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Output Feedback Stabilization of Systems with Nonlinearity of Unmeasured States

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Abstract— **This paper deals with global stabilization of a class of nonlinear dynamic systems using output feedback. The systems considered in the paper have nonlinear functions of unmeasured state variables, in addition to nonlinear functions of the system outputs. A nonlinear reduced order observer is proposed to estimate the unknown system state in the system. Estimated state variables are used for control design, together with the system output. Certain conditions are identified for the proposed observer and control design. Under those conditions, the proposed control design ensures the global stability of the closed-loop control system.**

I. INTRODUCTION

Global asymptotic stabilization of nonlinear dynamic systems via output feedback has attracted considerable attention in the last decade and beyond. A number of important results have been obtained when the nonlinear functions in the systems are the functions of the measured outputs. Among the early results, it is shown in [1] that the restriction on the growth rate of the nonlinear functions can be removed for the dynamic systems in the output feedback form. Subsequently, systematic design methods are shown in [2], [3] for the nonlinear systems with unknown parameters, and recent results on disturbance rejection and output regulation can be found in [4], [5]. When the nonlinearity in the system involves unmeasured state variables, the restriction on the growth rate of nonlinearity is often imposed for the proposed control design [6]. When the unmeasured states correspond to the states for the zero dynamics, a common restriction is the input to state stability of the zero dynamics, viewing the output as the input to the zero dynamics [7]. Linear growth rate of the other state variables in nonlinear functions is required in the global stabilization of nonlinear systems by output feedback[8], [9]. As pointed out in [6], output feedback stabilization is not possible even for some class of nonlinear systems which can be globally stabilized by static state feedback. In general, the separation principle, enjoyed by linear system design, does not work for nonlinear systems. Furthermore, except for nonlinear systems linear in unknow states such as the systems in the output feedback form, observer design methods [10], [11] are locally convergent, and therefore cannot be used for global stabilization.

In this paper, we consider a class of nonlinear systems with nonlinear functions of the unmeasured state variables. A number of structural conditions have been identified such that global stabilization can be achieved by output feedback control design. At least a polynomial growth rate is allowed for the nonlinear functions of the unmeasured states. Reduced order observer design is exploited for estimation for unmeasured state variables based on a state transformation. Another state transformation is introduced to allow control design to start directly from an estimated state. The proposed control design ensures the global asymptotic stabilization of the class of the nonlinear systems, and extends the class of nonlinear systems which can be stabilized by output feedback.

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II. PROBLEM FORMULATION

Consider a single-input-single-output nonlinear system

$$\begin{aligned} \dot{x} &= Ax + \phi(c^T x) + f\varphi(d^T x) + b\sigma(c^T x)u, \\ y &= c^T x \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the control input, $\phi : \mathbb{R} \to \mathbb{R}^n$, is a known nonlinear smooth vector field, $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth nonlinear function, $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinear function and $\sigma(y) \neq 0$, $\forall y \in \mathbb{R}$, $b, c, d, f \in \mathbb{R}^n$ are constant vectors, and $A \in \mathbb{R}^{n\times n}$ is a constant matrix.

We have the following assumptions of the system.

Assumption 1.

1.1 $\{c^T, A\}$ is observable.
1.2 $\{d^T, A\}$ is observable.
1.3 A SISO linear system characterized by $\{A, f, c^T\}$ has relative degree 1, and is minimum phase.
1.4 A SISO linear system characterized by $\{A, b, d^T\}$ has relative degree n.

Remark 1: Assumption 1 specifies the conditions of the nonlinear system which are useful to reveal a number of structural properties under linear state transformations for the convenience of observer and control design.

Since $\{d^T, A\}$ is observable, we define a state transformation

$$\xi = T_1 x$$

where $T_1 = [d, A^T d, \ldots, (A^{n-1})^T d]^T$, and it is nonsingular. Under the coordinate ξ, we have $y = c^T T_1^{-1}\xi := l^T\xi$, and we define $\psi(y) := T_1\phi(y)$.

