\}<\infty$ follows mainly from non-degeneracy of the diffusion term in the equation for $(\pi(t))$. The assumption (A3) is typical in diffusion approximations (see the assumption A22 in [11]) and as was justified in [11] it holds for uniformly elliptic diffusion terms. Applying Ito's formula to the function $f_i(x_1,\ldots,x_d)=(\pi_i e^{X_i})/(\sum\pi_j e^{X_j})$, a stochastic differential equation for $\pi_i(t)$ is obtained. The Brownian motion coefficient (row) is in the form

$$\begin{aligned} r_i(t) &= (r_i(t)-(\pi_i(t))^2)\sigma_i(z(t),\theta^0)\\ &\quad -\sum_{j\neq i}^{d}\pi_i(t)\pi_j(t)\sigma_j(z(t),\theta^0)\\ &= \pi_i(t)\sum_{j=1}^{d}\tilde{\pi}_j^i(t)\sigma_j(z(t),\theta^0) \end{aligned} \tag{II.1}$$

with $\tilde{\pi}_i^i(t)=1-\pi_i(t)$ and $\tilde{\pi}_j^i(t)=-\pi_j(t)$ for $i\neq j$. The matrix $r(t)r^T(r)$ is not uniformly elliptic. It is, however, uniformly elliptic on the subspace orthogonal to the vector $\mathbb{1}=(1,1,\ldots,1)$.

Consider now the so-called discounted cost functional

$$J_{\pi_2}^{\beta\theta^0}(V)=E_{\pi_2}\left\{\sum_{i=1}^{\infty}e^{-\beta\tau_i}\left[\ln\left(\pi(\tau_{i-1})e^{X(\tau_i)-X(\tau_{i-1})}\right)\right.\right.$$
$$\left.\left.+\ln e(\pi^-(\tau_i),\pi(\tau_i))\right]\right\}\ . \tag{II.2}$$

Let

$$w^{\beta\theta^0}(\pi,z)=\sup_V J_{\pi z}^{\beta\theta^0}(V)\ . \tag{II.3}$$

Theorem 1: Under (A1)–(A3), $w^{\beta\theta^0}$ is a bounded function continuous on S_δ^0 and $S\setminus S_\delta^0$ and is the unique solution to the following Bellman equation:

$$w^{\beta\theta^0}(\pi,z)=\sup_\tau E_{\pi z}\left\{e^{-\beta\tau\wedge\sigma}\left[\ln(\pi e^{X(\tau\wedge\sigma)})\right.\right.$$
$$\left.\left.+\ Mw^{\beta\theta^0}(\pi(\tau\wedge\sigma),z(\tau\wedge\sigma))\right]\right\}\ , \tag{II.4}$$

with

$$Mw(\pi,z)=\sup_{\pi^1\in S_{\delta'}}\left[\ln e(\pi,\pi^1)+w(\pi^1,z)\right]\ . \tag{II.5}$$

This result is verified in [7].

We now rewrite the Bellman equation (II.4) in terms of a bounded (by Proposition 3 of [7]) function $h^{\beta\theta^0}$. Thus

$$h^{\beta\theta^0}(\pi,z)=\sup_\tau E_{\pi z}\left\{e^{-\beta\tau\wedge\sigma}\left[\ln\left(\pi e^{X(\tau\wedge\sigma)}\right)\right.\right.$$
$$\left.+Mh^{\beta\theta^0}(\pi(\tau\wedge\sigma),z(\tau\wedge\sigma))\right]$$
$$\left.-\inf_{\pi'\in S z'\in D}w^{\beta\theta^0}(\pi',z')(1-e^{-\beta\tau\wedge\sigma})\right\}\ . \tag{II.6}$$

An important property for the growth optimal portfolio is the following result whose proof is given in [7].

