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Interpolation Theory for Structure-preserving Model Reduction

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Abstract— **We develop a general framework for interpolation-based model reduction that includes rational Krylov-based methods as a special case. This new broader framework allows retention of special structure in the reduced order models such as symmetry, second order structure, internal delays, and infinite dimensional subsystems.**

I. Introduction

Dynamical systems are the basic framework for modeling and control of complex systems of scientific interest or industrial value. Direct numerical simulation of these models may be the only possibility for accurate prediction or control of complex physical phenomena. The need for accuracy leads to ever greater detail in the model and hence to large-scale, complex dynamical systems, whose simulation can make unmanageably large demands on computational resources, creating a crucial need for efficient model utilization. This is the primary motivation for model reduction. The cost of simulation is strongly tied to the the underlying state space dimension. Then, the goal of model reduction can be interpretted as replacement of the original system with a dynamical system evolving in a lower dimensional state space, yet having (insofar as is possible) the same input/output response characteristics as the original system. The resulting reduced-order model can then be used reliably to replace the original system model as a component in a larger simulation or control context.

Rational Krylov-based projection methods have emerged as effective strategies for the reduction of large-scale linear dynamical systems that are presented in certain standard settings [2], [18], [20]. Rational Krylov methods are numerically stable, well suited to large scale computation, and they share well understood approximation properties that rational interpolants have to meromorphic functions. The model reduction framework we present here includes rational Krylov-based methods as a special case but has a far broader range of applicability allowing for the retention of special structure in the reduced order models such as internal delays, and infinite dimensional subsystems in addition to symmetry and second order structure.

We emphasize that the goal here is to show how structure preserving interpolatory model reduction can be achieved in a much more general setting than the regular first- or second-order state-space systems. The question of how to choose good or optimal interpolation points in this broader setting will be addressed in a separate work by the authors.

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II. Problem Setting

Linear dynamical systems are typically described in the standard state-space settings

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t), \qquad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t), \tag{1}$$

or

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{G}\dot{\mathbf{x}}(t) + \mathbf{K}\,\mathbf{x}(t) = \mathbf{B}\,\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t), \tag{2}$$

where $\mathbf{E}, \mathbf{A}, \mathbf{M}, \mathbf{G}, \mathbf{K} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$; $\mathbf{x}(t) \in \mathbb{R}^n$ is the *state* vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is the *input* vector, and $\mathbf{y}(t) \in \mathbb{R}^p$ is the *output* vector of the system.

While these two frameworks are quite general, in many cases dynamical systems will have a natural description that takes a different form than either (1) or (2). For example, consider a dynamical system modeling the forced vibration of a viscoelastic structure observed through sensors that are sensitive to a combination of displacement and velocity at the attachment points. This system will have a state-space description in the form of

$$\begin{aligned}\mathbf{M}\,\ddot{\mathbf{x}}(t) + \int_0^t \mathbf{R}(t-\tau)\,\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}\,\mathbf{x}(t) &= \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}_0\mathbf{x}(t) + \mathbf{C}_1\dot{\mathbf{x}}(t)\end{aligned} \tag{3}$$

The matrices $\mathbf{M}$, $\mathbf{K}$ are $n \times n$ real, symmetric, positive-definite matrices and $\mathbf{B}$ is an $n \times m$ matrix, $\mathbf{C}_0$ and $\mathbf{C}_1$ are $p \times n$ matrices, and $\mathbf{R}(t)$ is a symmetric matrix-valued function on $[0, \infty)$ that is absolutely integrable in the sense that $\int_0^\infty \|\mathbf{R}(\tau)\|\, d\tau < \infty$. Model reduction methods that are applied in the standard settings (1) or (2) (such as Krylov subspace methods [2], [18] or balanced truncation [25], [24]) are unable to handle functional and delay differential equations of the form described in (3). Indeed, the model in (3) is fundamentally infinite dimensional due to the hereditary damping term; hence converting it into either of the frameworks of (1) or (2) requires use of an infinite dimensional state space and interpretation of (1) or (2) as an operator evolution equation. Moreover, effective reduced-order models for (3) should properly take into account the structure of distributed system properties. That is, we should seek reduced order models having similar structure:

