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The Autonomous Regulator Problem for Linear, Time-delay Systems: a Geometric Approach

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***Abstract*— The aim of this paper is to show the applicability of geometric techniques to a regulation problem for linear, time-delay systems. Given a plant whose dynamics equations include delays, the problem we consider consists in finding a feedback regulator which guarantees asymptotic stability of the regulation loop and asymptotic command following of the reference signal generated by an exosystem, for any initial condition of the overall system. By associating to the time-delay plant a corresponding abstract system with coefficients in a ring, it is possible to place our investigation in a finite dimensional algebraic context, where intuition and results obtained in the classical case, that is without delays, may be exploited.**

I. Introduction

Time-delay dynamical systems of various kind frequently appear in industrial applications, where delays are unavoidable effects of the transportation of materials, and, more generally, in control applications where information is dispatched along slow or very long communication lines, like in tele-operated systems, networked systems, large Integrated Communication Control Systems or ICCS. The study of control problems concerning time-delay systems has, for that reason, attracted the attention of several authors and motivated, in the last years, large research efforts (see the Proceedings of the IFAC Workshops on Time-delay Systems [1], [2], [3], [4], [5], [6] and the books [7], [8] for an account of the recent literature).

Among the various approaches developed for dealing with time-delay systems and related control problems, the one based on the use of geometric methods, in the spirit of [9] and [10], has proved to be particularly effective in many situations, as shown in [11], [12] and the references therein. Application of geometric methods to time-delay systems relies on the possibility of associating naturally to any linear, time-delay system an abstract system with coefficients in a suitable ring. In this way, control problems arising in the time-delay framework can be equivalently formulated, and possibly solved, in an algebraic framework where input/output behaviours have finite dimensional state space realizations. Here, after recalling in Section II the relationship between time-delay systems and systems with coefficients in a ring and some basic notions and results of the geometric approach, we study the so-called Multivariable Autonomous Regulator Problem for linear, time-delay systems with geometric methods. A description of the problem for linear systems without delays and further references can be found in [13] and [14].

Given a linear, time-delay plant Σ_{pd}, the problem, formally stated in Section III, consists in finding a feedback regulator Σ_{rd} which guarantees asymptotic stability of the regulation loop and asymptotic command following of the reference signal generated by an exosystem Σ_{ed}, for any initial condition of the overall system. By extending the results known in the classical case, we find sufficient conditions for the solution of the considered problem in the ring framework. The solution then can be re-interpreted in the original delay-differential framework. The results obtained show the applicability and the efficacy of the geometric approach combined with methods of algebra and ring theory in solving control problems for time-delay systems.

II. Preliminary Results

Let us consider a linear, time invariant, time-delay system Σ_d defined by equations of the form

$$\Sigma_d = \begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i x(t-ih) + \sum_{i=0}^{b} B_i u(t-ih), \\ y(t) = \sum_{i=0}^{c} C_i x(t-ih), \end{cases} \tag{1}$$

where, denoting by $\mathbb{R}$ the field of real numbers, $x(\cdot)$ belongs to the space $\mathbb{R}^n$, $u(\cdot)$ belongs to the input space $\mathbb{R}^m$, $y(\cdot)$ belongs to the output space $\mathbb{R}^p$, A_i, with $i = 0, 1, \ldots, a$, B_i, with $i = 0, 1, \ldots, b$, C_i, with $i = 0, 1, \ldots, c$, are matrices of suitable dimensions with entries in $\mathbb{R}$, and $h \in \mathbb{R}^+$ is a given time delay.

According to a well-known procedure (see e.g. [11]), the system Σ_d can be associated to a new system Σ, defined over a ring. More precisely, by introducing the delay operator δ, defined, for any function $f(t)$, by $\delta f(t) = f(t-h)$, we can rewrite equations (1) as

$$\begin{cases} \dot{x}(t) = \sum_{i=0}^{a} A_i \delta^i x(t) + \sum_{i=0}^{b} B_i \delta^i u(t), \\ y(t) = \sum_{i=0}^{c} C_i \delta^i x(t). \end{cases} \tag{2}$$