Assumption 2. The nonlinear function ψ satisfies the following condition

$$\frac{\partial \psi_i(l^T\xi)}{\partial \xi_j} = 0$$

for $i = 1, \ldots, n-1$ and $j > i$.

Remark 2: The condition specified in Assumption 2 is similar to triangular condition imposed on the strict feedback form [3], and it is required for the control design.

There is not other restriction on the nonlinear vector field $\phi(y)$. However, we have an assumption of the nonlinear function $\varphi(\cdot)$.

Assumption 3. The following expression holds for any $\rho_1, \rho_2 \in \mathbb{R}$

$$|\varphi(\rho_1 + \rho_2) - \varphi(\rho_1)| \le \gamma_0(\rho_2) + |\rho_1|\gamma_1(\rho_1)\gamma_2(\rho_2)$$

where $\gamma_i : \mathbb{R} \to \mathbb{R}$ for $i = 0, 1, 2$ are functions with $\gamma_i(\rho) \ge 0$ for any $\rho \in \mathbb{R}$, and $\gamma_1(\cdot)$ is smooth. Furthermore, if for a function $\rho(t) : \mathbb{R}^+ \to \mathbb{R}$ with $|\rho(t)| < \rho_3 e^{-\rho_4 t}$ for positive reals constants ρ_3 and ρ_4, then there exist some positive real constants ρ_5 and ρ_6 such that $\gamma_0(\rho(t)) \le \rho_5 e^{-\rho_6 t}$ and $\gamma_2(\rho(t)) \le \rho_5 e^{-\rho_6 t}$.

Remark 3: Many common nonlinear functions, including polynomials, satisfy the conditions specified in Assumption 3. For example, for $\varphi(\rho) = \rho^2$, we have $|\varphi(\rho_1 + \rho_2) - \varphi(\rho_1)| \le \rho_2^2 + |\rho_1|2|\rho_2|$.

The problem considered in this paper is to design an output feedback control law to ensure the global and asymptotic stability of the system.

Remark 4: If we have $d = c$ in (1) or $\varphi \equiv 0$, the system can be transformed to the standard output feedback form considered in [2], [3] by a linear transformation. The difficulty in the stabilization problem considered in this paper is due to the nonlinear function φ of some unmeasured state variables.

III. Observer Design

Output feedback control often depends on the observer design to provide an estimate of the system state variables. For the nonlinear systems in the standard output feedback form, observers can be easily designed due to the linearity in the observer errors, and the control design is then followed by observer backstepping. For the system considered in this paper, there is an additional nonlinear function of unmeasured system state, and therefore the observer backstepping technique proposed for systems in the standard output feedback form cannot be applied. In this section, we propose a new observer for state observation.

Consider a state transformation

$$z = \mathcal{A}\mathcal{O}x := T_2 x$$

where

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ a_1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \dots & 1 \end{bmatrix}, \mathcal{O} = \begin{bmatrix} c^T \\ c^T A \\ \vdots \\ c^T A^{n-1} \end{bmatrix},$$

with a_i, for $i = 1, \dots, n$, being the coefficients of the characteristic polynomial of A. It can be obtained that

$$\begin{aligned} \dot{z} &= A_o z + \bar{\phi}(z_1) + \bar{f}\varphi(d^T T_2^{-1} z) + \bar{b}\sigma(z_1)u, \\ y &= z_1 \end{aligned} \tag{2}$$

where A_o is the left companion matrix of the characteristic polynomial of A, and $\bar{\phi} = T_2\phi$, $\bar{f} = T_2 f$ and $\bar{b} = T_2 b$. Note that Assumption 1.3 ensures $\bar{f}_1 \ne 0$.