$$\begin{aligned}\mathbf{M}_r\,\ddot{\mathbf{x}}(t) + \int_0^t \mathbf{R}_r(t-\tau)\,\dot{\mathbf{x}}(\tau)d\tau + \mathbf{K}_r\,\mathbf{x}(t) &= \mathbf{B}_r\mathbf{u}(t)\\ \mathbf{y}_r(t) &= \mathbf{C}_{0,r}\mathbf{x}(t) + \mathbf{C}_{1,r}\dot{\mathbf{x}}(t)\end{aligned} \tag{4}$$

where $\mathbf{M}_r$ and $\mathbf{K}_r$ are now $r \times r$ real, symmetric, positive-definite matrices with $r \ll n$ and $\mathbf{B}_r$ is an $r\times m$ matrix, $\mathbf{C}_{0,r}$

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and $\mathbf{C}_{1,r}$ are $p \times r$ matrices, and $\mathbf{R}_r(t)$ is an $r \times r$ matrix-valued function with $\int_0^\infty \|\mathbf{R}_r(\tau)\| \, d\tau < \infty$. Recasting (3) into the standard setting (1) strips out usable structure and can lead to unnecessary and dramatic increases in state space dimension.

In this paper, we present tools that allow for efficient, high-fidelity model reduction for linear dynamical systems that have a natural state space formulation distinct from (1) and (2). We consider cases with $n \gg m, p$ which often arises when the system of interest has been derived from a (spatial) semidiscretization of a partial differential equation describing local instantaneous equilibrium throughout a region of space with loads (sources and sinks) and observations that remain localized (in space).

The framework we present will allow more generality than is expressed in (3); e.g., we can accomodate multiple delays or memory convolution occurring within the state equations in higher derivatives, state variables can be coupled through infinite dimensional subsystems. This general framework can be accommodated by focussing on multiple-input/multiple-output (MIMO) systems with a transfer function $\mathcal{H}(s)$ having a (known) decomposition:

$$\mathcal{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s) + \mathcal{D}. \tag{5}$$

We assume that the factors $\mathcal{C}(s) \in \mathbb{C}^{p\times n}$ and $\mathcal{B}(s) \in \mathbb{C}^{n\times m}$ are analytic in the right half plane; that $\mathcal{K}(s) \in \mathbb{C}^{n\times n}$ is both analytic and full rank throughout the right half plane; and the feed forward term $\mathcal{D} \in \mathbb{R}^{p\times m}$ is constant. Our goal here is to generate, for some $r \ll n$, a reduced-order system with state space dimension r having a decomposition of the same form as (5):

$$\mathcal{H}_r(s) = \mathcal{C}_r(s)\mathcal{K}_r(s)^{-1}\mathcal{B}_r(s) + \mathcal{D}_r \tag{6}$$

with $\mathcal{C}_r(s) \in \mathbb{C}^{p\times r}$, $\mathcal{B}_r(s) \in \mathbb{C}^{r\times m}$, $\mathcal{K}_r(s) \in \mathbb{C}^{r\times r}$ and $\mathcal{D}_r \in \mathbb{R}^{p\times m}$ chosen so that $\mathcal{H}_r(s)$ will *exactly* interpolate $\mathcal{H}(s)$ at selected points $\sigma_1, \sigma_2, \ldots, \sigma_l \in \mathbb{C}$: $\mathcal{H}_r(\sigma_i) = \mathcal{H}(\sigma_i)$ for $i = 1, \ldots, l$.