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Now, by formally substituting the operator δ by the indeterminate Δ, we can consider the matrices

$$A = \sum_{i=0}^{a} A_i \Delta^i, \quad B = \sum_{i=0}^{b} B_i \Delta^i, \quad C = \sum_{i=0}^{c} C_i \Delta^i,$$

having their elements in the ring of polynomials $\mathbb{R}[\Delta]$, and we can associate to Σ_d the system Σ defined over the ring $\mathbb{R}[\Delta]$ by the set of equations

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{3}$$

where, by abuse of notation, we denote by $x(\cdot)$ an element of the free state module $X = (\mathbb{R}[\Delta])^n$, by $u(\cdot)$ an element of the free input module $U = (\mathbb{R}[\Delta])^m$, and by $y(\cdot)$ an element of the free output module $Y = (\mathbb{R}[\Delta])^p$.
The system Σ, derived as described above, is an exemplification of system with coefficients in a ring.

Abstract systems with coefficients in a ring R have been considered by many authors (see [15], [16], [17], [18], [19], [20] and the references therein) and their study has, in general, provided a better insight into the properties of classical dynamical systems with coefficients in $\mathbb{R}$, which in particular is a ring, as well as it has been useful in dealing with systems with integer coefficients, time-delay systems and families of parameter depending systems.

Note that Σ_d and Σ are quite different objects from a dynamical point of view. However, they share the structural properties that depend on the defining matrices and, in particular, they have the same signal flow graph. This fact implies that control problems concerning the input/output behavior of Σ_d can be formulated naturally in terms of the input/output behavior of Σ. Solutions found in the framework of systems over rings can often be interpreted in the original delay-differential framework, providing in this way a solution to the problem at issue. The advantage of working in the ring framework consists in the possibility of employing algebraic tools and of dealing with finite dimensional modules instead of infinite dimensional vector spaces, like the state space of Σ_d (see [21] for comments on this point). In particular, using the associated systems with coefficients in a ring, it is possible to extend to time-delay systems the methods and tools of the geometric approach: see [11] for an account of the geometric approach for systems with coefficients in a ring.
Since a ring cannot, in general, be endowed with a natural metric structure, when using systems over rings stability must be defined in a formal way by introducing the concept of Hurwitz set.

Definition 1: Given a ring R, a subset $S \subseteq R[z]$ of polynomials with coefficients in R in the indeterminate z is said an Hurwitz set if

(i) it is multiplicatively closed;
(ii) it contains at least an element of the form $z - \alpha$, with $\alpha \in R$;
(iii) it contains all the factors of all its elements.

Given a Hurwitz set S, a system Σ of the form (3) with coefficients in R is said S-stable if $\det(zI - A)$ belongs to S.
When systems over a ring are used to model time-delay systems and $R = \mathbb{R}[\Delta]$, the chosen Hurwitz set is the following (see [22]):

$$\mathcal{H} = \{p(z, \Delta) \in R[z], \text{ such that } p(\gamma, e^{-\gamma h}) \neq 0 \text{ for all } \gamma \in \mathbb{C}, \text{ with } \mathrm{Re}\,\gamma \geq 0\},$$

where $\mathbb{C}$ denotes the field of complex numbers.
Stability of the time-delay system Σ_d in the delay differential framework corresponds to $\mathcal{H}$-stability of the associated system Σ in the ring framework.
The basic notions of the geometric approach we will need in the following are briefly recalled below.

Definition 2 ([23]): Given a system Σ, defined over a ring R by equations of the form (3), a submodule $\mathcal{V}$ of its state module X is said to be

(i) (A, B)-invariant, or controlled invariant, if and only if $A\mathcal{V} \subseteq \mathcal{V} + Im\,B$;
(ii) feedback invariant if and only if there exists an R-linear map $F : X \to U$ such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.

Any feedback F as in (ii) above is called a friend of $\mathcal{V}$.
While controlled invariance is a purely geometric property, feedback invariance is a notion related to system dynamics and it is equivalent to invariance with respect to a closed loop dynamics. For systems with coefficients in a ring, an (A, B)-invariant submodule $\mathcal{V}$ is not necessarily of feedback type and therefore it cannot always be made invariant with respect to a closed loop dynamics, as it happens in the special case of systems with coefficients in the field of real numbers $\mathbb{R}$. The geometric notion of (A, B)-invariance is weaker than the dynamic notion of feedback type invariance, which is the most important in applications and the most difficult to check. Here, we are only interested in remarking that equivalence between them holds if $\mathcal{V}$ is a direct summand of X (see [24] for further comments).
In general it is not easy to characterize direct summands, but if R is a Principal Ideal Domain (PID), as in case $R = \mathbb{R}[\Delta]$, this can be done by using the notion of closure as described below.

