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Unscented Filtering for Interval-Constrained Nonlinear Systems

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***Abstract*— This paper addresses the state-estimation problem for nonlinear systems with an interval constraint on the state vector. Approximate solutions to this problem are reviewed and compared with new algorithms, which are based on the unscented Kalman filter. An illustrative example is discussed.**

I. INTRODUCTION

The classical Kalman filter (KF) provides optimal state estimates under Gaussian disturbances and linear model assumptions [2]. In practice, however, the dynamics and disturbances may be such that the state vector is known to satisfy an inequality [7] or an equality [11] constraint. For example, in a chemical reaction, the species concentrations are nonnegative [14]. Additional examples of systems with inequality-constrained states arise in aeronautics [9]. The equality-constrained case is addressed in [11–13] and is outside the scope of this paper. However, Gaussian noise and an inequality-constrained state vector are mutually exclusive assumptions even for linear systems [5, 6]. Therefore, for such systems, KF does not guarantee that its estimates satisfy the inequality constraint. In such cases, as well as for nonlinear systems, we wish to obtain state estimates that satisfy inequality constraints. In this paper, we are specifically concerned with interval constraints.

Various approximate algorithms have been developed for inequality-constrained linear state estimation. One of the most popular techniques is the moving horizon estimator (MHE) [5], which formulates the state-estimation problem as a non-recursive constrained quadratic program. The truncation procedure [8] reshapes the probability density function computed by KF, which is assumed to be Gaussian and is given by the state estimate and the error covariance, at the inequality constraint edges. Finally, if the state estimates do not satisfy the inequality constraint, then they are projected onto the boundary of the constraint region by the projection approach [8, 9].

For nonlinear systems, algorithms based on MHE are employed [1, 6]. However, since these techniques are non-recursive, they are computationally expensive and difficult to use in some real-time applications [14]. For such cases, the constrained extended KF (CEKF) [6, 14], which is a special case of MHE with unitary moving horizon and is called recursive nonlinear dynamics data reconciliation (RNDDR) in [14], is presented as a simpler and less computationally demanding algorithm. Motivated by the improved performance [4] of the unscented KF (UKF) [3] over the extended KF (EKF) [2], the unscented RNDDR, which is referred to as the sigma-point interval unscented Kalman filter (SIUKF) in this paper, is presented in [14].

The present paper addresses the state-estimation problem for interval-constrained nonlinear systems. We review UKF and SIUKF and present approximate solutions to this problem based on UKF as follows. We combine either the unscented transform (UT) [3] or the interval-constrained UT (ICUT) [14], which are used during the forecast step of UKF and SIUKF, respectively, together with one of the following data-assimilation approaches, namely, (i) the classical KF update [2, 3], (ii) the constrained Kalman update of CEKF [10, 14], (iii) the sigma-point constrained update of SIUKF [14], (iv) the classical KF update followed by either the truncation procedure [8], or (v) the projection approach [9]. Then we obtain six new algorithms, namely, the constrained UKF (CUKF), the constrained interval UKF (CIUKF), the interval UKF (IUKF), the truncated UKF (TUKF), the truncated IUKF (TIUKF), and the projected UKF (PUKF); see Table I. These algorithms are compared to UKF and SIUKF in terms of accuracy and processing time by means of a continuously stirred tank reactor [1] example. Our goal is to obtain nonnegative state estimates. A detailed version of this paper appears as [10, Chapter 6].

TABLE I: Interval-constrained state estimators based on the unconstrained UKF. We make explicit the procedure used during the forecast step (column-wise), as well as the type of KF update (KFU) used during data-assimilation (DA) (row-wise). Inside parentheses, we cite the section in which the method is either reviewed or presented.

DA \ forecast	UT [3]	ICUT [14]
classical KF update [2]	UKF (III)	IUKF (V-D)
CEKF update [14]	CUKF (V-B)	CIUKF (V-C)
sigma-point constrained update [14]	–	SIUKF (V-A)
KFU plus truncation [8]	TUKF (V-E)	TIUKF (V-F)
KFU plus projection [9]	PUKF (V-G)	–

II. STATE ESTIMATION FOR NONLINEAR SYSTEMS

For the stochastic nonlinear discrete-time dynamic system

$$x_k = f(x_{k-1},\ u_{k-1},\ k-1) + w_{k-1}, \qquad (2.1)$$
$$y_k = h(x_k,\ k) + v_k, \qquad (2.2)$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{N} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}^m$ are, respectively, the process and observation models, the state-estimation problem can be described as follows. Assume that,

