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Growth Optimal Portfolio Under Proportional Transaction Costs With Obligatory Diversification

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***Abstract*— A continuous time long run growth optimal portfolio with proportional cost consisting of the sum of a fixed proportional cost and a cost proportional to the volume of transactions is considered. An obligatory portfolio diversification is introduced according to which it is required to invest at least a fixed small portion of the wealth in each asset.**

I. Introduction

Assume that on a given complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$, consider three independent processes: a d-dimensional standard Brownian motion $(B(t))$, a d-dimensional compensated Poisson random measure $\tilde{N}(dt,du)$ and a time homogeneous Markov process $(z(t))$ with values in a finite space D and transition matrix P_t at time t. Consider also d assets with the ith asset price $S_i(t)$ at time t. It is assumed that the evolution of $S_i(t)$ is of the form

$$\delta_i(t) = \delta_i(0) e^{X_i^{\theta^0}}(t) \tag{I.1}$$

where $X_i(0) = 0$ and $X^{\theta^0}(t) = (X_1(t), \ldots, X_d(t))$ is a solution to the following Lévy stochastic differential equation:

$$dX(t) = \alpha(z(t),\theta^0)\,dt + \sigma(z(t),\theta^0)\,dB(t) + \int_{\mathbb{R}^n} \gamma(z(t),\theta^0,u)\,\tilde{N}(dt,du)\,. \tag{I.2}$$

θ^0 is a parameter that belongs to a compact space Θ. In [2], the continuity of the optimal ergodic cost with respect to $\theta \in \Theta$ is verified. It is assumed that α, σ, and γ are continuous bounded functions of θ, and

$$\sup_\theta \int_{\mathbb{R}} \gamma_{i_k}^2(z,\theta,u)\,\nu_k(du) < \infty$$

for $i,k = 1,2,\ldots,d$ with ν_k being the Lévy measure corresponding to $\tilde{N}(dt,du)$. In this paper, the dependence of $X(t)$ on θ^0 is neglected.

In what follows, denote by $\pi_i(t)$ the portion of the wealth process invested in the ith asset. Let $e^{X(t)} = \left(e^{X_1(t)}, \ldots, e^{X_d(t)}\right)$ and for $\pi, \zeta \in [0,\infty)^d$,

$$\pi \diamond \zeta = (\pi_1\zeta_1, \pi_2\zeta_2, \ldots, \pi_d\zeta_d)$$

and

$$g(\zeta) = \left(\frac{\zeta_1}{\sum \zeta_i}, \frac{\zeta_2}{\sum \zeta_i}, \ldots, \frac{\zeta_d}{\sum \zeta_i}\right)\,.$$

If the portfolio strategy in the time interval $[0,T]$ is not changed, the portions of the wealth invested in the assets at time T are of the form

$$\pi(T) = g\left(\pi(0) \diamond e^{X(T)}\right)\,. \tag{I.3}$$

By the form of (I.2), it is clear that the pair $(\pi(t), z(t))$ is a Markov process with transition operator Π_t.

It is assumed that

(A1) The solution to (I.2) with the initial condition $X(0) = x$ has a continuous density for each fixed $z(t) = z$ with respect to the Lebesgue measure l^d at time $t > 0$ – i.e., for a Borel set $A \subset \mathbb{R}^d$,

$$P_{X_z}\{X^z(t) \in A\} = \int_A P^z(x,x')\,l^d(dx') \tag{I.4}$$

where $X^z(t)$ is a solution to (I.2) with $z(t) \equiv z$, and $P^z(x,x')$ is a continuous function of x and x'.

Sufficient conditions for (A1) can be found in [1], [4], [6], [10].

Since $(z(t))$ is a finite state continuous time, time homogeneous Markov process, its evolution can be described in the following form:

$$\begin{aligned} \tau_1 &= \inf\{s \geq 0 \colon z(s) \neq z(0)\} \\ \tau_{n+1} &= \inf\{s \geq 0 \colon z(s+\tau_n) \neq z(\tau_n)\} \end{aligned}$$

for $z(0) = z$

$$\begin{aligned} P_z\{\tau_1 \leq t\} &= \int_0^t n(z,s)\,ds \\ P_z\{P_{z(\tau_n)}\{t_{n+1}^0 \leq t\}\} &= E_z\left\{\int_0^t n(z(\tau_n),s)\,ds\right\} \\ P_z\{z(\tau_1) = z'\} &= P(z,z')\,. \end{aligned} \tag{I.5}$$

By direct calculation the following result is obtained:

Lemma 1: Given (A1) for a Borel set $B \in \mathbb{R}^d$ and $z' \in D$,

$$P_{x,z}\{X(t) \in B, z(t) = z'\} = \int_A p_t(x,x',z,z')\,l^d(dx') \tag{I.6}$$

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where

$$p_t(x,x',z,z') = \sum_{k=1}^{\infty}\sum_{z_1\in D}\sum_{z_2\in D}\cdots\sum_{z_{k-1}\in D}\int_0^t n(z,s_1)$$
$$\int_{\mathbb{R}^d} P^z_{s_1}(x,x_1)P(z,z_1)\int_0^{t-s_1} n(z_1,s_2)$$
$$\int_{\mathbb{R}^d} P^{z_1}_{s_2}(x_1,x_2)P(z_1,z_2)\ldots$$
$$\ldots\int_0^{t-s_1-\ldots-s_{n-1}} n(z_{k-1},s_n)$$
$$\int_{\mathbb{R}^d} P^{z_{n-1}}_{s_n}(x_{n-1},x_n)P(z_{n-1},z')P^{z'}_{t-s_1-\ldots-s_n}(x_n,y)$$
$$\int_{t-s_1-\ldots-s_n}^{\infty} n(z',u)\,du\,l(dx_n)\,ds_n\,l(dx_{n-1})\,ds_{n-1}\ldots l(dx_1)\,ds_1\,. \tag{I.7}$$

The following continuity property will be crucial in further investigations:

Proposition 1: Under (A1), the operator Π_t is continuous in variation norm for $(\pi,z)\in S_\delta\times D$ – i.e., for $(\pi_{(n)},z)\to(\pi,z)\in S_\delta\times D$, and $(\pi_{(n)},n\geq 1)$ is a sequence in S_δ, it follows that

$$\sup_{A\in B(S\times D)}\left|\Pi_t(\pi_{(n)},z,A)-\Pi_t(\pi,z,A)\right|\to 0 \tag{I.8}$$

as $n\to\infty$, with

$$S=\{v=(v_1,\ldots,v_d),v_i\geq 0,\sum v_i=1\}$$

and

$$S_\delta=\{v\in S,v_i\geq\delta,i=1,2,\ldots,d\}$$

for $0<\delta<1/d$.