Definition 3 ([25]): Let $\mathcal{V} \subseteq \mathcal{W} \subseteq R^n$. The closure of $\mathcal{V}$ in $\mathcal{W}$ is the submodule

$$cl_{\mathcal{W}}(\mathcal{V}) = \{x \in \mathcal{W} \text{ for which there exists } a \in R,\ a \neq 0, \text{ such that } ax \in \mathcal{V}\}.$$

Remark that $cl_{\mathcal{W}}(\mathcal{V})$ is the smallest closed submodule containing $\mathcal{V}$ and that it has the same dimension of $\mathcal{V}$. When no confusion arises, we drop the subscript in denoting closures.

Proposition 1 ([25]): Let R be a PID, then a submodule $\mathcal{V} \subseteq R^n$ is a direct summand of X if and only if $\mathcal{V}$ is closed.

Given a system Σ, it is often of interest to consider the maximum (A, B)-invariant submodule contained in the Kernel of the output map C, which is generally denoted by $\mathcal{V}^*$. For systems with coefficients in a Principal Ideal Domain R, $\mathcal{V}^*$ is of feedback type if and only if it is closed (see [24]). A procedure for constructing $\mathcal{V}^*$ can be found in ([26]).

III. Statement of the Problem

In order to formally state the problem we want to tackle, let us consider a linear, time-delay, to-be-controlled system defined by equations of the form

$$\Sigma_{pd} = \begin{cases} \dot{x}_p(t) = \sum_{i=0}^{a_p} A_{pi} x_p(t-ih) + \\ \qquad + \sum_{i=0}^{b_p} B_{pi} u(t-ih) \\ y(t) = \sum_{i=0}^{c_p} C_{pi} x_p(t-ih), \end{cases} \tag{4}$$

where the state $x_p(\cdot)$, the input $u(\cdot)$ and the output $y(\cdot)$ respectively belong to $\mathbb{R}^n$, $\mathbb{R}^m$, $\mathbb{R}^p$, the matrices A_{pi}, with $i=0,1,\ldots,a_p$, B_{pi}, with $i=0,1,\ldots,b_p$, C_{pi}, with $i=0,1,\ldots,c_p$, have compatible dimensions and entries in $\mathbb{R}$, the scalar $h \in \mathbb{R}^+$ is a given time delay. Indeed, according to the procedure introduced in Section II, we will henceforth consider the following, equivalent writing for (4)

$$\begin{cases} \dot{x}_p(t) = \sum_{i=0}^{a_p} A_{pi}\delta^i x_p(t) + \sum_{i=0}^{b_p} B_{pi}\delta^i u(t) \\ y(t) = \sum_{i=0}^{c_p} C_{pi}\delta^i x_p(t) \end{cases}$$

Let us also consider the exogenuos generator Σ_{ed} of reference signals, of the form

$$\Sigma_{ed} = \begin{cases} \dot{x}_e(t) = A_e x_e(t), \\ r(t) = E_e x_e(t), \end{cases} \tag{5}$$

where the state $x_e(\cdot)$ and the output $r(\cdot)$ respectively belong to $\mathbb{R}^{n_e}$ and $\mathbb{R}^p$, and where the matrices A_e and E_e have compatible dimensions and entries in $\mathbb{R}$.
Then, let the system Σ_d be defined as the parallel connection of Σ_{pd} and Σ_{ed} shown in Figure 1. Namely, let the output $e \in \mathbb{R}^p$ of Σ_d be defined as the regulation error: i.e., the difference between the reference signal r and the controlled output y.