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for all $k \geq 1$, the known data are the measurements $y_k \in \mathbb{R}^m$, the inputs $u_{k-1} \in \mathbb{R}^p$, and the probability density functions (PDFs) $\rho(x_0)$, $\rho(w_{k-1})$ and $\rho(v_k)$, where $x_0 \in \mathbb{R}^n$ is the initial state vector, $w_{k-1} \in \mathbb{R}^n$ is the process noise, and $v_k \in \mathbb{R}^m$ is the measurement noise. Next, define the profit function

$$J(x_k) \triangleq \rho(x_k|(y_1, \ldots, y_k)), \tag{2.3}$$

which is the value of the conditional PDF of the state vector $x_k \in \mathbb{R}^n$ given the past and present measured data $y_1, \ldots, y_k$. Under the stated assumptions, the maximization of (2.3) is the state-estimation problem, while the maximizer of J is the optimal state estimate.

The solution to this problem is complicated by the fact that, for nonlinear systems, $\rho(x_k|(y_1, \ldots, y_k))$ is not completely characterized by its mean $\hat{x}_{k|k}$ and covariance $P^{xx}_{k|k} \triangleq \mathcal{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^{\mathrm{T}}]$. We thus use an approximation based on the classical Kalman filter (KF) for linear systems [2] to provide a suboptimal solution to the nonlinear case, specifically, the unscented Kalman filter (UKF) [3]. To accomplish that, UKF propagates only approximations to $\hat{x}_{k|k}$ and $P^{xx}_{k|k}$ using the initial mean $\hat{x}_{0|0}$ and the covariance $P^{xx}_{0|0} \triangleq \mathcal{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^{\mathrm{T}}]$ of $\rho(x_0)$, which are assumed to be known. We assume that the maximizer of J is $\hat{x}_{k|k}$. Furthermore, we assume that the mean and covariance of $\rho(w_{k-1})$ and $\rho(v_k)$ are known and equal to zero and Q_{k-1}, R_k, respectively. Also, w_{k-1} and v_k are assumed to be uncorrelated.

III. Unscented Kalman Filter

Instead of analytically or numerically linearizing (2.1)-(2.2) and using the KF equations [2], UKF employs the unscented transform (UT) [3], which is a numerical procedure for approximating the mean and covariance of a random vector obtained from a nonlinear transformation.

UKF is a two-step estimator whose *forecast* step is given by

$$\gamma_0 = \frac{\lambda}{n+\lambda}, \quad \gamma_j \triangleq \frac{1}{2(n+\lambda)}, \; j = 1, \ldots, 2n, \tag{3.1}$$

$$\mathcal{X}_{k-1|k-1} = \hat{x}_{k-1|k-1} 1_{1\times(2n+1)} + \sqrt{n+\lambda}\begin{bmatrix} 0_{n\times 1} & (P^{xx}_{k-1|k-1})^{1/2} & -(P^{xx}_{k-1|k-1})^{1/2}\end{bmatrix}, \tag{3.2}$$

$$\mathcal{X}_{j,k|k-1} = f(\mathcal{X}_{j,k-1|k-1}, \; u_{k-1}, \; k-1), \; j = 0, \ldots, 2n, \tag{3.3}$$

$$\hat{x}_{k|k-1} = \sum_{j=0}^{2n} \gamma_i \, \mathcal{X}_{j,k|k-1}, \tag{3.4}$$

$$P^{xx}_{k|k-1} = \sum_{j=0}^{2n} \gamma_j \, [\mathcal{X}_{j,k|k-1} - \hat{x}_{k|k-1}][\mathcal{X}_{j,k|k-1} - \hat{x}_{k|k-1}]^{\mathrm{T}} + Q_{k-1}, \tag{3.5}$$

$$\mathcal{X}_{k|k-1} = \hat{x}_{k|k-1} 1_{1\times(2n+1)} + \sqrt{n+\lambda}\begin{bmatrix} 0_{n\times 1} & (P^{xx}_{k|k-1})^{1/2} & -(P^{xx}_{k|k-1})^{1/2}\end{bmatrix}, \tag{3.6}$$