We will construct reduced-order models via projection. That is, we will specify matrices $\mathbf{V}_r \in \mathbb{C}^{n\times r}$ and $\mathbf{W}_r \in \mathbb{C}^{n\times r}$ such that $\mathbf{W}_r^T\mathbf{V}_r$ is invertible; this choice can be associated with a rank r (oblique) projector given by $\mathbf{V}_r(\mathbf{W}_r^T\mathbf{V}_r)^{-1}\mathbf{W}_r^T$. The reduced order model $\mathcal{H}_r(s)$ of (6) is then obtained by defining

$$\mathcal{K}_r(s) = \mathbf{W}_r^T\mathcal{K}(s)\mathbf{V}_r, \quad \mathcal{B}_r(s) = \mathbf{W}_r^T\mathcal{B}(s), \quad \text{and} \quad \mathcal{C}_r(s) = \mathcal{C}(s)\mathbf{V}_r. \tag{7}$$

Since $\mathcal{D}$ and $\mathcal{D}_r$ are small matrices, no order reduction is necessary for them and the choice $\mathcal{D}_r = \mathcal{D}$ is both common and convenient. In the case that $\mathcal{D}_r = \mathcal{D}$, the interpolation conditions $\mathcal{H}_r(\sigma_i) = \mathcal{H}(\sigma_i)$ produce identical conditions to the case where $\mathcal{D}_r = \mathcal{D} = \mathbf{0}$, so we will treat that simpler case first. Choosing $\mathcal{D}_r \neq \mathcal{D}$ produces no advantage in terms of order reduction but may allow some advantage in achieving higher fidelity. We consider interpolation conditions in the case $\mathcal{D}_r \neq \mathcal{D}$ in the final section.

III. Interpolatory model reduction

Often reduced order models are constructed presupposing $\mathcal{D}_r = \mathcal{D}$. We consider this case first. We write $\mathcal{D}_\sigma^\ell f$ to denote the ℓ^{th} derivative of the univariate function $f(s)$ evaluated at $s = \sigma$ with the usual convention for $\ell = 0$, $\mathcal{D}_\sigma^0 f = f(\sigma)$. Also, we write $\mathrm{Ran}(\mathbf{Z})$ to denote the range of the matrix $\mathbf{Z}$.

Theorem 3.1: For $\mathcal{H}(s)$ and $\mathcal{H}_r(s)$ as defined, respectively, in (5) and (6), assume that $\mathcal{D}_r = \mathcal{D}$. Suppose matrices $\mathbf{V}_r \in \mathbb{C}^{n\times r}$ and $\mathbf{W}_r \in \mathbb{C}^{n\times r}$ are given such that $\mathbf{W}_r^T\mathbf{V}_r$ is invertible, and $\mathcal{K}_r(s)$, $\mathcal{B}_r(s)$ and $\mathcal{C}_r(s)$ are obtained as in (7). Suppose further that $\mathcal{B}(s)$, $\mathcal{C}(s)$, and $\mathcal{K}(s)$ are analytic at a point $\sigma \in \mathbb{C}$ and both $\mathcal{K}(\sigma)$ and $\mathcal{K}_r(\sigma) = \mathbf{W}_r^T\mathcal{K}(\sigma)\mathbf{V}_r$ have full rank. Let nonnegative integers M and N be given as well as nontrivial vectors, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^p$.

- If $\mathcal{D}_\sigma^i[\mathcal{K}(s)^{-1}\mathcal{B}(s)]\mathbf{b} \in \mathrm{Ran}(\mathbf{V}_r)$ for $i = 0, \ldots, N$ then
$$\mathcal{H}^{(\ell)}(\sigma)\mathbf{b} = \mathcal{H}_r^{(\ell)}(\sigma)\mathbf{b} \quad \text{for} \quad \ell = 0, \ldots, N.$$
- If $\left(\mathbf{c}^T\mathcal{D}_\sigma^j[\mathcal{C}(s)\mathcal{K}(s)^{-1}]\right)^T \in \mathrm{Ran}(\mathbf{W}_r)$ for $j = 0, \ldots, M$, then
$$\mathbf{c}^T\mathcal{H}^{(\ell)}(\sigma) = \mathbf{c}^T\mathcal{H}_r^{(\ell)}(\sigma) \quad \text{for} \quad \ell = 0, \ldots, M.$$
- If $\mathcal{D}_\sigma^i[\mathcal{K}(s)^{-1}\mathcal{B}(s)]\mathbf{b} \in \mathrm{Ran}(\mathbf{V}_r)$ for $i = 0, \ldots, N$ and $\left(\mathbf{c}^T\mathcal{D}_\sigma^j[\mathcal{C}(s)\mathcal{K}(s)^{-1}]\right)^T \in \mathrm{Ran}(\mathbf{W}_r)$ for $j = 0, \ldots, M$ then
$$\mathbf{c}^T\mathcal{H}^{(\ell)}(\sigma)\mathbf{b} = \mathbf{c}^T\mathcal{H}_r^{(\ell)}(\sigma)\mathbf{b} \text{ for } \ell = 0, \ldots, M{+}N{+}1.$$