Problem 1: Given the linear, time-delay to-be-controlled system Σ_{pd}, described by (4), and the linear exogenous system Σ_{ed}, described by (5), the Multivariable Autonomous Regulator Problem consists in finding a time-delay feedback regulator Σ_{rd}, of the form

$$\Sigma_{rd} = \begin{cases} \dot{x}_r(t) = \sum_{i=0}^{a_r} A_{ri} x_r(t-ih) + \sum_{i=0}^{b_r} B_{ri} e(t-ih), \\ u(t) = \sum_{i=0}^{c_r} C_{ri} x_r(t-ih) + \sum_{i=0}^{d_r} D_{ri} e(t-ih), \end{cases} \tag{6}$$

where the state $x_r(\cdot)$, the input $e(\cdot)$, the output $u(\cdot)$ respectively belong to $\mathbb{R}^{n_r}$, $\mathbb{R}^p$, $\mathbb{R}^m$, and where the matrices A_{ri}, with $i=0,1,\ldots,a_r$, B_{ri}, with $i=0,1,\ldots,b_r$, C_{ri}, with $i=0,1,\ldots,c_r$, D_{ri}, with $i=0,1,\ldots,d_r$, have compatible dimensions and entries in $\mathbb{R}$, which achieves asymptotic stability of the regulation loop and asymptotic tracking of the reference signal for any initial condition of the composite system Σ_d.
The complete block diagram for the Multivariable Autonomous Regulator Problem is shown in Figure 1.

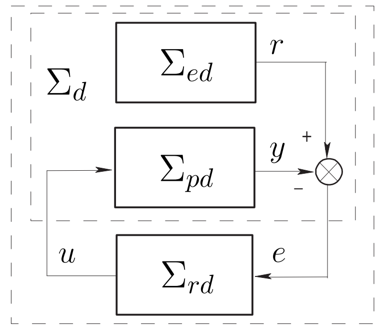

Fig. 1. Block diagram of the Multivariable Autonomous Regulator Problem for linear, time-delay systems.

IV. Statement of the Problem in Geometric Terms for Systems Over a Ring

In order to solve Problem 1, we will reformulate it in the ring framework and we will apply the methods and tools of the geometric approach to systems over rings.
According to the technique recalled in Section II, let the systems Σ_e and Σ_p be defined, respectively, over the ring $\mathbb{R}[\Delta]$ by the equations

$$\Sigma_e = \begin{cases} x_e(t+1) = A_e x_e(t), \\ r(t) = E_e x_e(t). \end{cases} \tag{7}$$

and

$$\Sigma_p = \begin{cases} x_p(t+1) = A_p x_p(t) + B_p u(t), \\ y(t) = C_p x_p(t), \end{cases} \tag{8}$$

with

$$A_p = \sum_{i=0}^{a_p} A_{pi}\Delta^i,\ B_p = \sum_{i=0}^{b_p} B_{pi}\Delta^i,\ C_p = \sum_{i=0}^{c_p} C_{pi}\Delta^i. \tag{9}$$

Hence, the system Σ over the ring $\mathbb{R}[\Delta]$, associated to Σ_d, is defined by the equations

$$\Sigma = \begin{cases} x(t+1) = Ax(t) + Bu(t), \\ e(t) = Ex(t), \end{cases} \tag{10}$$

with

$$A = \begin{bmatrix} A_p & O \\ O & A_e \end{bmatrix}, \quad B = \begin{bmatrix} B_p \\ O \end{bmatrix}, \quad E = \begin{bmatrix} -C_p & E_e \end{bmatrix}. \tag{11}$$

The regulator Σ_r over the ring $\mathbb{R}[\Delta]$, associated to a feedback regulator Σ_{rd}, which possibly solves the Multivariable Autonomous Regulator Problem in the delay-differential framework, is defined by the equations

$$\Sigma_r = \begin{cases} x_r(t+1) = A_r x_r(t) + B_r e(t), \\ u(t) = C_r x_r(t) + D_r e(t). \end{cases} \tag{12}$$

where

$$\begin{matrix} A_r = \sum_{i=0}^{a_r} A_{ri}\Delta^i & B_r = \sum_{i=0}^{b_r} B_{ir}\Delta^i \\ C_r = \sum_{i=0}^{c_r} C_{ir}\Delta^i & D_r = \sum_{i=0}^{d_r} D_{ri}\Delta^i. \end{matrix} \tag{13}$$

Now, let the so-called *autonomous extended system* $\hat{\Sigma}$ be defined as the closed loop connection of Σ and Σ_r, so that, with $\hat{x} = \begin{bmatrix} x_p^\top & x_e^\top & x_r^\top \end{bmatrix}^\top$, its equations are

$$\hat{\Sigma} = \begin{cases} \hat{x}(t+1) = \hat{A}\,\hat{x}(t), \\ e(t) = \hat{E}\,\hat{x}(t), \end{cases} \tag{14}$$

where

$$\hat{A} = \begin{bmatrix} A_p - B_p D_r C_p & B_p D_r E_e & B_p C_r \\ O & A_e & O \\ -B_r C_p & B_r E_e & A_r \end{bmatrix}, \tag{15}$$

$$\hat{E} = \begin{bmatrix} -C_p & E_e & O \end{bmatrix}. \tag{16}$$

In the light of the above correspondences, the Multivariable Autonomous Regulator Problem is recast in geometric terms in the framework of systems over a ring as follows.