Proof: Note that

$$\Pi_t(\pi,z,A) = E_{\pi z}\left\{\mathbb{1}_A\left(\pi_1(t),\ldots,\pi_{d-1}(t),1-\sum_{i=1}^{n-1}\pi_i(t),z(t)\right)\right\}$$

so the density of $(\pi_1(t),\ldots,\pi_{d-1}(t))$ is used. Consider the following transformations of $\mathbb{R}^d$:

$$\exp\begin{bmatrix}x_1\\x_2\\\vdots\\x_d\end{bmatrix}=\begin{bmatrix}e^{x_1}\\e^{x_2}\\\vdots\\e^{x_d}\end{bmatrix} \tag{I.9}$$

for $\pi=(\pi_1,\pi_2,\ldots,\pi_d)\in S$,

$$D_\pi\begin{bmatrix}x_1\\x_2\\\vdots\\x_d\end{bmatrix}=\begin{bmatrix}\pi_1x_1\\\pi_2x_2\\\vdots\\\pi_dx_d\end{bmatrix} \tag{I.10}$$

and for $[x_1,x_2,\ldots,x_d]\in[0,\infty)^d\setminus\{0\}$,

$$G\begin{bmatrix}x_1\\x_2\\\vdots\\x_{d-1}\\x_d\end{bmatrix}=\begin{bmatrix}\frac{x_1}{\sum x_i}\\\frac{x_2}{\sum x_i}\\\vdots\\\frac{x_{d-1}}{\sum x_i}\\\sum x_i\end{bmatrix}. \tag{I.11}$$

The transformation $GD_\pi\exp$ transforms $X(t)=(X_1(t),\ldots,X_d(t))$ into $\pi_1(t),\pi_2(t),\ldots,\pi_{d-1}(t),\sum_{i=1}^d\pi_ieX_i(t))$ and the determinant of the Jacobian of its inverse is of the form

$$\frac{1}{y_1y_2\cdots y_d}\frac{1}{\left(1-\sum_{i=1}^{d-1}y_i\right)}.$$

Consequently, the density $f^\pi(y_1,\ldots,y_d)$ of $\left(\pi_1(t),\pi_2(t),\ldots,\pi_{d-1}(t),\sum_{i=1}^d\pi_ie^{X_i(t)}\right)$ is of the form, assuming that $z(0)=z$, $z(t)=z'$, $i=1$ is fixed,

$$\frac{P_t\left(\ln\frac{1}{\pi_1}y_1y_d,\ldots,\ln\frac{1}{\pi_{d-1}}y_{d-1}y_d,\ln\frac{1}{\pi_d}\left(1-\sum_{i=1}^{d-1}y_i\right)y_d,z,z'\right)}{y_1y_2\cdots y_d\left(1-\sum_{i=1}^{d-1}y_i\right)}. \tag{I.12}$$

By the Scheffé Theorem [11], the pointwise convergence of transition densities implies convergence in L^1. Consequently, whenever $\pi_{(n)}\to\pi\in S_\delta$, it follows, by continuity of P_t, that

$$f^{\pi_{(n)}}(y_1,\ldots,y_d)\to f^\pi(y_1,\ldots,y_d)$$

pointwise and the convergence is also in L^1. Therefore

$$\int_0^\infty f^{\pi_{(n)}}(y_1,\ldots,y_d)\,l(dy_d)\to\int_0^\infty f^\pi(y_1,\ldots,y_d)\,l(dy_d) \tag{I.13}$$

in $L^1([0,\infty]^{d-1})$. Finally, by (I.13), for $z,z'\in D$,

$$\sup_{A\in B(S)}\left|\Pi_t(\pi_{(n)},z,A\times\{z'\})-\Pi_t(\pi,z,A\times\{z'\})\right|$$
$$\leq\int_{[0,\infty)^{d-1}}\left|\int_0^\infty f^{\pi_{(n)}}(y_1,\ldots,y_d)\,l(dy_d)-\int_0^\infty f^\pi(y_1,\ldots,y_d)\,l(dy_d)\right|l^{d-1}(dy_1,\ldots,dy_{d-1})\to 0$$

from which (I.8) follows. ∎

Using the same arguments as in the proof of Proposition 1, there is

Corollary 1: If the transition density $p_t^\theta(x,x',z,z')$ for $(X^\theta(t)),(z(t))$ depends in a continuous way on θ, x, and x', then for $(\theta_n,\pi_{(n)})\to(\theta,\pi)\in\Theta\times S_\delta$ and $z\in D$,

$$\sup_{A\in B(S\times D)}\left|\Pi_t^{\theta_n}\left(\pi_{(n)},z,A)-\Pi_t^\theta(\pi,z,A)\right)\right|\to 0\,. \tag{I.14}$$

Having shown the properties of the uncontrolled process $\pi(t)$, now the control problem is introduced. Assume that there are proportional transaction costs consisting of a fixed proportional managing cost and a cost proportional to the volume of the transactions. Let ζ_i^-, $i=1,2,\ldots,d$ denote the amount of wealth process invested in the ith asset. Clearly, $W^-=\sum_{i=1}^d\zeta_i^-$ is the wealth before a possible transaction. Changing the portfolio to $(\zeta_1,\zeta_2,\ldots,\zeta_d)$ requires paying immediately transaction costs of the form

$$kW^- + \sum_{i=1}^{d} c_i^1(\zeta_i - \zeta_i^-)^+ + c_i^2(\zeta_i - \zeta_i^-)^- \tag{I.15}$$

with $k > 0$ corresponding to a fixed managing cost. Short selling or short borrowing are not allowed and it is assumed that the portfolio is self-financing. Therefore, the wealth W after a transaction is equal to

$$W^- - ksW^- - \sum_{i=1}^{d} c_i^1(\zeta_i - \zeta_i^-)^+ + c_i^2(\zeta_i - \zeta_i^-)^- . \tag{I.16}$$

Let $\pi_i^- = \zeta_i^-/W^-$ and $\pi_i = \zeta_i/W$ be respectively the portion of wealth invested in the ith asset before and after a transaction. From (I.16), it follows that

$$1 - k - \sum_{i=1}^{d} c_i^1 \left(\pi_i \frac{W}{W^-} - \pi_i^-\right)^+ + c_i^2 \left(\pi_i \frac{W}{W^-} - \pi_i^-\right) = \frac{W}{W^-}$$

or

$$c\left(\pi \frac{W}{W^-} - \pi^-\right) + \frac{W}{W^-} = 1 ,$$

with

$$c(v) = k + \sum_{i=1}^{d} \left(c_i^1 v_i^+ + c_i^2 v_i^-\right) .$$

In what follows, it is assumed that $0 < c_i^1, c_i^2 < 1-k$.