Due to the additional nonlinear function φ, the observer with linear observer error cannot be applied for (2). However, we exploit the reduced observer design technique for this system, and propose the following observer design.

$$\begin{aligned} \dot{w} &= Fw + \phi_w(y) + b_w\sigma(y)u \\ \hat{z}_{2:n} &= w + f_w y \end{aligned} \tag{3}$$

where the subscript $(2 : n)$ denotes the vector form by the second to the nth elements of the original vector, and $\hat{z}_{2:n}$ denotes the estimate of $z_{2:n}$, $F \in \mathbb{R}^{(n-1)\times(n-1)}$ is the left companion matrix of the characteristic polynomial $s^{n-1} + \frac{\bar{f}_2}{\bar{f}_1}s^{n-2} + \ldots + \frac{\bar{f}_n}{\bar{f}_1} = 0$,

$$f_w = \left[\frac{\bar{f}_2}{\bar{f}_1}, \dots, \frac{\bar{f}_n}{\bar{f}_1}\right]^T,$$

$$b_w = \bar{b}_{2:n} - \bar{b}_1 f_w,$$

and

$$\phi_w(y) = \bar{\phi}_{2:n}(y) - \bar{\phi}_1(y)f_w - (a_{2:n} - a_1 f_w)y$$

with $a = [a_1, a_2, \dots, a_n]^T$.

For the properties of this reduced order observer, we have the following lemma.

Lemma 1: The reduced order observer (3) provides an exponentially convergent estimate of unmeasured system state variables of (2).

Proof. Let us introduce another state transformation

$$\eta = T_3 z$$

where

$$T_3 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -\frac{\bar{f}_2}{\bar{f}_1} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\bar{f}_n}{\bar{f}_1} & 0 & \dots & 1 \end{bmatrix}.$$

A direct evaluation gives

$$\dot{\eta}_{2:n} = F\eta_{2:n} + \phi_w(y) + b_w\sigma(y)u \tag{4}$$

Note that the nonlinear function φ does not appear in the dynamics of $\eta_{2:n}$ in (4), due to the special choice of the transformation matrix T_3. Now, let $\tilde{\eta}_{2:n} = \eta_{2:n} - w$. From (3) and (4), the dynamics of $\tilde{\eta}_{2:n}$ is obtained as

$$\dot{\tilde{\eta}}_{2:n} = F\tilde{\eta}_{2:n}.$$

From the inverse transformation of T_3, we have

$$z_{2:n} = \eta_{2:n} + f_w y$$

Therefore, we have the expression of $\hat{z}_{2:n}$ in (3). Let $\tilde{z} = z - \hat{z}$, to denote the observer error. We have $\tilde{z}_{2:n} = \tilde{\eta}_{2:n}$, and hence

$$\dot{\tilde{z}}_{2:n} = F\tilde{z}_{2:n} \tag{5}$$

Assumption 1.3 implies that F is Hurwitz, and therefore the observer error exponentially converge to zero. This concludes the proof.

IV. Control Design

Even with state feedback, there are not many design methods to ensure global stability of nonlinear dynamic systems. One of the well known classes of nonlinear systems is the strict feedback form, for which backstepping can be applied to design a globally stabilizing controller. For linear systems, state feedback stabilization methods can be easily extended to output feedback, by applying the separation principle, that is, to replace the state variables in the state feedback control design by its estimate from an observer. However, the separation principle does not hold for nonlinear systems in general. In fact, it can be shown that the stability cannot be guaranteed in general by replacing an exponentially convergent state estimate in the control law for nonlinear systems in the strict feedback from.

In this section, we will introduce a control design method starting from the estimated state variables for the class of the nonlinear systems considered in this paper. For the convenience of the control design, we transform the system (1) to the state variable ξ. It can be obtained as

$$\begin{aligned} \dot{\xi} &= A_c\xi + \psi(y) + g\varphi(\xi_1) + h\sigma(y)u, \\ y &= l^T\xi \end{aligned} \tag{6}$$

where A_c is the lower companion matrix of the characteristic polynomial of A, $g = T_1 f$, and $h = T_1 b$. Based on Assumption 1.4, we have $h_i = 0$ for $i = 1, \ldots, n-1$ and $h_n \neq 0$. Hence from the structure of A_c and h, we can write the dynamics for the individual states as