$$\mathcal{Y}_{j,k|k-1} = h(\mathcal{X}_{j,k|k-1}, \; k), \; j = 0, \ldots, 2n, \tag{3.7}$$

$$\hat{y}_{k|k-1} = \sum_{j=0}^{2n} \gamma_j \, \mathcal{Y}_{j,k|k-1}, \tag{3.8}$$

$$P^{yy}_{k|k-1} = \sum_{j=0}^{2n} \gamma_j \, [\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}][\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}]^{\mathrm{T}} + R_k, \tag{3.9}$$

$$P^{xy}_{k|k-1} = \sum_{j=0}^{2n} \gamma_i \, [\mathcal{X}_{j,k|k-1} - \hat{x}_{k|k-1}][\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}]^{\mathrm{T}}, \tag{3.10}$$

where $(\cdot)^{1/2}$ is the Cholesky square root, $\lambda > -n$, $\mathcal{X}_j$ is the jth column of the sigma-point matrix $\mathcal{X} \in \mathbb{R}^{n\times(2n+1)}$ with weights γ_j satisfying $\sum_{j=0}^{2n} \gamma_j = 1$, $P^{xx}_{k|k-1}$ is the forecast error covariance, $P^{yy}_{k|k-1}$ is the innovation covariance, $P^{xy}_{k|k-1}$ is the cross covariance, and $P^{xx}_{k|k}$ is the data-assimilation error-covariance, and whose *data-assimilation* step is given by the classical KF update, that is,

$$K_k = P^{xy}_{k|k-1}(P^{yy}_{k|k-1})^{-1}, \tag{3.11}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - \hat{y}_{k|k-1}), \tag{3.12}$$

$$P^{xx}_{k|k} = P^{xx}_{k|k-1} - K_k P^{yy}_{k|k-1} K_k^{\mathrm{T}}, \tag{3.13}$$

where $K_k \in \mathbb{R}^{n\times m}$ is the Kalman gain matrix. Henceforth, the notation $\hat{x}_{k|k-1}$ indicates an estimate of x_k at time k based on information available up to and including time $k-1$. Likewise, $\hat{x}_{k|k}$ indicates an estimate of x_k at time k using information available up to and including time k.

IV. State Estimation for Interval-Constrained Nonlinear Systems

Assume that, for all $k \geq 0$, the state vector x_k satisfies the interval constraint

$$d_k \;\leq\; x_k \;\leq\; e_k, \tag{4.1}$$

where $d_k \in \mathbb{R}^n$ and $e_k \in \mathbb{R}^n$ are assumed to be known and, for $j = 1, \ldots, n$, $d_{j,k} < e_{j,k}$. Also, if $x_{j,k}$ is left-unbounded or right-unbounded, then we set $d_{j,k} = -\infty$ or $e_{j,k} = \infty$, respectively. Thus, the objective of the interval-constrained state-estimation problem is to maximize (2.3) subject to (4.1). That is, we look for the maximizer of J that satisfies (4.1).

In addition to nonlinear dynamics, the solution to this problem is complicated due the inclusion of an interval constraint. We thus extend approximate algorithms derived in the linear scenario to provide suboptimal estimates in the nonlinear case.

V. Interval-Constrained UKFs

A. Sigma-Point Interval Unscented Kalman Filter

The sigma-point interval unscented Kalman filter (SIUKF) uses the interval-constrained UT (ICUT) [14] to generate sigma points satisfying $d_{k-1} \leq \mathcal{X}_{j,k-1|k-1} \leq e_{k-1}$, $j = 0, \ldots, 2n$. In this case, $\mathcal{X}_{k-1|k-1}$ is chosen as

$$\begin{aligned}\mathcal{X}_{k-1|k-1} = \hat{x}_{k-1|k-1} 1_{1\times(2n+1)} + &\Big[\, 0_{n\times 1} \quad \theta_{1,k-1}\mathrm{col}_1[(P^{xx}_{k-1|k-1})^{1/2}] \;\ldots \\ &\theta_{n,k-1}\mathrm{col}_n[(P^{xx}_{k-1|k-1})^{1/2}] \\ &-\theta_{n+1,k-1}\mathrm{col}_{n+1}[(P^{xx}_{k-1|k-1})^{1/2}] \;\ldots \\ &-\theta_{2n,k-1}\mathrm{col}_{2n}[(P^{xx}_{k-1|k-1})^{1/2}]\Big],\end{aligned} \tag{5.1}$$