It appears that starting from the portfolio $(\pi_1^-, \pi_2^-, \ldots, \pi_d^-)$ the portfolio $(\pi_1, \pi_2, \ldots, \pi_d)$ is available. Naturally it follows that (see Lemma 1 of [12], or [13])

Lemma 2: There is a unique continuous function $e\colon S \times S \to (0, 1-k]$ such that for $\pi^-, \pi \in S$ there is the equality

$$c(\pi e(\pi^-, \pi) - \pi^-) + e(\pi^-, \pi) = 1 . \tag{I.17}$$

The function e is bounded away from zero and

$$e(\pi, \pi')e(\pi', \pi'') < e(\pi, \pi'') ,$$

which means that it is not profitable to make two instantaneous portfolio changes. The wealth process W^- after the change of portfolio from π^- to π is diminished to $e(\pi^-, \pi)W^- = W$.

Denote by $W^-(t)$, $W(t)$, $\pi^-(t)$, $\pi(t)$, the wealth process before and after transaction or the portfolio before and after transaction at time t respectively. The purpose is to maximize the following long run wealth growth rate:

$$y^{\theta^0}((\pi(t)) = \liminf_{T\to\infty} \frac{1}{T} E_{\pi z}\{\ln W(T)\} . \tag{I.18}$$

Since $k > 0$, the strategy is of impulsive form – i.e. it is a sequence $V = (\tau_n, \pi^n)$ consisting of transaction times (stopping times τ_n for $n = 1, 2, \ldots$) and portfolios π^n which are chosen at time τ_n. The following equalities are satisfied

$$W(t) = W(\tau_n) \sum_{i=1}^{d} \pi_i(\tau_n) e^{X_i(t) - X_i(\tau_n)} \tag{I.19}$$

$$\pi(t) = g\left(\pi(\tau_n) \diamond e^{X(t) - X(\tau_n)}\right) \tag{I.20}$$

for $\tau_n < t < \tau_{n+1}$, and

$$W(\tau_n) = e(\pi(\tau_n^-), \pi^n) W^-(\tau_n) . \tag{I.21}$$

Additionally, the portfolio $\pi(t)$ is not allowed to be too close to the boundary of the simplex S. An obligatory diversification of the portfolio is introduced. Let $0 < \delta < \delta' < 1/d$, and

$$S_\delta^0 = \{v \in S\colon v_i > \delta \text{ for } i = 1, 2, \ldots, d\} .$$

As soon as the portfolio $(\pi(t))$ enters the set $S \setminus S_\delta^0$, it is changed by choosing a new portfolio from the set $S_{\delta'}$. Both parameters δ and δ' are assumed to be fixed in the paper. The following remark justifies the use of obligatory portfolio diversification:

Remark 1: Assume that there is a unique invariant measure μ^{θ^0} for Markov process $(z(t))$. Under the assumptions, the law of large numbers for the martingale

$$\int_0^t \sigma(z(s), \theta)\, dB(s) + \int_0^t \int_{\mathbb{R}^d} \gamma(z(s), \theta, u) \tilde{N}(ds, du)$$

is applicable and therefore

$$\lim_{t\to\infty} \frac{1}{t} X_i(t) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \alpha_i(z(s), \theta^0)\, ds = \sum_{z' \in D} \alpha_i(z', \theta^0)\, \mu^{\theta^0}(z') = r_i^{\theta^0} \quad \mathbb{P} \text{ a.e.}$$

Consequently, $X_i(t)$ is of order $t r_i^{\theta^0}$. If $r_i^{\theta^0} > r_j^{\theta^0}$, for $j \neq i$, then provided that $\pi_i(0) > 0$ and the portfolio is not changed, it follows that

$$\pi_i(t) = \frac{\pi_i(0) e^{X_i(t)}}{\sum_j \pi_j(0) e^{X_j(t)}} \to 1 \quad \mathbb{P} \text{ a.e.}$$

as $t \to \infty$, while $\pi_j(t) \to 0$ for $j \neq i$ as $t \to \infty$ $\mathbb{P}$ a.e.

In other words, assuming that the $r_i^{\theta^0}$ are not the same for $i = 1, 2, \ldots, d$, the process $\pi(t)$ in the limit converges to the boundary of S, provided that $\pi(0) > 0$. As a consequence,

$$\liminf_{T\to\infty} \frac{1}{T} E\left[\ln W(T)\right] = \max_{i=1,2,\ldots,d} r_i^{\theta^0} .$$

This is the value of the wealth process that can be guaranteed over the long run. It may happen however that it is more profitable to change the portfolio regularly than just wait for the guaranteed value.

The portfolios from the boundary of S are unacceptable from a risk sensitivity point of view. To eliminate risk, usually a portfolio is diversified. Therefore, in the paper an obligatory diversification is required.

A model with fixed proportional transaction costs ($k > 0, c \equiv 0$) was studied by Morton and Pliska in [9]. A simple one asset Black Scholes model with fixed proportional plus proportional transaction costs was considered in [5], and the control was restricted to a diversification boundary and the choice of a new portfolio when this boundary was reached. In the paper [12], general discrete and continuous time models with an obligatory diversification were studied. For a continuous time model, a certain transaction delay was introduced, which played an important role in the proofs. This paper generalizes [12] in various directions. A more specific asset growth model based on Lévy noise is considered. As in [12], the vanishing discount approach is used. To obtain continuity results time discretization is used. The main result, existence of the smooth solutions to the ergodic Bellman equation is obtained by the continuity properties of the transaction operator and finiteness of the space D.