Proof: Due to the page limitations, the proof is omitted. The proof will be included in the full paper.

Remark 3.1: The interpolation conditions that underlie rational Krylov methods for model reduction are contained as a special case of Theorem 3.1 after defining

$$\mathcal{K}(s) = s\mathbf{E} - \mathbf{A}, \quad \mathcal{B}(s) = \mathbf{B}, \quad \text{and} \quad \mathcal{C}(s) = \mathbf{C}.$$

Remark 3.2: Theorem 3.1 tells precisely what vectors to include in the reducing subspaces $\mathbf{V}_r$ and $\mathbf{W}_r$ in order to solve the interpolation problem via projection for the general dynamical systems of the form as described in (5).

IV. Recursive Generation of Interpolating Bases

Using Theorem 3.1, recurrences may be derived to generate projecting subspaces that force interpolation as described above. This is illustrated in the following result:

Theorem 4.1: Suppose we know expansions for $\mathcal{K}(s)$, $\mathcal{B}(s)$, and $\mathcal{C}(s)$ about $s = \sigma$:

$$\mathcal{K}(\sigma+\varepsilon) = \sum_{\ell=0}^{\infty} \varepsilon^\ell \, \mathcal{K}_\ell, \quad \mathcal{B}(\sigma+\varepsilon) = \sum_{\ell=0}^{\infty} \varepsilon^\ell \, \mathcal{B}_\ell, \quad \text{and} \quad \mathcal{C}(\sigma+\varepsilon) = \sum_{\ell=0}^{\infty} \varepsilon^\ell \, \mathcal{C}_\ell,$$

and let nontrivial vectors $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^p$ be given.

Define $\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N\}$ and $\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_M\}$ by solving recursively:

$$\begin{aligned}
\mathcal{K}_0\ \mathbf{f}_0 &= \mathcal{B}_0\mathbf{b}\\
\mathcal{K}_0\ \mathbf{f}_1 &= \mathcal{B}_1\mathbf{b} - \mathcal{K}_1\ \mathbf{f}_0\\
\mathcal{K}_0\ \mathbf{f}_2 &= \mathcal{B}_2\mathbf{b} - \mathcal{K}_1\ \mathbf{f}_1 - \mathcal{K}_2\ \mathbf{f}_0\\
&\ \vdots\\
\mathcal{K}_0\ \mathbf{f}_N &= \mathcal{B}_N\mathbf{b} - \textstyle\sum_{i=1}^{N}\mathcal{K}_i\ \mathbf{f}_{N-i}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{g}_0^T\ \mathcal{K}_0 &= \mathbf{c}^T\mathcal{C}_0\\
\mathbf{g}_1^T\ \mathcal{K}_0 &= \mathbf{c}^T\mathcal{C}_1 - \mathbf{g}_0^T\ \mathcal{K}_1\\
\mathbf{g}_2^T\ \mathcal{K}_0 &= \mathbf{c}^T\mathcal{C}_2 - \mathbf{g}_1^T\ \mathcal{K}_1 - \mathbf{g}_0^T\ \mathcal{K}_2\\
&\ \vdots\\
\mathbf{g}_M^T\ \mathcal{K}_0 &= \mathbf{c}^T\mathcal{C}_M - \textstyle\sum_{j=1}^{M}\mathbf{g}_{M-j}^T\ \mathcal{K}_j
\end{aligned}$$