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + \psi_1(y) + g_1\varphi(\xi_1) \\ &\vdots \\ \dot{\xi}_{n-1} &= \xi_n + \psi_{n-1}(y) + g_{n-1}\varphi(\xi_1) \\ \dot{\xi}_n &= h_n\sigma(y)u + \psi_n(y) + g_n\varphi(\xi_1) \\ &\quad -\sum_{i=1}^{n} a_i\xi_{n-i+1} \end{aligned} \tag{7}$$

Based on the state estimate for z, we have the estimate for ξ given by

$$\hat{\xi} = T_1T_2^{-1}\begin{bmatrix} y \\ \hat{z}_{2:n} \end{bmatrix}$$

Let $\tilde{\xi} = \xi - \hat{\xi}$ and it can be obtained that

$$\tilde{\xi} = T_1T_2^{-1}\tilde{z} = T_1T_2^{-1}\begin{bmatrix} 0 \\ \tilde{z}_{2:n} \end{bmatrix}$$

and

$$\dot{\tilde{\xi}} = T_1T_2^{-1}\begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}T_2T_1^{-1}\tilde{\xi} := G\tilde{\xi}$$

Since ξ is not available, control design will be carried out with $\hat{\xi}$. For the control design, we introduce the following notations:

$$\begin{aligned} \zeta_1 &:= \hat{\xi}_1 \\ \zeta_i &:= \hat{\xi}_i - \alpha_{i-1}(\hat{\xi}_1, \ldots, \hat{\xi}_{i-1}, y) \text{ for } i = 2, \ldots, n \end{aligned} \tag{8}$$

where α_i are the stabilizing functions to be designed. In order to express the control designed for Step n in the same form as the other steps, we define $\xi_{n+1} := h_n\sigma(y)u - \sum_{i=1}^{n} a_i\xi_{n-i+1}$ and $\hat{\xi}_{n+1} := h_n\sigma(y)u - \sum_{i=1}^{n} a_i\hat{\xi}_{n-i+1}$. Thus, we can denote $\zeta_{n+1} = \hat{\xi}_{n+1} - \alpha_n$ and $\tilde{\xi}_{n+1} = -\sum_{i=1}^{n} a_i\tilde{\xi}_{n-i+1}$. For the convenience of notations, we introduce

$$\begin{aligned} \bar{g}_1 &:= g_1 \\ \bar{g}_i &:= g_i - \sum_{j=1}^{i-1}\left(\frac{\partial\alpha_{i-1}}{\partial\hat{\xi}_j} + l_j\frac{\partial\alpha_{i-1}}{\partial y}\right)g_j \end{aligned} \tag{9}$$

for $i = 2, \ldots, n$.

The control design will be carried out in n steps.

Step 1. We start the control design from ζ_1. Its dynamics are given by

$$\begin{aligned} \dot{\zeta}_1 &= \dot{\xi}_1 - \dot{\tilde{\xi}}_1 \\ &= \xi_2 + \psi_1(y) + g_1\varphi(\xi_1) - G_1\tilde{\xi} \\ &= \zeta_2 + \alpha_1 + \psi_1(y) + g_1\varphi(\xi_1) - G_1\tilde{\xi} + \tilde{\xi}_2 \end{aligned}$$

where G_1 denotes the first row of matrix G. From Assumption 3, we have

$$\begin{aligned} & \zeta_1 g_1(\varphi(\xi_1) - \varphi(\hat{\xi}_1)) \\ = & \zeta_1 g_1(\varphi(\hat{\xi}_1 + \tilde{\xi}_1) - \varphi(\hat{\xi}_1)) \\ \leq & |\zeta_1 g_1|(\gamma_0(\tilde{\xi}_1) + |\hat{\xi}_1|\gamma_1(\hat{\xi}_1)\gamma_2(\tilde{\xi}_1)) \\ \leq & k\zeta_1^2 g_1^2 + \frac{1}{4k}\gamma_0^2(\tilde{\xi}_1) + k\zeta_1^2 g_1^2\hat{\xi}_1^2\gamma_1^2(\hat{\xi}_1) + \frac{1}{4k}\gamma_2^2(\tilde{\xi}_1) \\ = & k\zeta_1^2 g_1^2(1 + \hat{\xi}_1^2\gamma_1^2(\hat{\xi}_1)) + \frac{1}{4k}\gamma_0^2(\tilde{\xi}_1) + \frac{1}{4k}\gamma_2^2(\tilde{\xi}_1) \end{aligned}$$