II. Discrete Time Approximations

The transition operator Π_t of the Markov process $(\pi(t), z(t))$ has a nice continuity property (see Proposition 1). To use this property for a continuous time model, consider a time discretization. Let

$$\sigma = \inf\{s \geq 0 \colon \pi(s) \in S \setminus S_\delta^0\}$$

and

$$\sigma_n = \inf\{2^{-n}s, s = 0, 1, 2, \ldots, \pi(2^{-n}s) \in S \setminus S_\delta^0\},$$

i.e. σ and σ_n are the first exit times of S_δ^0 for a continuous time or discrete time $2^{-n}s$ Markov processes $(\pi(t)), (\pi(2^{-n}s))$ respectively. Clearly, $\sigma_n \geq \sigma$. It is assumed furthermore that

(A2) $\sup_{z \in D} \sup_{\pi \in S_\delta^0} E_{\pi_2}\{\sigma\} < \infty$

(A3) $\sup_{z \in D} \sup_{\pi \in S_\delta^0} E_{\pi_2}\{\sigma_n - \sigma\} \to 0$ as $n \to \infty$.

Remark 2: It is clear in view of Remark 1 that if $r_i^{\theta^0}$ are not the same for all i, then $\sigma < \infty$ $\mathbb{P}$ a.e. The fact that $E_{\pi z}\{\sigma\} < \infty$ follows mainly from nondegeneracy of the diffusion term in the equation for $(\pi(t))$. The assumption (A3) is typical in diffusion approximations (see the assumption A22 in [7]) and as was justified in [7] it holds for uniformly elliptic diffusion terms. Applying Ito's formula to the function $f_i(x_1, \ldots, x_d) = (\pi_i e^{X_i})/(\sum \pi_j e^{X_j})$, a stochastic differential equation is obtained for $\pi_i(t)$. The Brownian motion coefficient (row) is in the form

$$\begin{aligned} r_i(t) &= (r_i(t) - (\pi_i(t))^2)\sigma_i(z(t), \theta^0) \\ &\quad - \sum_{j \neq i}^{d} \pi_i(t)\pi_j(t)\sigma_j(z(t), \theta^0) \\ &= \pi_i(t) \sum_{j=1}^{d} \tilde{\pi}_j^i(t)\sigma_j(z(t), \theta^0) \end{aligned} \tag{II.1}$$

with $\tilde{\pi}_i^i(t) = 1 - \pi_i(t)$ and $\tilde{\pi}_j^i(t) = -\pi_j(t)$ for $i \neq j$. The matrix $r(t)r^T(r)$ is not uniformly elliptic. It is, however, uniformly elliptic on the subspace orthogonal to the vector $\mathbb{1} = (1, 1, \ldots, 1)$.

Thus

Lemma 3: If $\sigma(z, \theta^0)\sigma(z, \theta^0)^T$ is uniformly elliptic, i.e. there is $\varepsilon > 0$ such that for $a \in \mathbb{R}^d$, $z \in D$,

$$a^T \sigma(z, \theta^0)\sigma(z, \theta^0)^T a \geq \varepsilon a^T a$$

then under (A2) $r(t)r(t)^T$ is uniformly elliptic for $a \in \mathbb{R}^d$ such that $\sum_{r=1}^d a_i = 0$ and $\pi(t) \in S_\delta$ – i.e. there is $\varepsilon' > 0$ such that for $a \in \mathbb{R}^d$, $\sum_{i=1}^d a_i = 0$, $\pi(t) \in S_\delta$,

$$a^T r(t)r(t)^T a \geq \varepsilon' a^T a \tag{II.2}$$

with ε' uniform for $t > 0$, $\pi(t) \in S_\delta$, $z(t) \in D$.

Proof: Assume that for a sequence of nonzero vectors $a^n \in \mathbb{R}^d$, t_n, $(a^n)^T r(t_n) r(t_n)^T a^n \to 0$ as $n \to \infty$ with $\sum_{i=1}^d a_i^n = 0$. Since $a^n / \sqrt{(a^n)^T a^n}$ is in the unit sphere, one can choose subsequences, for simplicity again denoted by n such that $a^n / \sqrt{(a^n)^T a^n} \to b$, $t_n \to t$, where b is on the unit sphere and t is finite. If $t = \infty$, the process $\pi(t)$ is leaving S_δ so that requirement that $\pi(t) \in S_\delta$ is not satisfied. Letting $n \to \infty$,

$$\frac{(a^n)^T}{\sqrt{(a^n)^T a^n}} r(t_n) r(t_n)^T \frac{a^n}{\sqrt{(a^n)^T a^n}} \to 0$$

it follows that $b^T(r(\bar{t})r(\bar{t})b = 0$ with $\bar{t} = t$ or t^- depending on the form of t_n. Since

$$\begin{aligned} b^T r(\bar{t}) &= \sum_{i=1}^d b_i \pi_i(\bar{t}) \sum_{j=1}^d \tilde{\pi}_j^i(\bar{t})\sigma_j(z(t), \theta^0) \\ &= \sum_{j=1}^d \left(\sum_{i=1}^d b_i \pi_i(\bar{t}) \tilde{\pi}_j^i(\bar{t}) \right) \sigma_j(z(\bar{t}), \theta^0) \end{aligned}$$

by uniform ellipticity of $\sigma\sigma^*$, it follows that

$$\sum_{i=1}^d b_i \pi_i(\bar{t}) \tilde{\pi}_j^i(\bar{t}) = 0 \quad \text{for } j = 1, 2, \ldots, d. \tag{II.3}$$

The vectors $(\pi_i(\bar{t})\tilde{\pi}_j^i(\bar{t}))_{i=1,2,\ldots,d}$ are orthogonal to $\mathbb{1} = (1, 1, \ldots, 1)$ and since the matrix $\pi_i(\bar{t})\tilde{\pi}_j^i(\bar{t})$ is of rank $d - 1$, vector $\mathbb{1}$ is the unique (up to multiplicative constant) orthogonal vector. Consequently, $b = c\mathbb{1}$, with $c = 1/\sqrt{d}$, and that

$$\frac{a_i^n}{\sqrt{(a^n)^T a^n}} - \frac{1}{\sqrt{d}} \to 0$$

for $i = 1, 2, \ldots, d$.