Then

- if span$\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N\} \subset$ Ran$(\mathbf{V}_r)$ then $\mathcal{H}^{(\ell)}(\sigma)\mathbf{b} = \mathcal{H}_r^{(\ell)}(\sigma)\mathbf{b}$ for $\ell = 0, \ldots, N$;
- if span$\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_M\} \subset$ Ran$(\mathbf{W}_r)$ then $\mathbf{c}^T\mathcal{H}^{(\ell)}(\sigma) = \mathbf{c}^T\mathcal{H}_r^{(\ell)}(\sigma)$ for $\ell = 0, \ldots, M$; and
- if both span$\{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N\} \subset$ Ran$(\mathbf{V}_r)$ and span$\{\mathbf{g}_0, \mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_M\} \subset$ Ran$(\mathbf{W}_r)$ then $\mathbf{c}^T\mathcal{H}^{(\ell)}(\sigma)\mathbf{b} = \mathbf{c}^T\mathcal{H}_r^{(\ell)}(\sigma)\mathbf{b}$ for $\ell = 0, \ldots, M+N+1$.

Remark 4.1: Theorem 4.1 illustrates how to construct the reducing subspaces numerically in a recursive way.

A. Second- and higher-order Dynamical Systems

Consider the second-order dynamical system of the form

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{G}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \qquad (8)$$

where $\mathbf{M}, \mathbf{G}, \mathbf{K} \in \mathbb{R}^{n\times n}$ are symmetric positive definite matrices, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$.

Second order systems of the form (8) arise naturally in analyzing many physical phenomena, such as structural vibration, electrical circuits, and micro-electro-mechanical systems; see, for example, [14], [26], [4], [13], [32], [12], [14], [22], [6], and references therein. $\mathbf{M}$, $\mathbf{G}$, and $\mathbf{K}$ are called, respectively, the mass, damping and stiffness matrices. The transfer function $\mathcal{H}(s)$ from inputs $\mathbf{u}(t)$ to outputs $\mathbf{y}(t)$ is given by

$$\mathcal{H}(s) = \mathbf{C}(s^2\mathbf{M} + s\mathbf{G} + \mathbf{K})^{-1}\mathbf{B}.$$

In many cases, the original system dimension n is too large for efficient simulation and control purposes. Therefore, the goal is to generate, for some $r \ll n$, an r^{th} order reduced second-order system of the form

$$\begin{aligned}
\mathbf{M}_r\ddot{\mathbf{x}}_r(t) + \mathbf{G}_r\dot{\mathbf{x}}_r(t) + \mathbf{K}_r\mathbf{x}_r(t) &= \mathbf{B}_r\mathbf{u}(t),\\
\mathbf{y}_r(t) &= \mathbf{C}_r\mathbf{x}_r(t) \qquad (9)
\end{aligned}$$

where $\mathbf{M}_r, \mathbf{G}_r, \mathbf{K}_r \in \mathbb{R}^{r\times r}$, $\mathbf{B}_r \in \mathbb{R}^{r\times m}$ and $\mathbf{C}_r \in \mathbb{R}^{p\times r}$ so that $\mathbf{y}_r(t)$ approximates $\mathbf{y}(t)$ for a wide range of inputs $\mathbf{u}(t)$.

Since converting (8) into the first-order framework and applying reduction in that first-order setting destroys the structure, the goal is to apply reduction directly in the second-order framework. To achieve this, one constructs a matrix $\mathbf{V}_r \in \mathbb{R}^{n\times r}$ such that the associated reduced-order model in (9) is given by

$$\begin{aligned}
&\mathbf{M}_r = \mathbf{V}_r^T\mathbf{M}\mathbf{V}_r, \quad \mathbf{G}_r = \mathbf{V}_r^T\mathbf{G}\mathbf{V}_r, \quad \mathbf{K}_r = \mathbf{V}_r^T\mathbf{K}\mathbf{V}_r,\\
&\mathbf{B}_r = \mathbf{V}_r^T\mathbf{B}, \qquad \text{and} \qquad \mathbf{C}_r = \mathbf{C}\mathbf{V}_r \qquad .
\end{aligned}$$