Theorem 2: Under (A1)–(A4) there exist a constant λ^{θ^0} and a continuous bounded function w^{θ^0} such that

$$w^{\theta_0}(\pi,z)=\sup_\tau E_{\pi z}\left\{\ln\left(\pi e^{X(\tau\wedge\sigma)}\right)-\lambda^{\theta^0}(\tau\wedge\sigma)\right.$$
$$\left.+Mw^{\theta_0}(\pi(\tau\wedge\sigma),z(\tau\wedge\sigma))\right\}\ . \tag{II.7}$$

Moreover,

$$\lambda^{\theta_0} = \sup_V J^{\theta^0}(V) \,, \tag{II.8}$$

i.e., λ^{θ_0} is the optimal value of the cost functional (I.11) and the strategy $\hat{V} = (\hat{\tau}_n, \hat{\pi}^n)$ such that

$$\hat{\tau} = \inf\left\{ s \geq 0 \colon w^{\theta_0}(\pi(s), z(s)) = Mw^{\theta_0}(\pi(s), z(s)) \right\} . \tag{II.9}$$

$$\begin{aligned} \hat{\tau}_1 &= \hat{\tau} \\ \hat{\tau}_{n+1} &= \hat{\tau}_n + \hat{\tau} \circ \theta_{\tau_n} \end{aligned} \tag{II.10}$$

and

$$\hat{\pi}^n = \hat{\pi}(\pi^-(\hat{\tau}_n, z(\hat{\tau}_n))$$

where $\hat{\pi} \colon S \times D \to S_{\delta'}$ is a Borel function such that

$$Mw^{\theta_0}(\pi, z) = \ln e(\pi, \hat{\pi}(\pi, z)) + w^{\theta_0}(\hat{\pi}(\pi, z), z)$$

is optimal.

Proof: Note that

$$\inf_{K' \in S z' \in D} \beta w^{\beta\theta^0}(\pi', z')$$

is bounded so that there is a constant λ^{θ^0} and a sequence $\beta_n \downarrow 0$ such that

$$\inf_{\pi' \in S z' \in D} \beta_n w^{\beta\theta^0}(\pi', z') \to \lambda^{\theta^0}$$

as $n \to \infty$. Furthermore, by (A2),

$$E_{\pi z}\left\{ \frac{1}{\beta_n}\left(1 - e^{-\beta_n \tau\wedge\sigma}\right) \right\} \to E_{\pi z}\{\tau \wedge \sigma\}$$

as $n \to \infty$, and the limit is uniform in τ, π, and z. By Proposition 3 of [7], the functions $h^{\beta\theta^0}$ are bounded. Therefore, $Mh^{\beta\theta^0}(\pi, z)$ is uniformly continuous in $\pi \in D$ (use the continuity of e). One can therefore choose a subsequence of β_n, for simplicity again denoted by β_n, such that

$$Mh^{\beta_n\theta^0}(\pi, z) \to h^{\theta^0}(\pi, z) \tag{II.11}$$

uniformly, where $h^{\theta^0}(\pi, z)$ is a continuous function of π. Therefore, by (II.6), there is a continuous function w^{θ^0} such that

$$\sup_{\pi \in S}\sup_{z \in D}\left| h^{\beta_n\theta^0}(\pi, z) - w^{\theta^0}(\pi, z) \right| \to 0$$

as $n \to \infty$. From (II.11), it follows that

$$Mh^{\beta_n\theta^0}(\pi, z) \to Mw^{\theta^0}(\pi, z)$$

uniformly in $\pi \in S$, $z \in D$.

Finally, w^{θ_0} is a solution to (II.7). Equality (II.8) and the form of optimal strategy $\hat{V}$ follows from standard arguments. ■

III. CONTINUITY OF THE OPTIMAL COST

In this section the dependence of optimal values of the cost functional (I.11) on the parameter $\theta \in \Theta$ is studied. The solution to the equation (I.2) with the parameter θ will be now denoted by X^θ. By continuity of the parameters with respect to θ and Doob's inequality (see 1.42 of [16]), it follows that for each $T > 0$, whenever $\theta_n \to \theta$,

$$E\left\{ \sup_{s \leq T}\left| X^{\theta_n}(s) - X^\theta(s) \right| \right\} \to 0 \tag{III.1}$$

as $n \to \infty$. Denote by $\pi^\theta(t)$ the process $\pi(t)$ corresponding to the growth rate $(X^\theta(t))$. Let

$$\sigma^\theta = \inf\left\{ s \geq 0 \colon \pi^\theta(s) \in S \setminus S^0_\delta \right\} .$$

Assume that

(A5) $\sigma^{\theta_n} \to \sigma^\theta$ P a.e. whenever $\theta_n \to \theta$.