where for $i = 1, \ldots, n$ and $j = 1, \ldots, 2n$

$$\theta_{j,k-1} \triangleq \min\left(\mathrm{col}_j(\Theta)\right), \tag{5.2}$$

$$\Theta_{(i,j)} \triangleq \begin{cases} \sqrt{n+\lambda}, & \text{if } S_{(i,j)} = 0, \\ \min\left(\sqrt{n+\lambda}, \; \frac{e_{i,k-1} - \hat{x}_{i,k-1|k-1}}{S_{(i,j)}}\right), & \text{if } S_{(i,j)} > 0, \\ \min\left(\sqrt{n+\lambda}, \; \frac{d_{i,k-1} - \hat{x}_{i,k-1|k-1}}{S_{(i,j)}}\right), & \text{if } S_{(i,j)} < 0, \end{cases} \tag{5.3}$$

$$S \triangleq \begin{bmatrix} (P^{xx}_{k-1|k-1})^{1/2} & -(P^{xx}_{k-1|k-1})^{1/2} \end{bmatrix}, \tag{5.4}$$

with weights, for $j = 1, \ldots, 2n$,

$$\gamma_{0,k-1} \triangleq b_{k-1}, \quad \gamma_{j,k-1} \triangleq a_{k-1}\,\theta_{j,k-1} + b_{k-1} \tag{5.5}$$

satisfying $\sum_{j=0}^{2n} \gamma_{j,k-1} = 1$, where

$$a_{k-1} \triangleq \frac{2\lambda - 1}{2(n+\lambda)\left(\sum_{j=1}^{n}\theta_{j,k-1} - (2n+1)\sqrt{n+\lambda}\right)}, \tag{5.6}$$

$$b_{k-1} \triangleq \frac{1}{2(n+\lambda)} - \frac{2\lambda - 1}{(2\sqrt{n+\lambda})\left(\sum_{j=1}^{n}\theta_{j,k-1} - (2n+1)\sqrt{n+\lambda}\right)}. \tag{5.7}$$

Note that, for UT, $\theta_{j,k-1} = \sqrt{n+\lambda}$, for all $j = 1, \ldots, 2n$; see (3.2). Figure 1 illustrates how the sigma points of ICUT are chosen compared to UT. Note that whenever a sigma point violates (4.1), it is projected onto the nearest surface $x_{k-1} = d_{k-1}$ or $x_{k-1} = e_{k-1}$. In so doing, unlike UT, the sigma points are not necessarily symmetric around $\hat{x}_{k-1|k-1}$ such that their weighted sample mean and covariance may not capture $\hat{x}_{k-1|k-1}$ and $P^{xx}_{k-1|k-1}$.

(a) (b)

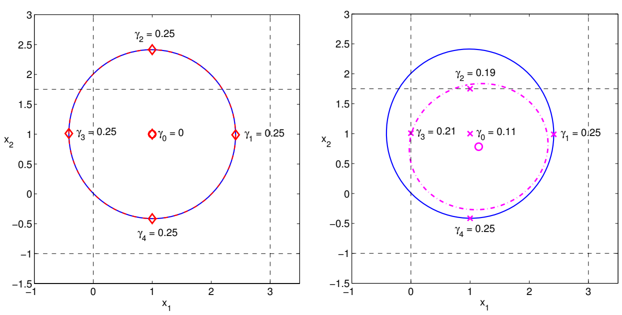


Fig. 1. Sigma points of (a) UT ($\diamond$) in comparison with those of (b) ICUT ($\times$) and related weights for an example where $x_{k-1} \in \mathbb{R}^2$, $\hat{x}_{k-1|k-1} = [\,1 \quad 1\,]^{\mathrm{T}}$, $P^{xx}_{k-1|k-1} = I_{2\times 2}$, $d_{k-1} = [\,0 \quad -1\,]^{\mathrm{T}}$, $e_{k-1} = [\,3 \quad 1.75\,]^{\mathrm{T}}$, and $\lambda = 0$. The circle ($\circ$) represents the weighted mean of sigma points, the dot-dashed ($-\cdot-$) line denotes the corresponding covariance, and (——) is the actual covariance. For UT, these lines coincide.