Problem 2: Given the system Σ described by (10) and (11), the Multivariable Autonomous Regulator Problem consists in finding a feedback regulator Σ_r, described by (12), such that there exists a closed $\hat{A}$-invariant submodule $\hat{\mathcal{W}} \subseteq \hat{X}$ of $\hat{\Sigma}$, which satisfies the conditions

(i) $\hat{\mathcal{W}} \subseteq \ Ker\,\hat{E}$,
(ii) the dynamics induced on the quotient module $\hat{X}/\hat{\mathcal{W}}$ by that of $\hat{\Sigma}$ is $\mathcal{H}$-stable.

Note that, since $\hat{\mathcal{W}}$ is a closed submodule, $\hat{\mathcal{W}}$ is a direct summand of $\hat{X}$. Therefore, the quotient $\hat{X}/\hat{\mathcal{W}}$ is a free R-submodule of the form R^q, for some q and the induced dynamics can be represented by a suitable, $q \times q$ matrix Z, with entries in R. $\mathcal{H}$-stability of the induced dynamics means that $\det(zI - Z)$ belongs to $\mathcal{H}$.
Moreover, condition (i) of Problem 2 means that any state trajectory originating from a state on $\hat{\mathcal{W}}$ evolves inside $\hat{\mathcal{W}} \subseteq \ Ker\,\hat{E}$. Hence it is invisible at the output of $\hat{\Sigma}$ (perfect tracking).
Condition (ii) corresponds, in the time delay framework, to the fact that any state trajectory originating from a state external to $\hat{\mathcal{W}}$ asymptotically tends to $\hat{\mathcal{W}}$, so that asymptotic tracking is assured.

V. Solution of the Problem

In the classical case, suitable stabilizability and detectability assumptions have to be made in order to solve the problem (see [13]). In the ring framework, this corresponds essentially to assume $\mathcal{H}$-stabilizabilty of the pair (A,B) and $\mathcal{H}$-detectability of the pair (A,E). Since the use of static feedback and of static output injection would be too restrictive, these notions mean that there exists a dynamic extension (A',B') of (A,B) and a feedback F such that $\det(zI-(A'+B'F))$ belongs to $\mathcal{H}$ and, respectively, that there exists a dynamic extension (A'',E'') of (A,E) and an output injection G such that $\det(zI-(A''+GE''))$ belongs to $\mathcal{H}$. In order to simplify notations, we will assume in the following, that, if necessary, dynamic extensions have already been made. This allows us to assume, as basic conditions for solving the Multivariable Autonomous Regulator Problem, that

- *Stabilizability*: there exists a feedback F such that $\det(zI-(A+BF))$ belongs to $\mathcal{H}$,
- *Detectability*: there exists an output injection G such that $\det(zI-(A+GE))$ belongs to $\mathcal{H}$.

In order to state sufficient conditions for the solvability of Problem 2 in terms of the autonomous extended system $\hat{\Sigma}$, we introduce the free submodule $\hat{\mathcal{P}} \subseteq \hat{X}$, spanned in $\hat{X}$ by the columns of the matrix

$$\hat{P} = \begin{bmatrix} I & O \\ O & O \\ O & I \end{bmatrix}, \tag{17}$$

which refers to the partition considered in (15). The submodule $\hat{\mathcal{P}}$ is $\hat{A}$-invariant. In fact, we have

$$\hat{A}\hat{P} = \hat{P}\hat{S} \tag{18}$$

with

$$\hat{S} = \begin{bmatrix} A_p - B_p D_r C_p & B_p C_r \\ -B_r C_p & A_r \end{bmatrix}.$$

This also shows that the dynamics induced by the matrix $\hat{A}$ on the submodule $\hat{\mathcal{P}}$ coincides with that of the regulation loop.
Now we can state sufficient conditions for solvability of Problem 2, referred to $\hat{\Sigma}$.