In view of Remark 2 and Lemma 3 of [7] the above convergence is natural for the uniformly elliptic diffusion term in (I.2). By Corollary 1, there is the convergence (in variation norm) of the transition operators Π^{θ_n} to Π^θ for $\theta_n \to \theta$. Therefore, repeating the arguments of section 2, it follows that the value function $w^{\beta\theta}$ corresponding to the discounted cost functional (II.2) is a continuous function of θ. Define the function

$$h^{\beta\theta}(\pi, z) = w^{\beta\theta}(\pi, z) - \inf_{\pi' \in S z' \in D} w^\beta(\pi', z') .$$

As was already pointed out in Proposition 3 of [7], $h^{\beta\theta}$ is a function uniformly bounded in β and θ. There is the following continuity result:

Theorem 3: Given (A1)–(A5), the optimal values λ^θ of the cost functional J^θ are continuous functions of the parameter θ, namely $\lambda^{\theta_n} \to \lambda^\theta$ whenever $\theta_n \to \theta$.

Proof: For each θ, by Theorem 2, there is a constant λ^θ (optimal value of J^θ) and a continuous function w^θ such that

$$\begin{aligned} w^\theta(\pi, z) = \sup_\tau E_{\pi z}\Big\{ &\ln\left(\pi e^{X^\theta(\tau\wedge\sigma^\theta)}\right) - \lambda^\theta(\tau \wedge \sigma^\theta) \\ &+ Mw^\theta(\pi(\tau \wedge \sigma^\theta), z(\tau \wedge \sigma^\theta), z(\tau \wedge \sigma^\theta)) \Big\} . \end{aligned} \tag{III.2}$$

Since $h^{\beta\theta}$ was bounded, the function w^θ is also bounded in $\theta \in \Theta$. By the assumption, λ^θ is also bounded. Let $\theta_n \to \theta$. Choosing a suitable subsequence h_k, $\lambda^{\theta_{h_k}} \to \lambda$ and $Mw^{\theta_{h_k}}(\pi, z) \to h^\theta(\pi, z)$ uniformly. Therefore, using (III.1), (A5), and (A2), it follows that

$$\begin{aligned} w^{\theta_{n_k}}(\pi, z) \to w^\theta(\pi, z) = \sup_\tau E_{\pi z}\Big\{ &\ln\left(\pi e^{X^\theta(\tau\wedge\sigma^\theta)}\right) \\ &- \lambda(\tau \wedge \sigma^\theta) + h^\theta(\pi(\tau \wedge \sigma^\theta), z(\tau \wedge \sigma^\theta)) \Big\} . \end{aligned}$$

Therefore, also $Mw^{\theta_{n_k}}(\pi, z) \to Mw^\theta(\pi, z)$, and

$$\begin{aligned} w^\theta(\pi, z) = \sup_\tau E_{\pi z}\Big\{ &\ln\left(\pi e^{X^\theta(\tau\wedge\sigma^\theta)}\right) \\ &- \lambda(\tau \wedge \sigma^\theta) + Mw^\theta(\pi(\tau \wedge \sigma^\theta), z(\tau \wedge \sigma^\theta)) \Big\} . \end{aligned} \tag{III.3}$$

By the uniqueness of λ, since from (III.3) it is clear that $\lambda = \lambda^\theta$, from any sequence λ^{θ_n} there is a subsequence converging to λ^θ. This means that $\lambda^{\theta_n} \to \lambda^\theta$ which is the claim of Theorem 3. ■

Remark 3: The continuity property shown in Theorem 3 is fundamental to use an adaptive control approach to the model – i.e., consider that θ^0 is unknown and using either a Bayesian (see [5] or [4]) or parametric approach (see [6] or [2], [3]). In both approaches, either a finite class of nearly optimal controls (see [6] and [5]) or occupation measures techniques were used.

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