Summing over i in the last convergence, and taking into account that $\sum_{i=1}^d a_i^n = 0$, it follows that $-1/\sqrt{d} = 0$, a contradiction. Consequently, (II.2) is satisfied. ■

Consider now the following optimal stopping problem

$$\begin{aligned} w_T^{\beta\theta}(\pi, z) = \sup_\tau E_{\pi z} \Big\{ & e^{-\beta \tau \wedge \sigma \wedge T} \Big[\ln \Big(\pi e^{X(\tau \wedge \sigma \wedge T)} \Big) \\ & + F(\pi(\tau \wedge \sigma \wedge T), z(\tau \wedge \sigma \wedge T)) \Big] \Big\} \end{aligned} \tag{II.4}$$

with $\beta \geq 0$, positive integer T, and function F, which is continuous and bounded. Let J_n be the family of discretized stopping times taking values in the set $\{2^{-n}s, s = 0, 1, 2, \ldots\}$. The following discretized version of (II.4) is obtained

$$\begin{aligned} w_{T,n}^{\beta\theta}(\pi, z) = \sup_{\tau \in J_n} E_{\pi z} \Big\{ & e^{-\beta \tau \wedge \sigma_n \wedge T} \Big[\ln \Big(\pi e^{X(\tau \wedge \sigma_n \wedge T)} \Big) \\ & + F(\pi(\tau \wedge \sigma_n \wedge T), z(\tau \wedge \sigma_n \wedge T)) \Big] \Big\}. \end{aligned} \tag{II.5}$$

Note that since

$$-\sum_{i=1}^{d}|X_i(\tau\wedge\sigma_n\wedge T)| \le \sum_{i=1}^{d}\pi_i X_i(\tau\wedge\sigma\wedge T) \le \ln\left(\pi e^{X(\tau\wedge\sigma\wedge T)}\right) \le \max_i X_i(\tau\wedge\sigma\wedge T) \le \sum_{i=1}^{d}|X_i(\tau\wedge\sigma\wedge T)| \quad \text{(II.6)}$$

the values $w_T^{p\theta}$ and $w_{T,n}^{\beta\theta}$ are bounded. Moreover, there is

Proposition 2: Under (A1)–(A3), $w_{T,n}^{\beta\theta}(\pi,z)$ is a continuous function of $\pi \in S_\delta^0$, $\pi \in S\setminus S_\delta^0$, and $(w_T, n^{\beta\theta}, n\ge 1)$ converges as $n\to\infty$ to $w_T^{\beta\theta}$ uniformly on compact subsets of S^0 so that $w_T^{\beta\theta}$ is continuous on S_δ^0 and on $S\setminus S_\delta^0$.

Proof: Consider the following sequence of functions:

$$\begin{aligned}
h_{T2^n}(\pi,z) &= F(\pi,z)\\
h_{T2^n-1}(\pi,z) &= \max\left\{F(\pi,z), E_{\pi z}\left\{e^{-2^{-n}}\left[\ln\left(\pi e^{X(2^{-n})}\right) + F(\pi(2^{-n}),z(2^{-n}))\right]\right\}\mathbb{1}_{S_\delta^0}(\pi)\right.\\
&\quad + F(\pi,z)\mathbb{1}_{S\setminus S_\delta^0}(\pi)\\
h_m(\pi,z) &= \max\left\{F(\pi),E_{\pi z}\left\{e^{-2^{-N}}\left[\ln\left(\pi e^{X(2^{-N})}\right) + h_{m+1}(\pi(2^{-n}),z(2^{-n}))\right]\right\}\mathbb{1}_{S_\delta^0}(\pi)\right.\\
&\quad + F(\pi,z)\mathbb{1}_{S\setminus S_\delta^0}(\pi)
\end{aligned} \quad \text{(II.7)}$$

Clearly, $w_{T,n}^{\beta\theta}(\pi,z) = h_0(\pi,z)$. By Proposition 1 each function h_m is continuous separately on S_δ^0 and $S\setminus S_\delta^0$. Consequently, the same is true for the function $w_{T,n}^{\beta\theta}$. Now, for any stopping time τ, define a stopping time $\tau_n \in J_n$ in the following way

$$\tau_n = (k+1)2^n$$

whenever $\tau\in(k2^n,(k+1)2^n]$, and

$$\tau_n = k2^n$$

for $\tau = k2^n$ with $k = 0,1,2,\ldots$. Then

$$\begin{aligned}
&\Big| E_{\pi z}\left\{e^{-\beta\tau\wedge\sigma\wedge T}\left[\ln\left(\pi e^{X(\tau\wedge\sigma\wedge T)}\right.\right.\right.\\
&\qquad + F(\pi(\tau\wedge\sigma\wedge T), z(\tau\wedge\sigma\wedge T)))]\}\\
&\qquad - E_{\pi z}\left\{e^{-\beta\tau_n\wedge\sigma_n\wedge T}\left[\ln\left(\pi e^{X(\tau_n\wedge\sigma_n\wedge T)}\right.\right.\right.\\
&\qquad + F(\pi(\tau_n\wedge\sigma_n\wedge T), z(\tau_n\wedge\sigma_n\wedge T)))]\}\Big|\\
&\le E_{\pi z}\left\{e^{-\beta\tau\wedge\sigma\wedge T}\chi_{\substack{\tau\le\sigma\\ \tau\le T}}\right.\\
&\quad \left[\sup_{s\in[0,2^{-n}]}\left|E_{\pi(t),z(t)}\left\{1-e^{-\beta s}\ln(\pi(t)e^{X(s)}\right\}\right|\right.\\
&\quad \left.\left. + \sup_{s\in[0,2^{-n}]}\left|E_{\pi(t),z(t)}\left\{F(\pi(0),z(0)) - e^{-\beta s}F(\pi(s),z(s))\right\}\right|\right]\right\}\\
&\quad + E_{\pi z}\left\{e^{-\beta\tau\wedge\sigma\wedge T}\chi_{\sigma\le\tau}\chi_{\sigma\le T}\chi_{\sigma\le\sigma_n}\right.\\
&\quad \left\{\left[\ln(\pi(0)e^{X(\sigma)} - e^{\beta(\sigma_n-\sigma)}\ln(\pi(\sigma))e^{X(\sigma_n)}\right]\right.\\
&\quad \left.\left. + \left(F(\pi(\sigma),z(\sigma)) - e^{-\beta(\sigma_n-\sigma)}F(\pi(\sigma_n),z(\sigma_n))\right)\right\}\right\}\\
&= I_n + II_n\,.
\end{aligned} \quad \text{(II.8)}$$

The convergence $I_n\to 0$ follows from the following two properties of the process $\pi(t)$:

Lemma 4: For any compact set $K\subset S^0$, $\varepsilon>0$, $T>0$, there is a compact set $K'\subset S^0$ such that

$$\sup_{z\in D}\sup_{\pi\in K} P_{\pi z}\left\{\pi(t)\notin K' \text{ for some } [0,T]\right\} < \varepsilon\,. \quad \text{(II.9)}$$

Lemma 5: Let $B_\delta(\pi) = \{\pi'\in S,\ \|\pi'-\pi\|<\delta\}$. For any $\varepsilon>0$, $\delta>0$, compact set $K'\subset S^0$ there is $h_0>0$ such that for $h\le h_0$

$$\sup_{z\in D}\sup_{\pi\in K'} P_{\pi z}\{\pi(h)\notin B_\delta(\pi)\} < \varepsilon\,. \quad \text{(II.10)}$$

the proofs of these two lemmas can be shown based on Lemma 2 of [8] and Lemma 2.5 of [3]. The convergence $II_n\to 0$ follows from (A3). Consequently, by (II.8), $w_{T,n}^{\beta\theta}(\pi,z)\to w_T^{\beta\theta}(\pi,z)$ uniformly in π from compact subsets of S^0. ■

III. Discounted Growth Optimal Portfolio

Assume in what follows that an impulsive strategy $V = (\tau_n,\pi^n)$ contains obligatory and nonobligatory transactions – i.e., whenever $\pi(t)$ enters $S\setminus S_\delta^0$, it is required to make an obligatory transaction to $\pi'\in S_\delta$ and when $\pi(t)\in S_\delta^0$ a transaction can be made but it is not required. Consider now the so-called discounted cost functional

$$J_{\pi_2}^{\beta\theta^0}(V) = E_{\pi_2}\left\{\sum_{i=1}^{\infty} e^{-\beta\tau_i}\left[\ln\left(\pi(\tau_{i-1})e^{X(\tau_i)-X(\tau_{i-1})}\right) + \ln e(\pi^-(\tau_i),\pi(\tau_i))\right]\right\}\,. \quad \text{(III.1)}$$

Let

$$w^{\beta\theta^0}(\pi,z) = \sup_V J_{\pi z}^{\beta\theta^0}(V)\,. \quad \text{(III.2)}$$

Theorem 1: Under (A1)–(A3), $w^{\beta\theta^0}$ is a bounded function continuous on S_δ^0 and $S\setminus S_\delta^0$ and is the unique solution to the following Bellman equation:

$$w^{\beta\theta^0}(\pi,z) = \sup_\tau E_{\pi z}\left\{e^{-\beta\tau\wedge\sigma}\left[\ln(\pi e^{X(\tau\wedge\sigma)}) + Mw^{\beta\theta^0}(\pi(\tau\wedge\sigma), z(\tau\wedge\sigma))\right]\right\}, \quad \text{(III.3)}$$

with

$$Mw(\pi,z) = \sup_{\pi^1\in S_{\delta'}}\left[\ln e(\pi,\pi') + w(\pi^1,z)\right]. \quad \text{(III.4)}$$

Proof: Let, for a continuous bounded function w on $S_{\delta'}$,

$$G^{\beta\theta^0}w(\pi,z) = \sup_\tau E_{\pi_2}\left\{e^{-\beta\tau\wedge\sigma}\left[\ln\left(\pi e^{X(\tau\wedge\sigma)}\right) + Mw(\pi(\tau\wedge\sigma), z(\tau\wedge\sigma))\right]\right\}. \quad \text{(III.5)}$$

By Proposition 2, the mapping

$$\pi \to \sup_\tau E_{\pi z}\left\{e^{-\beta\tau\wedge\sigma\wedge T}\left[\ln\left(\pi e^{X(\tau\wedge\sigma\wedge T)}\right) + Mw(\pi(\tau\wedge\sigma\wedge T), z(\tau\wedge\sigma\wedge T))\right]\right\}$$

is continuous for $\pi\in S^0_\delta$ and $\pi\in S\setminus S^0_\delta$ and any positive integer T. By (A2), it follows that $G^{\beta\theta^0}w(\pi,z)$ is continuous for $\pi\in S^0_\delta$ and $\pi\in S\setminus S^0_\delta$. Let

$$q^{\beta\theta^0}(\pi,z) = E_{\pi z}\left\{\sum_{i=1}^\infty e^{-\beta\tau_i}\left[\ln\left(\pi(\tau_{i-1})e^{X(\tau_i)-X(\tau_{i-1})}\right) + \ln e(\pi^-(\tau_i),\tilde\pi)\right]\right\}, \quad \text{(III.6)}$$

where

$$\tau_1 = \sigma$$
$$\tau_{n+1} = \tau_n + \sigma\Theta_{\tau_n}$$
$$\pi_n = \tilde\pi$$

where $\tilde\pi\in S_{\delta'}$ is fixed and Θ_t denotes the Markov shift operator. Consider now the following sequence of functions

$$\begin{aligned} q_0^{\beta\theta^0}(\pi,z) &= q^{\beta\theta^0}(\pi,z) \\ q_1^{\beta\theta^0}(\pi,z) &= G^{\beta\theta^0}q_0^{\beta\theta^0}(\pi,z) \\ q_n^{\beta\theta^0}(\pi,z) &= G^{\beta\theta^0}q_{n-1}^{\beta\theta^0}(\pi,z). \end{aligned} \quad \text{(III.7)}$$

Note that $q^{\beta\theta^0}(\pi,z)$ is the value of the cost functional $J^{\beta\theta^0}_{\pi z}$ corresponding to obligatory transactions to a fixed portfolio $\tilde\pi\in S_{\delta'}$. The value $q_n^{\beta\theta}(\pi,z)$ is the value of the cost functional with the strategy which consists of optimal first n transactions and then afterwards only obligatory transactions to $\tilde\pi$. Therefore, it is clear that the sequence $q_n^{\beta\theta^0}(\pi,z)$ is increasing. Consequently there is a limit $\hat q^{\beta\theta^0}(\pi,z)$ and letting $n\to\infty$ in (III.7) it follows that

$$\hat q^{\beta\theta^0}(\pi,z) = G^{\beta\theta^0}\hat q^{\beta\theta}(\pi,z). \quad \text{(III.8)}$$

Since for any bounded function f, the function $\pi\to Mf(\pi,z)$ is continuous (by the continuity of $e(\pi,\pi')$), using Proposition 2 as in the beginning of the proof the continuity of $\hat q^{\beta\theta^0}(\pi,z)$ for $\pi\in S^0_\delta$ and $\pi\in S\setminus S^0_\delta$ is used. The function $\hat q^{\beta\theta^0}$ is therefore a solution to (III.3) with suitable continuity properties. Iterating (III.3) it follows that $\hat q^{\beta\theta^0}$ coincides with $w^{\beta\theta^0}$, which completes the proof. ∎

Remark 3: In the proof of Theorem 1, a smoothing property of the operator M is used. Alternatively, one could use the fact that due to a fixed proportional transaction cost it is not optimal to have too many transactions in a finite time interval. This method allows one to prove a version of Theorem 1 for processes $(z(t))$ taking values in a general (not necessarily finite) state space whose transition probability is continuous in variation.