SIUKF is a two-step estimator whose forecast step is given by (5.1)-(5.7) together with (3.3) and

$$\hat{x}_{k|k-1} = \sum_{j=0}^{2n} \gamma_{j,k-1}\mathcal{X}_{j,k|k-1}, \tag{5.8}$$

$$P^{xx}_{k|k-1} = \sum_{j=0}^{2n} \gamma_{j,k-1}[\mathcal{X}_{j,k|k-1} - \hat{x}_{k|k-1}][\mathcal{X}_{j,k|k-1} - \hat{x}_{k|k-1}]^{\mathrm{T}} + Q_{k-1}, \tag{5.9}$$

and (3.6), and whose data-assimilation step is given by

$$\hat{\mathcal{X}}_{j,k|k} = \mathop{\arg\min}_{\{\mathcal{X}_{j,k}:\ d_k \le \mathcal{X}_{j,k} \le e_k\}} J_1(\mathcal{X}_{j,k}), \quad j = 0, \ldots, 2n, \tag{5.10}$$

$$\hat{x}_{k|k} = \sum_{j=0}^{2n} \gamma_{j,k-1}\,\hat{\mathcal{X}}_{j,k|k}, \tag{5.11}$$

$$P^{xx}_{k|k} = \sum_{j=0}^{2n} \gamma_{j,k-1}\left[\hat{\mathcal{X}}_{j,k|k} - \hat{x}_{k|k}\right]\left[\hat{\mathcal{X}}_{j,k|k} - \hat{x}_{k|k}\right]^{\mathrm{T}}, \tag{5.12}$$

where $J_1(\mathcal{X}_{j,k}) \triangleq \Big[(y_k - h(\mathcal{X}_{j,k}, k))^{\mathrm{T}} R_k^{-1}(y_k - h(\mathcal{X}_{j,k}, k)) + (\mathcal{X}_{j,k} - \mathcal{X}_{j,k|k-1})^{\mathrm{T}}(P^{xx}_{k|k-1})^{-1}(\mathcal{X}_{j,k} - \mathcal{X}_{j,k|k-1})\Big]$, and each column of $\hat{\mathcal{X}}_{k|k} \triangleq [\hat{\mathcal{X}}_{0,k|k} \quad \hat{\mathcal{X}}_{1,k|k} \quad \ldots \quad \hat{\mathcal{X}}_{2n,k|k}] \in \mathbb{R}^{n\times(2n+1)}$ is the solution of a nonlinear constrained optimization problem. We refer to (5.10)-(5.11) as the sigma-point constrained update.

SIUKF enforces (4.1) during both the forecast and data-assimilation steps. Moreover, not only $\hat{x}_{k|k-1}$ and $\hat{x}_{k|k}$, but also $P^{xx}_{k|k-1}$ and $P^{xx}_{k|k}$, assimilate the interval-constraint information.

B. Constrained Unscented Kalman Filter

In this section, we present the constrained unscented Kalman filter (CUKF). Combining UT for forecast and an inequality-constrained update for data-assimilation, CUKF is the straightforward unscented-based extension of the constrained extended Kalman filter [6, 14], which corresponds to the moving horizon estimator with a unitary window length [6].

Similar to UKF, CUKF is a two-step estimator. The forecast step is given by (3.1)-(3.10). To enforce (4.1), we replace (3.12) of UKF by the constrained optimization problem

$$\hat{x}_{k|k} = \mathop{\arg\min}_{\{x_k:\ d_k \le x_k \le e_k\}} J_2(x_k) \tag{5.13}$$

where $J_2(x_k) \triangleq \Big[(y_k - h(x_k, k))^{\mathrm{T}} R_k^{-1}(y_k - h(x_k, k)) + (x_k - \hat{x}_{k|k-1})^{\mathrm{T}}(P^{xx}_{k|k-1})^{-1}(x_k - \hat{x}_{k|k-1})\Big]$, such that the data-assimilation step is given by (3.11), (5.13), (3.13). Note that, under Gaussian and linear assumptions, maximizing (2.3) is equivalent to minimizing J_2 [11, Lemma 4.1]. Note that the information provided by (4.1) is not assimilated into the error covariance $P^{xx}_{k|k}$ in (3.13).