In [30], Su and Craig has shown that one can directly reduce the second-order matrices in a structure preserving setting, meanwhile matching the moments around $\sigma = \infty$. Recently, Bai and Su [5] further improved this work by introducing the so-called *second-order Krylov subspaces* and *second-order Arnoldi procedure*. These methods use in effect, a two-stage recurrence in $\mathbb{R}^n$ to generate the effect of the usual one-stage Krylov recurrence in $\mathbb{R}^{2n}$. In addition to being numerically effective and robust due to a Arnoldi-like structure, the method of [5] has also extended the structure-preserving moment matching property of [30] to interpolation around arbitrary points $\sigma \in \mathbb{C}$. For more work on the structure-preserving second-order model reduction, [6], [28], [12], [8], [9], [17], [23], [11].

Clearly, systems of the form (8) fit in our generalized transfer function framework after defining

$$\mathcal{C}(s) = \mathbf{C}, \quad \mathcal{K}(s) = s^2\mathbf{M} + s\mathbf{G} + \mathbf{K} \quad \text{and} \quad \mathcal{B}(s) = \mathbf{B}.$$

Hence we can apply Theorems 3.1 and 4.1. To preserve the symmetry and positive definiteness of $\mathbf{M}$, $\mathbf{G}$ and $\mathbf{K}$, we apply one sided reduction, i.e. we choose $\mathbf{W}_r = \mathbf{V}_r$ in Theorem. 3.1. Hence for this special case of $\mathcal{H}(s)$ in (8) and one-sided reduction, the recurrence in Theorem 3.1 simplifies to a three-term recurrence:

Algorithm 4.1: A recurrence for reduction of second-order systems:

1) Choose σ and the tangential direction $\mathbf{b}$
2) Define $\mathcal{K}_0 = \sigma^2\mathbf{M} + \sigma\mathbf{G} + \mathbf{K}$, $\mathcal{K}_1 = 2\sigma\mathbf{M} + \mathbf{G}$, and $\mathcal{K}_2 = \mathbf{M}$
3) Set $\mathbf{f}_{-1} = \mathbf{0}$. Solve $\mathcal{K}_0\mathbf{f}_0 = \mathbf{B}\mathbf{b}$.
4) $for \quad j = 1 : N$
   - Solve $\mathcal{K}_0\mathbf{f}_j = -\mathcal{K}_1\mathbf{f}_{j-1} - \mathcal{K}_2\mathbf{f}_{j-2}$,
5) $\mathbf{V}_r = [\mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_N]$.

The transfer function of the resulting reduced second-order model will interpolate that of the original model together with its derivatives at the selected frequency σ and will preserve the original structure.

Remark 4.2: We note that Algorithm 4.1 can handle multi-input and multi-output case. The algorithm contains the second-order recursion of [5] as a special case; namely for the single-input/single-output second-order model $\mathcal{H}(s)$ in (8), i.e. $\mathbf{B}$ and $\mathbf{C}$ are, respectively, column and row vectors, Algorithm 4.1 yields the method of [5].

Remark 4.3: Combining Theorems 3.1 and 4.1, Algorithm 4.1 can be extended to provide interpolation at multiple interpolation points $\sigma_1, \sigma_2, \ldots, \sigma_r$. This is analogous to applying rational Krylov projection as opposed (rational) Arnoldi in the generic setting of first-order dynamical systems.

Remark 4.4: We can easily extend this discussion to higher-order systems:

$$\mathbf{A}_0 \frac{d^\ell \mathbf{x}}{dt^\ell} + \mathbf{A}_1 \frac{d^{\ell-1}\mathbf{x}}{dt^{\ell-1}} + \cdots + \mathbf{A}_\ell \mathbf{x}(t) = \mathbf{B}\mathbf{u}(t)$$

and

$$\mathbf{y}(t) = \mathbf{C}_1 \frac{d^{\ell-1}\mathbf{x}}{dt^{\ell-1}} + \cdots + \mathbf{C}_\ell \mathbf{x}(t)$$

This will yield a $\ell + 1$-term recursion similar to Algorithm 4.1.