where we have used the fact that $|ab| \leq ka^2 + \frac{1}{4k}b^2$ for any positive real a, b, k. Based on the above, we design the first stabilizing function as

$$\begin{aligned} \alpha_1 &= -(r_1 + k)\zeta_1 - \psi_1(y) - g_1\varphi(\hat{\xi}_1) \\ &\quad -k\zeta_1\bar{g}_1^2(1 + \gamma_1^2(\hat{\xi}_1)) \end{aligned} \tag{10}$$

where r_1 is among the set of positive real design parameters $\{r_i\}$ for $i = 1, \ldots, n$.

Step 2. The dynamics of ζ_2 is described by

$$\begin{aligned} \dot{\zeta}_2 &= \dot{\hat{\xi}}_2 - \dot{\alpha}_1(\hat{\xi}_1, y) \\ &= \dot{\xi}_2 - G_2\tilde{\xi} - \frac{\partial\alpha_1}{\partial\hat{\xi}_1}\dot{\hat{\xi}}_1 - \frac{\partial\alpha_1}{\partial y}l_1\dot{\xi}_1 \\ &= \dot{\xi}_2 - \left[\frac{\partial\alpha_1}{\partial\hat{\xi}_1} + \frac{\partial\alpha_1}{\partial y}l_1\right]\dot{\xi}_1 - G_2\tilde{\xi} + \frac{\partial\alpha_1}{\partial\hat{\xi}_1}G_2\tilde{\xi} \end{aligned}$$

Remark 5: To obtain the above expression, we have used $\dot{y} = \frac{\partial\alpha_1}{\partial y}l_1\dot{\xi}_1$ not $\dot{y} = \sum_{j=1}^{n}\frac{\partial\alpha_1}{\partial y}l_j\dot{\xi}_j$. This is due to the triangular condition specified in Assumption 2. From Assumption 2, we have $\frac{\partial\psi_1(y)}{\partial y} = 0$ if $l_2 \neq 0$. Similarly we will use $\dot{y} = \sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y}l_j\dot{\xi}_j$ at Step i, because if

$l_i \neq 0$, then from Assumption 2 we must have $\frac{\partial \psi_j(y)}{\partial y} = 0$ for $j = 1, \ldots, i-1$. The differences of $\dot{y}$ shown in the expressions do not mean that $\dot{y}$ is different at different steps. In fact, for a given y, the expression is unique, and many terms are actually zero. We use different expressions of $\dot{y}$ in different step for the convenience of notations for a generic expression of y.

Using the dynamics of ξ_1 and ξ_2, and $\xi = \hat{\xi} + \tilde{\xi}$, we have

$$\begin{aligned}\dot{\zeta}_2 &= \zeta_3 + \alpha_2 + \psi_2(y) + \bar{g}_2\varphi(\xi_1) \\ &\quad -[\frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} l_1](\hat{\xi}_2 + \psi_1(y)) \\ &\quad +\tilde{\xi}_3 - [\frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} l_1]\tilde{\xi}_2 \\ &\quad -G_2\tilde{\xi} + \frac{\partial \alpha_1}{\partial \hat{\xi}_j} G_1\tilde{\xi} \qquad (11)\end{aligned}$$

The stabilizing function α_2 is designed as

$$\begin{aligned}\alpha_2 &= -\zeta_1 - (r_2 + k)\zeta_2 - \psi_2(y) - \bar{g}_2\varphi(\hat{\xi}_1) \\ &\quad +[\frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} l_1](\hat{\xi}_2 + \psi_1(y)) \\ &\quad -k\zeta_2\bar{g}_2^2(1 + \hat{\xi}_1^2\gamma_1^2(\hat{\xi}_1)) \\ &\quad -k\zeta_2[\frac{\partial \alpha_1}{\partial \hat{\xi}_1} + \frac{\partial \alpha_1}{\partial y} l_1]^2 - k\zeta_2[\frac{\partial \alpha_1}{\partial \hat{\xi}_1}]^2 \qquad (12)\end{aligned}$$