The following property of $w^{\beta\theta^0}$ will be important later:

Corollary 2: For $\pi,\pi'\in S^0_{\delta'}$ and $z\in D$, the following inequalities are satisfied

$$\left|w^{\beta\theta^0}(\pi,z) - w^{\beta\theta^0}(\pi',z)\right| \le \left|\ln e(\pi,\pi')\right| + \left|\ln e(\pi',\pi)\right| \quad \text{(III.9)}$$

and for $\pi\in S$, $\pi'\in S^0_{\delta'}$,

$$\ln e(\pi,\pi') + w^{\beta\theta^0}(\pi',z) \le w^{\beta\theta^0}(\pi,z) = Mw^{\beta\theta^0}(\pi,z). \quad \text{(III.10)}$$

Proof: For $\pi,\pi'\in S^0_\delta$,

$$w^{\beta\theta^0}(\pi,z) \ge Mw^{\beta\theta^0}(\pi,z) \ge \ln e(\pi,\pi') + w^{\beta\theta^0}(\pi',z),$$

and

$$w^{\beta\theta^0}(\pi',z) \ge \ln e(\pi',\pi) + w^{\beta\theta^0}(\pi,z),$$

and (III.10) is immediate from the definition of M. ∎

To study a long time growth optimal function $J(V)$ of the form (I.18), a uniform ergodicity of the Markov process (z_t) is imposed, namely:

(A4) there is $T>0$ and $\Delta<1$ such that

$$\sup_{z,z'\in D}\sup_{A\subset D}\left|P_T(z,A) - P_T(z',A)\right| = \Delta < 1.$$

Let

$$h^{\beta\theta^0}(\pi,z) = w^{\beta\theta^0}(\pi,z) - \inf_{\substack{\pi'\in S\\ z'\in D}} w^{\beta\theta^0}(\pi',z'). \quad \text{(III.11)}$$

Proposition 3: Under (A1)–(A4),

$$\sup_{\substack{\pi\in S\\ z\in D}} h^{\beta\theta^0}(\pi,z) \le \frac{M}{1-\Delta}, \quad \text{(III.12)}$$

with constant M independent on β and θ^0.

Proof: The Bellman equation (III.3) can be also written in the following equivalent form for every $T\ge 0$

$$w^{\beta\theta^0}(\pi,z) = \sup_\tau E_{\pi z}\left\{e^{-\beta\tau\wedge\sigma\wedge T}\left[\ln\pi e^{X(\tau\wedge\sigma\wedge T)} + \chi_{\tau\wedge\sigma>T}Mw^{\beta\theta^0}(\pi(\tau\wedge\sigma), z(\tau\wedge\sigma)) + \chi_{\tau\wedge\sigma\ge T}w^{\beta\theta^0}(\pi(T), z(T))\right]\right\}. \quad \text{(III.13)}$$

Iterating (III.13), it follows that

$$w^{\beta\theta^0}(\pi,z) = \sup_V E_{\pi z}\left\{\sum_{i=1}^{\infty} e^{-\beta\tau_i}\left(\ln\pi(\tau_{i-1})e^{(X(\tau_i)-X(\tau_{i-1}))} + \ln e(\pi^-(\tau_i),\pi(\tau_i))\chi_{\tau_i\le T} + e^{-\beta T}w^{\beta\theta^0}(\pi(T),z(T))\right\}\ . \quad \text{(III.14)}$$

Fix $\bar\pi \in S^0_\delta$. It is claimed that there is a constant C such that for any $\pi\in S$, $z\in D$, $\theta^0\in\Theta$,

$$w^{\beta\theta^0}(\pi,z) \le C + w^{\beta\theta^0}(\bar\pi,z)\ . \quad \text{(III.15)}$$

Assume that starting from (π,z), an optimal time $\hat\tau\wedge\sigma(\pi)$ for the first transaction is determined to choose portfolio $\bar\pi$. Starting from $(\bar\pi,z)$ make only obligatory transactions until $\hat\tau\wedge\sigma(\pi)$, each time choosing portfolio $\bar\pi$ and at $\hat\tau\wedge\sigma(\pi)$ choose again portfolio $\bar\pi$. By $\sigma(\pi)$ above, it denotes the first exit time from S^0_δ by $(\pi(t))$ starting from π. Then, from (III.9),

$$\begin{aligned} w^{\beta\theta^0}(\pi,z) - w^{\beta\theta^0}(\bar{p}i,z) \le E_{\pi z}\Big\{e^{-\beta\hat\tau\wedge\sigma(\pi)}\Big[\ln\pi e^{X(\hat\tau\wedge\sigma(\pi))} \\ +2K + w^{\beta\theta^0}(\bar\pi,z(\hat\tau\wedge\sigma(\pi)))\Big]\Big\} - (J_{\bar\pi,z}(\hat\tau\wedge\sigma(\pi)) \\ - K + E_{\bar\pi,z}\Big[w^{\beta\theta^0}(\bar\pi,z(\hat\tau\wedge\sigma(\pi))e^{-\beta\hat\tau\wedge\sigma(\pi)}\Big]\Big) \\ \le C_1 + 3K + C_2 = C \end{aligned}$$

where $J_{\bar\pi,z}(\hat\tau\wedge\sigma(\pi))$ is a cost functional of the strategy consisting of obligatory transactions to $\bar\pi$ and stopped at $\hat\tau\wedge\sigma(\pi)$ which by (A1)–(A2) is bounded uniformly by C_2. Furthermore,

$$K = \sup_{\substack{\pi\in S\\ \pi'\in S_\delta}} \left|\ln e(\pi,\pi')\right|\ , \quad \text{(III.16)}$$

and

$$\sup_{\substack{\pi\in S\\ z\in D}} E_{\pi z}\left\{e^{-\beta\tau\wedge\sigma}\left(\ln\pi e^{X(\tau\wedge\sigma)}\right)\right\} \le C_1\ ,$$

which verifies (III.15).