B. Delay Systems

Another important application for our generalized interpolatory model reduction setting is linear dynamical systems with an internal delay presented in state space form as:

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}_0\,\mathbf{x}(t) + \mathbf{A}_1\,\mathbf{x}(t-\tau) + \mathbf{B}\,\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) \tag{10}$$

with $\tau > 0$, $\mathbf{E}$, $\mathbf{A}_0$, $\mathbf{A}_1 \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$. We wish to produce a reduced order model having the same internal delay structure:

$$\mathbf{W}_r^T\mathbf{E}\mathbf{V}_r\,\dot{\mathbf{x}}_r(t) = \mathbf{W}_r^T\mathbf{A}_0\mathbf{V}_r\,\mathbf{x}_r(t) + \mathbf{W}_r^T\mathbf{A}_1\mathbf{V}_r\,\mathbf{x}_r(t-\tau) + \mathbf{W}_r^T\mathbf{B}\,\mathbf{u}(t) \tag{11}$$

$$\mathbf{y}_r(t) = (\mathbf{C}\mathbf{V}_r)\,\mathbf{x}_r(t)$$

Taking a Laplace transform, the transfer function of (10) is found to be

$$\mathcal{H}(s) = \mathbf{C}\left(s\,\mathbf{E} - \mathbf{A}_0 + e^{-\tau s}\,\mathbf{A}_1\right)^{-1}\mathbf{B} \tag{12}$$

$$= \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s) \tag{13}$$

with

$$\mathcal{C}(s) = \mathbf{C}, \;\; \mathcal{K}(s) = s\,\mathbf{E} - \mathbf{A}_0 + e^{-\tau s}\,\mathbf{A}_1, \quad \text{and} \;\; \mathcal{B}(s) = \mathbf{B} \tag{14}$$

Hence, the delay system in (10) perfectly fits in our generalized framework. One can use Theorem 3.1 with $\mathcal{C}(s)$ $\mathcal{K}(s)$ and $\mathcal{B}(s)$ as defined in (14) to obtain a reduced-order model as in (11) having the same internal delay structure as the original model (10). We note that this reduced model not only has the same structure as the original model but also *exactly* interpolates the original system at the selected interpolation points.

In order to apply interpolatory model reduction to the full-order model (10) without our generalized interpolation framework, one would need to approximate the exponential $e^{-\tau s}$ with a rational approximation. Commonly a rational approximation for the delay term, $e^{-\tau s}$, is used which then allows the use of a variety of standard (finite dimensional) system theoretic tools. For example, $e^{-\tau s} \approx \frac{p_\ell(-\tau s)}{p_\ell(\tau s)}$ with a common choice for ℓ-th order polynomials coming from Laguerre-Fourier series or Padé approximation.

Assume that the first order Padé approximation is used. In this case, $e^{-\tau s}$ is replaced by $\frac{1-\tau s/2}{1+\tau s/2}$, leading to the transfer function

$$H_{p,1}(s) = (\mathbf{C} + s\frac{\tau}{2}\mathbf{C})(\mathbf{M}s^2 + \mathbf{G}s + \mathbf{K})^{-1}\mathbf{B} \tag{15}$$

where $\mathbf{M} = \frac{\tau}{2}\mathbf{E}$, $\mathbf{G} = \mathbf{E} + \frac{\tau}{2}(-\mathbf{A}_0 + \mathbf{A}_1)$, and $\mathbf{K} = -(\mathbf{A}_0 + \mathbf{A}_1)$. Note that, due to the frequency dependency in the observation matrix, existing second-order model reduction approaches will not work for the two-sided projection; one will have to transform (15) into an equivalent first-order framework and perform the reduction there. Hence, using a Padé approximant has not only destroyed the delay structure but also caused to work with the matrices of double the size. In most cases, In order to obtain good full-order Padé approximation, one will need to go to higher order approximations. However, similar to the first-order Padé case, for an ℓ^{th} order Padé approximation, one will need to work with matrices of dimension $[500(\ell{+}1)]\times[500(\ell{+}1)]$, causing a big overhead in the model reduction process.