Step i. Similar to the procedures shown in Step 2, in the subsequent steps, for $i = 3, \ldots, n$, we have

$$\begin{aligned}\dot{\zeta}_i &= \zeta_{i+1} + \alpha_i + \psi_i(y) + \bar{g}_i\varphi(\xi_1) \qquad (13) \\ &\quad -\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j](\hat{\xi}_{j+1} + \psi_j(y)) \\ &\quad +\tilde{\xi}_{i+1} - \sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j]\tilde{\xi}_{j+1} \\ &\quad -G_i\tilde{\xi} + \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} G_j\tilde{\xi} \qquad (14)\end{aligned}$$

The stabilizing function α_i is obtained as

$$\begin{aligned}\alpha_i &= -\zeta_{i-1} - (r_i + k)\zeta_i - \psi_i(y) - \bar{g}_i\varphi(\hat{\xi}_1) \\ &\quad -\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j](\hat{\xi}_{j+1} + \psi_j(y)) \\ &\quad -k\zeta_i\bar{g}_i^2(1 + \hat{\xi}_1^2\gamma_1^2(\hat{\xi}_1)) \\ &\quad -k\zeta_i\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j]^2 \\ &\quad -k\zeta_i\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j}]^2 \qquad (15)\end{aligned}$$

When $i = n$, we have α_n as defined in (15). The control input is then designed by setting $\zeta_{n+1} = 0$, which results in

$$u = \frac{\alpha_n + \sum_{i=1}^n a_i\hat{\xi}_{n-i+1}}{h_n\sigma(y)} \qquad (16)$$

Remark 6: In the control design, we use k to be denote a generic positive real design constant, instead of using $k_{i,j}$ for different i and j in α_i, for $i = 1, \ldots, n$. Therefore, each k in α_i can have different values for the control design. The values of the design parameters k and r_i do not affect the stability, as it will shown in the next section.

V. Stability Analysis

We shall establish the stability of the closed-loop system using a Lyapunov function. With the design of α_1 in (10), we have

$$\begin{aligned}\dot{\zeta}_1 &= -(r_1 + k)\zeta_1 + \zeta_2 + g_1(\varphi(\xi_1) - \varphi(\hat{\xi})) \\ &\quad -k\zeta_1\bar{g}_1^2(1 + \hat{\xi}_1^2\gamma_1^2(\hat{\xi}_1)) - G_1\tilde{\xi} + \tilde{\xi}_2\end{aligned}$$

Let $V_1 = \frac{1}{2}\zeta_1^2$. It can be obtained that

$$\dot{V}_1 \leq -r_1\zeta_1^2 + \zeta_1\zeta_2 + \epsilon_1 \qquad (17)$$

where

$$\epsilon_1 = \frac{1}{4k}((|\tilde{\zeta}_2| + |G_1\tilde{\xi}|)^2 + \gamma_0^2(\hat{\xi}_1) + \gamma_0^2(\tilde{\xi}_1)).$$

With the design of the stabilizing function α_{i-1}, we have the dynamics of ζ_i, for $i = 2, \ldots, n$, expressed as

$$\begin{aligned}\dot{\zeta}_i &= -\zeta_{i-1} - (r_i + k)\zeta_i + \zeta_{i+1} + \bar{g}_i(\varphi(\xi_1) - \varphi(\hat{\xi}_1)) \\ &\quad -k\zeta_i\bar{g}_i^2(1 + \hat{\xi}_1^2\gamma_1^2(\hat{\xi})) \\ &\quad -k\zeta_i\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j]^2 \\ &\quad -k\zeta_i\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j}]^2 + \tilde{\xi}_{i+1} \\ &\quad -\sum_{j=1}^{i-1}[\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} + \frac{\partial \alpha_{i-1}}{\partial y} l_j]\tilde{\xi}_{j+1} \\ &\quad -G_i\tilde{\xi} + \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\xi}_j} G_j\tilde{\xi}\end{aligned}$$