Theorem 1: The Multivariable Autonomous Regulator Problem is solvable for the system $\hat{\Sigma}$ if, for some regulator (12), there exists an $\hat{A}$-invariant submodule $\hat{\mathcal{V}} \subseteq \hat{X}$, such that

(i) $\hat{\mathcal{V}} \subseteq Ker\,\hat{E}$,
(ii) $\hat{\mathcal{V}} \oplus \hat{\mathcal{P}} = \hat{X}$,
(iii) the dynamics restricted to the quotient module $\hat{X}/\hat{\mathcal{V}}$ is $\mathcal{H}$-stable.

Proof. Assume that, for some regulator (12), there exists an $\hat{A}$-invariant submodule $\hat{\mathcal{V}} \subseteq \hat{X}$ satisfying conditions (i)-(iii). Then, Problem 2 is solvable, since conditions (i)-(ii) of the problem statement are satisfied with $\hat{\mathcal{W}} = \hat{\mathcal{V}}$. ■

In order to provide checkable conditions for the solvability of Problem 2 directly referred to the regulated system Σ, let us denote by $\mathcal{P} \subseteq X$ the submodule spanned by the columns of the matrix

$$P = \begin{bmatrix} I \\ O \end{bmatrix}, \tag{19}$$

which refers to the partition considered in (11). Since

$$AP = PA_p,$$

$\mathcal{P}$ is A-invariant and the dynamics induced by A on $\mathcal{P}$ is that of the to-be-controlled system.
Then, the sufficient condition for solvability of Problem 2, referred to Σ, is expressed by the following

Theorem 2: The Multivariable Autonomous Regulator Problem is solvable if there exists an (A,B)-controlled invariant submodule $\mathcal{V} \subseteq X$ such that

(i) $\mathcal{V} \subseteq Ker\,E$,
(ii) $\mathcal{V} \oplus \mathcal{P} = X$.

Proof. Let $\mathcal{V} \subseteq X$ be an (A,B)-controlled invariant submodule satisfying conditions (i)–(ii). By condition (ii) $\mathcal{V}$ is of feedback type and it can be shown, as in [13], that there exists a friend F, partitioned into $[F_p\ F_e]$ according to (11), such that $A_p + B_pF_p$ is $\mathcal{H}$-stable. Now, let G be such that $(A+GE)$ is $\mathcal{H}$-stable. The regulator Σ_r of the form (12), with

$$A_r = A + BF + GE, \quad B_r = -G,$$
$$C_r = F, \qquad D_r = O,$$

then solves the problem. In fact, the matrices $\hat{A}$ and $\hat{E}$ have the form

$$\hat{A} = \begin{bmatrix} A & BF \\ -GE & A+BF+GE \end{bmatrix}, \quad \hat{E} = \begin{bmatrix} E & O \end{bmatrix}, \tag{20}$$

and conditions (i)–(iii) of Theorem 1 are satisfied by taking $\hat{\mathcal{V}}$ as the submodule spanned by the columns of the matrix

$$\begin{bmatrix} V \\ V \end{bmatrix},$$

where, by (ii), a basis matrix for $\mathcal{V}$ has the form

$$V = \begin{bmatrix} V_1 \\ I \end{bmatrix}.$$

This can be easily checked by performing the similarity transformation defined by

$$\hat{T} = \hat{T}^{-1} = \begin{bmatrix} I & O \\ I & -I \end{bmatrix}.$$

■

Theorem 2 gives a sufficient condition for solvability of the Multivariable Autonomous Regulator Problem in terms of the original data. However, no algorithm to compute $\mathcal{V}$ is given.