V. The case $\mathcal{D}_r \neq \mathcal{D}$

We consider now a full order model (5)

$$\mathcal{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s) + \mathcal{D}$$

and reduced order models having the form (6)

$$\mathcal{H}_r(s) = \mathcal{C}_r(s)\mathcal{K}_r(s)^{-1}\mathcal{B}_r(s) + \mathcal{D}_r.$$

Theorem 5.1: Suppose $2r$ distinct points are given in the right halfplane, $\{\mu_1, \mu_2, ...\mu_r\} \cup \{\sigma_1, \sigma_2, ...\sigma_r\}$, together with $2r$ nontrivial vectors, $\{\mathbf{c}_1, \mathbf{c}_2, ...\mathbf{c}_r\} \subset \mathbb{C}^p$ and $\{\mathbf{b}_1, \mathbf{b}_2, ...\mathbf{b}_r\} \subset \mathbb{C}^m$. Define matrices $\mathbf{V}_r \in \mathbb{C}^{n\times r}$ and $\mathbf{W}_r \in \mathbb{C}^{n\times r}$ such that

$$\mathbf{V}_r = [\mathcal{K}(\sigma_1)^{-1}\mathcal{B}(\sigma_1)\mathbf{b}_1, \,...,\, \mathcal{K}(\sigma_r)^{-1}\mathcal{B}(\sigma_r)\mathbf{b}_r\,]$$

and

$$\mathbf{W}_r^T = \begin{bmatrix} \mathbf{c}_1^*\mathcal{C}(\mu_1)\mathcal{K}(\mu_1)^{-1} \\ \vdots \\ \mathbf{c}_r^*\mathcal{C}(\mu_r)\mathcal{K}(\mu_r)^{-1} \end{bmatrix}$$

Assume that $\mathbf{W}_r^T\mathbf{V}_r$ is nonsingular and let $\mathbf{F}$ and $\mathbf{G}$ be solutions to

$$\mathbf{F}^T\mathbf{V}_r = [\mathbf{b}_1, \,...,\, \mathbf{b}_r\,] = \mathbf{B} \;\; \text{and} \;\; \mathbf{W}_r^T\mathbf{G} = \begin{bmatrix} \mathbf{c}_1^* \\ \vdots \\ \mathbf{c}_r^* \end{bmatrix} = \mathbf{C}^*$$

For any $\mathcal{D}_r \in \mathbb{C}^{p\times m}$, define

$$\begin{aligned} \mathcal{K}_r(s) &= \mathbf{W}_r^T(\mathcal{K}(s) - \mathbf{G}(\mathcal{D}_r - \mathcal{D})\mathbf{F}^T)\mathbf{V}_r, \\ \mathcal{B}_r(s) &= \mathbf{W}_r^T(\mathcal{B}(s) - \mathbf{G}(\mathcal{D}_r - \mathcal{D})), \\ \text{and} \quad \mathcal{C}_r(s) &= (\mathcal{C}(s) - (\mathcal{D}_r - \mathcal{D})\mathbf{F}^T)\mathbf{V}_r. \end{aligned} \tag{16}$$

Then with $\mathcal{H}_r(s) = \mathcal{C}_r(s)\mathcal{K}_r(s)^{-1}\mathcal{B}_r(s) + \mathcal{D}_r$ we have

$$\mathcal{H}(\sigma_i)\mathbf{b}_i = \mathcal{H}_r(\sigma_i)\mathbf{b}_i \quad \text{and} \quad \mathbf{c}_i^*\mathcal{H}(\mu_i) = \mathbf{c}_i^*\mathcal{H}_r(\mu_i)$$

for $i = 1, \,...,\, r$.

VI. Conclusions and Future Work

We have presented a general interpolatory framework model reduction of structured dynamical systems. The proposed framework is much broader than rational Krylov-based methods and allows retention of special structure in the reduced order models such as internal delays, infinite dimensional subsystems, symmetry, and second order structure. In addition to the proofs of theorems presented here, application of this new setting to model reduction of partitioned systems and descriptor system will be presented in a separate work.

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