Let $V_i = \frac{1}{2}\zeta_i^2$. The derivative of V_i is obtained as

$$\dot{V}_i = -\zeta_{i-1}\zeta_i - r_i\zeta_i^2 + \zeta_i\zeta_{i+1} + \epsilon_i \qquad (18)$$

where

$$\epsilon_i = \frac{1}{4k}(\sum_{j=1}^{i-1}(\tilde{\xi}_j^2 + |G_j\tilde{\xi}|^2) + (|\tilde{\xi}_{i+1}| + |G_i\tilde{\xi}|)^2).$$

Therefore, if we let $V = \sum_{i=1}^n V_i$, we have

$$\begin{aligned}\dot{V} &\leq -\sum_{i=1}^n r_i\zeta_i^2 + \sum_{i=1}^n \epsilon_i \\ &\leq -rV + \sum_{i=1}^n \epsilon_i\end{aligned}$$

where $r = \min\{r_1, \ldots, r_n\}$. It can be established that

$$\sum_{i=1}^n \epsilon_i \leq \rho_7 e^{-\rho_8 t}$$

for some positive real constants ρ_7 and ρ_8. Therefore, from the comparison lemma [12], we have

$$\begin{aligned} V(t) &\leq V(0)e^{-rt} + \int_0^t e^{-r(t-\tau)}\rho_7 e^{-\rho_8\tau} d\tau \\ &= V(0)e^{-rt} + \frac{\rho_7}{r-\rho_8}(e^{-\rho_8 t} - e^{-rt}) \end{aligned} \quad (19)$$

Hence we conclude V is bounded and $\lim_{t\to\infty} V(t) = 0$ which implies the boundedness of ζ_i and $\lim_{t\to\infty}\zeta_i(t) = 0$ for $i = 1,\ldots,n$. Now we need to establish the property of $\hat{\xi}$ for $i = 1,\ldots,n$. From $\lim_{t\to\infty}\hat{\xi}_1(t) = 0$, we have $\lim_{t\to\infty}\alpha_1(\hat{\xi}(t), y(t)) = 0$, based on the structure of α_1 and $\frac{\partial\alpha_i(\hat{\xi}_1, l^T\xi)}{\partial\xi_j} = 0$ for $j = 2,\ldots,n$ which is ensured by Assumption 3. Therefore, we have $\lim_{t\to\infty}\hat{\xi}_2(t) = 0$ as $\hat{\xi}_2 = \zeta_2 + \alpha_1$. By induction, staring with $i = 2$ to $i = n$, from $\lim_{t\to\infty}\hat{\xi}_j(t) = 0$ for $j = 1,\ldots,i$, we have $\lim_{t\to\infty}\alpha_i(\hat{\xi}_1(t),\ldots,\hat{\xi}_i(t), y(t)) = 0$ and then $\lim_{t\to\infty}\hat{\xi}_{i+1}(t) = 0$. Hence we conclude $\lim_{t\to\infty}\hat{\xi}_i(t) = 0$ for $i = 1\ldots,n$, and therefore $\lim_{t\to\infty}\xi(t) = 0$ as $\xi = \hat{\xi} - \tilde{\xi}$ and $\lim_{t\to\infty} x(t) = 0$.

To summarize the result established in this section, we have the following theorem.

Theorem 2: Under Assumptions 1 to 3, the output feedback control (16) based on the output y and observer state $\hat{z}$ obtained in (3) globally and asymptotically stabilize the dynamic system (1).