A possible strategy to search for $\mathcal{V}$, in some situations, is the following. Consider the maximum (A, B)-invariant submodule $\mathcal{V}^*$ for the system Σ, defined by (10), contained in $Ker\ E$ and assume it is closed. If $\mathcal{V}^* + \mathcal{P} = X$ holds, choose a basis $T = [T_1\ T_2\ T_3]$ of X, with $Im\,T_1 = \mathcal{V}^* \cap \mathcal{P}$ and $Im\,[T_1\ T_2] = \mathcal{V}^*$. Then, for any friend F of $\mathcal{V}^*$, the matrix $(A + BF)$ in the new basis takes the structure

$$A'_F = \begin{bmatrix} A'_{F_{11}} & A'_{F_{12}} & A'_{F_{13}} \\ O & A'_{F_{22}} & O \\ O & O & A'_{F_{33}} \end{bmatrix}. \tag{21}$$

If there exists a matrix M of suitable dimension such that the equality $A'_{F_{11}}M - MA'_{F_{22}} = -A'_{F_{12}}$ holds, it can be shown that the submodule $\mathcal{V} = \text{Im}\,(T_1 X + T_2)$ is an (A, B)-controlled invariant submodule satisfying conditions (i) and (ii) of Theorem 2 and hence the Multivariable Autonomous Regulator Problem is solvable.

VI. An Illustrative Example

Assume that the time-delay system Σ_{pd}, defined by the equations

$$\Sigma_{pd} = \begin{cases} \dot{x}_1(t) = -2x_1(t) + x_2(t-h) + u_1(t) + u_2(t-h) \\ \dot{x}_2(t) = -x_2(t) + u_2(t) \\ y(t) = x_1(t) \end{cases} \tag{22}$$

is required to track the reference signals generated by the exosystem Σ_{ed}, defined by the equations

$$\Sigma_{ed} = \begin{cases} \dot{x}_e(t) = 0 \\ r(t) = x_e(t) \end{cases} \tag{23}$$

Denoting by Σ_p and Σ_e the associated systems over the ring $R = \mathbb{R}[\Delta]$, with the notation of (11), we have that the regulated system Σ is described by the triple (A, B, E), where

$$A = \left[\begin{array}{cc|c} -2 & \Delta & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad B = \left[\begin{array}{cc} 1 & \Delta \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] \tag{24}$$

$$E = \left[\begin{array}{cc|c} -1 & 0 & 1 \end{array}\right]$$

The pair (A, E) is weakly observable. In this case,

$$\mathcal{V}^* = Ker\ E = span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

and $\mathcal{V}^* + \mathcal{P} = X$ holds with

$$\mathcal{P} = span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$$

In particular,

$$\mathcal{V} = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and $\mathcal{V}$ is a controlled invariant submodule such that

$$\mathcal{V} \cap \mathcal{P} = \{0\}, \quad \mathcal{V} + \mathcal{P} = R^3.$$

Then, $\mathcal{V} \oplus \mathcal{P} = X$ and the hypotheses of Theorem 2 are satisfied.

The feedback

$$F = \begin{bmatrix} 1 & -\Delta & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

where

$$F_p = \begin{bmatrix} 1 & -\Delta \\ 0 & -1 \end{bmatrix}$$

is such that $A_p + B_pF_p$ is $\mathcal{H}$- stable.

$$G = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

is such that $A + GE$ is $\mathcal{H}$-stable. Then, the regulator Σ_r defined by equations of the form (12) with

$$A_r = \begin{bmatrix} -1 & -\Delta & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{25}$$

$$C_r = \begin{bmatrix} 1 & -\Delta & 1 \\ 0 & -1 & 0 \end{bmatrix}, \qquad D_r = 0$$

solves the problem over the ring R.
The problem for the delay differential systems (22) is solved by the regulator

$$\Sigma_{rd} = \begin{cases} \dot{x}_{r1}(t) &= -x_{r1}(t) - x_{r2}(t-h) + x_{r3}(t) \\ \dot{x}_{r2}(t) &= -2x_{r2}(t) \\ \dot{x}_{r3}(t) &= x_{r1}(t) - x_{r3}(t) + e(t) \\ u_1(t) &= x_{r1}(t) + x_{r2}(t-h) + x_{r3}(t) \\ u_2(t) &= -x_{r2}(t). \end{cases} \tag{26}$$

VII. CONCLUSION

In this work, the classic formulation of the Multivariable Autonomous Regulator Problem has been extended to encompass the case where time delays are present.

The problem, then, has been recast in geometric terms in the equivalent context of finite-dimensional dynamical systems with coefficients in the ring of polynomials $\mathbb{R}[\Delta]$.

Algebraic and ring theoretic notions, as well as concepts, have been used in deriving sufficient conditions for the solution of the problem.

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