By (III.14), (III.15), (III.10), and (III.16),

$$\begin{aligned} w^{\beta\theta^0} - w^{\beta\theta^0} &\le \sup_V J^{\beta T}_{\pi z}(V) + C + E_{\pi z}\left[w^{\beta\theta^0}(\bar\pi,z(t))\right]e^{-\beta T} \\ &- \left(\sup_V J^{\beta T}_{\pi',z'}(V) - K + E_{\pi',z'}\left[w^{\beta\theta^0}(\bar\pi,z(T))\right]e^{-\beta T}\right) \\ &\le M + \left(E_{\pi z}\left[w^{\beta\theta^0}(\bar\pi,z(T))\right] - E_{\pi z'}\left[w^{\beta\theta^0}(\bar\pi,z(T))\right]\right)e^{-\beta T} \\ &= M + \Bigg[\sum_{z''\in D_1} w^{\beta\theta^0}(\bar\pi,z')(P_T(z,z'')-P_T(z',z'')) \\ &+ \sum_{z''\notin D_1} w^{\beta\theta^0}(\bar\pi,z')(P_T(z,z'')-P_T(z',z''))\Bigg]e^{-\beta t} \\ &\le M + \Delta\left(\sup_{z'} w^{\beta\theta^0}(\bar\pi,z') - \inf_{z'} w^{\beta\theta^0}(\bar\pi,z')\right)\ , \end{aligned}$$

where $D_1 = \{z''\in D\colon P_T(z,z'')\ge P_T(z',z'')\}$, from which (III.12) is obtained. ■

IV. Long View Growth Optimal Portfolio

Now rewrite the Bellman equation (III.3) in terms of a bounded (by Proposition 3) function $h^{\beta\theta^0}$. Thus

$$\begin{aligned} h^{\beta\theta^0}(\pi,z) = \sup_\tau E_{\pi z}\Big\{e^{-\beta\tau\wedge\sigma}\Big[\ln\left(\pi e^{X(\tau\wedge\sigma)}\right) \\ + Mh^{\beta\theta^0}(\pi(\tau\wedge\sigma),z(\tau\wedge\sigma))\Big] \\ - \inf_{\substack{\pi'\in S\\ z'\in D}} w^{\beta\theta^0}(\pi',z')(1-e^{-\beta\tau\wedge\sigma})\Big\}\ . \quad \text{(IV.1)} \end{aligned}$$

The main result of the paper can be formulated as follows:

Theorem 2: Under (A1)–(A4) there exist a constant λ^{θ^0} and a continuous bounded function w^{θ^0} such that

$$\begin{aligned} w^{\theta_0}(\pi,z) = \sup_\tau E_{\pi z}\Big\{\ln\left(\pi e^{X(\tau\wedge\sigma)}\right) - \lambda^{\theta^0}(\tau\wedge\sigma) \\ + Mw^{\theta_0}(\pi(\tau\wedge\sigma),z(\tau\wedge\sigma))\Big\}\ . \quad \text{(IV.2)} \end{aligned}$$

Moreover,

$$\lambda^{\theta_0} = \sup_V J^{\theta^0}(V)\ , \quad \text{(IV.3)}$$

i.e., λ^{θ_0} is the optimal value of the cost functional (I.18) and the strategy $\hat V = (\hat\tau_n,\hat\pi^n)$ such that

$$\hat\tau = \inf\left\{s\ge 0\colon w^{\theta_0}(\pi(s),z(s)) \le Mw^{\theta_0}(\pi(s),z(s))\right\}\ . \quad \text{(IV.4)}$$

$$\hat\tau_1 = \hat\tau$$
$$\hat\tau_{n+1} = \hat\tau_n + \hat\tau\circ\theta_{\tau_n} \quad \text{(IV.5)}$$

and

$$\hat\pi^n = \hat\pi(\pi^-(\hat\tau_n,z(\hat\tau_n))$$

where $\hat\pi\colon S\times D\to S_{\delta'}$ is a Borel function such that

$$Mw^{\theta_0}(\pi,z) = \ln e(\pi,\hat\pi(\pi,z)) + w^{\theta_0}(\hat\pi(\pi,z),z)$$

is optimal.

Proof: Note that

$$\inf_{\substack{K'\in S\\ z'\in D}} \beta w^{\beta\theta^0}(\pi',z')$$

is bounded so that there is a constant λ^{θ^0} and a sequence $\beta_n\downarrow 0$ such that

$$\inf_{\substack{\pi'\in S\\ z'\in D}} \beta_n w^{\beta\theta^0}(\pi',z') \to \lambda^{\theta^0}$$

as $n\to\infty$. Furthermore, by (A2),

$$E_{\pi z}\left\{\frac{1}{\beta_n}\left(1-e^{-\beta_n\tau\wedge\sigma}\right)\right\} \to E_{\pi z}\{\tau\wedge\sigma\}$$

as $n\to\infty$, and the limit is uniform in τ, π, and z. By Proposition 3, the functions $h^{\beta\theta^0}$ are bounded. Therefore, $Mh^{\beta\theta^0}(\pi,z)$ is uniformly continuous in $\pi\in D$ (use the continuity of e). One can therefore choose a subsequence of β_n, for simplicity again denoted by β_n, such that

$$Mh^{\beta_n\theta^0}(\pi,z) \to h^{\theta^0}(\pi,z) \quad \text{(IV.6)}$$

uniformly, where $h^{\theta^0}(\pi,z)$ is a continuous function of π. Therefore, by (IV.1), there is a continuous function w^{θ^0} such that

$$\sup_{\pi\in S}\sup_{z\in D}\left|h^{\beta_n\theta^0}(\pi,z)-w^{\theta^0}(\pi,z)\right|\to 0$$

as $n\to\infty$. From (IV.6), it follows that

$$Mh^{\beta_n\theta^0}(\pi,z)\to Mw^{\theta^0}(\pi,z)$$

uniformly in $\pi\in S$, $z\in D$.

Finally, w^{θ_0} is a solution to (IV.2). Equality (IV.3) and the form of optimal strategy $\hat{V}$ follows from standard arguments. ∎

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