VI. Example

Consider a nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + y^3 + x_2^2 + u \\ \dot{x}_2 &= x_1 + 2x_2^2 \\ y &= x_1 \end{aligned}$$

Comparing with the structure of (1), we have

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, d = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$f = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \phi(y) = \begin{bmatrix} y^3 \\ 0 \end{bmatrix}, \varphi(d^T x) = x_2^2, \sigma = 1.$$

It can be verified that all the three assumptions are satisfied. For the state transformation, we have $z = x$ and the reduced order observer is designed as

$$\begin{aligned} \dot{w} &= -2w - 3y - 2y^3 - 2u \\ \hat{z}_2 &= w + 2y \end{aligned}$$

and the observer error dynamics is given by $\dot{\tilde{z}}_2 = -2\tilde{z}_2$. For the state transformation ξ, we have $\xi_1 = x_2$ and $\xi_2 = x_1$, and the system is described by

$$\begin{aligned} \dot{\xi}_1 &= \xi_2 + 2\xi_1^2 \\ \dot{\xi}_2 &= \xi_1 + y^3 + \xi_1^2 + u \\ y &= \xi_2 \end{aligned}$$

For the nonlinear function $\varphi(\xi_1) = \xi_1^2$, we take $\gamma_1 = 1$. The control design starts from $\hat{\xi}_1$, for which we have $\dot{\hat{\xi}}_1 = \hat{z}_2$. From the control design proposed earlier, we have

$$\begin{aligned} \alpha_1 &= -(r_1 + k)\zeta_1 - 2\hat{\xi}_1^2 - 4k\zeta_1(1 + \hat{\xi}_1^2) \\ \alpha_2 &= -\zeta_1 - (r_2 + k)\zeta_2 - y^3 - \bar{g}_2\hat{\xi}_1^2 - k\zeta_2\bar{g}_2^2(1 + \hat{\xi}_1^2) \\ &\quad -2k\zeta_2[\frac{\partial\alpha_1}{\partial\hat{\xi}_1}]^2 \\ u &= \alpha_2 - \hat{\xi}_1 \end{aligned}$$

In the simulation study, we set $r_1 = r_2 = 1$ and $k = 0.01$. The simulation results are shown in Figures 1 and 2.

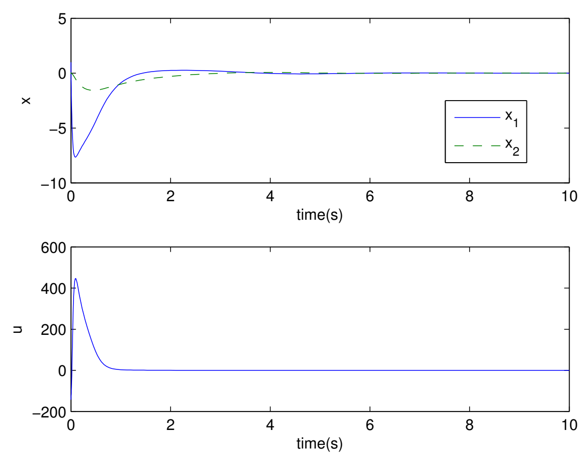

Fig. 1. The state variables and the input

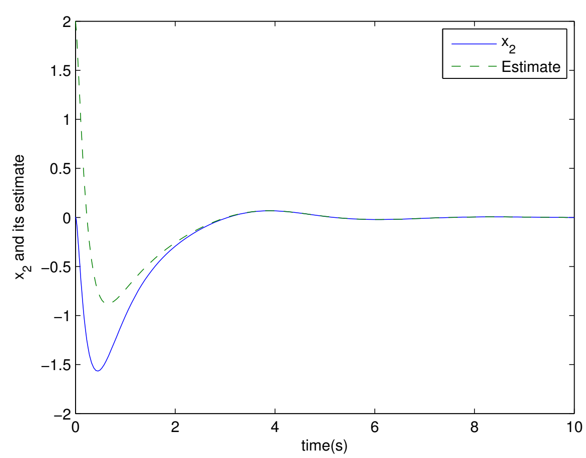

Fig. 2. Unmeasured state and its estimate

VII. Conclusions

We have proposed a control design method for global and asymptotic stabilization of a new class of nonlinear dynamics by output feedback. The reduced order observer proposed in the paper provides an exponentially convergent estimate of unmeasured system states, and control design is then carried out from an estimated state, unlike the observer backstepping [3] which starts from the system output. With due consideration of errors arisen from nonlinear functions of estimated state variables, a control input, which is a function of the system output and the estimated states, globally and asymptotically stabilize the nonlinear system.

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