C. Constrained Interval Unscented Kalman Filter

We present now the constrained interval unscented Kalman filter (CIUKF) as a simplified version of SIUKF. CIUKF is a two-step estimator whose forecast step is given by (5.1)-(5.7), (3.3), (5.8)-(5.9), and (3.6)-(3.7) together with

$$\hat{y}_{k|k-1} = \sum_{j=0}^{2n} \gamma_{j,k-1}\mathcal{Y}_{j,k|k-1}, \tag{5.14}$$

$$P^{yy}_{k|k-1} = \sum_{j=0}^{2n} \gamma_{j,k-1}[\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}][\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}]^{\mathrm{T}} + R_k \tag{5.15}$$

$$P^{xy}_{k|k-1} = \sum_{j=0}^{2n} \gamma_{j,k-1}[\mathcal{X}_{j,k|k-1} - \hat{x}_{k|k-1}][\mathcal{Y}_{j,k|k-1} - \hat{y}_{k|k-1}]^{\mathrm{T}}, \tag{5.16}$$

and whose data-assimilation step is given by (3.11), (5.13), (3.13).

Note that the forecast step of both CIUKF and SIUKF use ICUT, while the data-assimilation step of CIUKF and CUKF are equal. That is, the data-assimilation step of CIUKF is a single sigma-point special case of the data-assimilation step of SIUKF. However, unlike SIUKF, $P^{xx}_{k|k}$ of CIUKF and CUKF are not affected by (4.1).

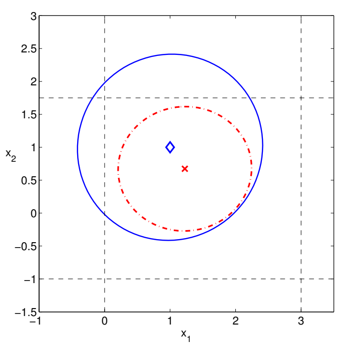
D. Interval Unscented Kalman Filter

The interval unscented Kalman filter (IUKF), which is a simplified version of CIUKF, is a two-step estimator whose forecast step is given by (5.1)-(5.7), (3.3), (5.8)-(5.9), (3.6)-(3.7), and (5.14)-(5.16) and whose data-assimilation step is given by (3.11)-(3.13). That is, its forecast equations are equal to the forecast equations of CIUKF, which uses ICUT, and its data-assimilation equations are equal to the data-assimilation equations of UKF. However, unlike in SIUKF and CIUKF, (4.1) is not enforced during data assimilation by IUKF.

E. Truncated Unscented Kalman Filter

Let $\hat{x}_{k|k}$ given by (3.12) and $P^{xx}_{k|k}$ given by (3.13) be, respectively, the pseudo mean and pseudo covariance of $\rho(x_k|(y_1,\dots,y_k))$ obtained from UKF; see Section III. We want to truncate $\rho(x_k|(y_1,\dots,y_k))$ at the n constraint edges given by the rows of (4.1) such that the pseudo mean $\hat{x}^{\mathrm{t}}_{k|k}$ of the truncated PDF is an interval-constrained state estimate with truncated error covariance $P^{xx\mathrm{t}}_{k|k}$. This procedure is called PDF truncation [8].

For example, consider the case where, even though $\hat{x}_{k|k} = [\,1 \quad 1\,]^{\mathrm{T}}$ satisfies the interval constraint (4.1) with parameters $d_k = [\,0 \quad -1\,]^{\mathrm{T}}$ and $e_k = [\,3 \quad 1.75\,]^{\mathrm{T}}$, $P^{xx}_{k|k} = I_{2\times 2}$ has significant area outside (4.1); as shown in the solid line of Figure 2. Therefore, $\hat{x}^{\mathrm{t}}_{k|k} = [\,1.23 \quad 0.67\,]^{\mathrm{T}}$ is obtained by shifting $\hat{x}_{k|k}$ towards the centroid of the truncated PDF and $P^{xx\mathrm{t}}_{k|k} = \mathrm{diag}(0.52,\ 0.45)$ is obtained by truncating $P^{xx}_{k|k}$ due the prior knowledge provided by (4.1).



2. The unconstrained estimate $\hat{x}_{k|k}$ ($\diamond$) satisfying the interval constraint (4.1) (— —) and its covariance $P^{xx}_{k|k}$ (——) with significant area outside (4.1) compared to the truncated estimate $\hat{x}^{\mathrm{t}}_{k|k}$ ($\times$) and covariance $P^{xx\mathrm{t}}_{k|k}$ ($-\cdot-$).

We present the truncated unscented Kalman filter (TUKF) as the unscented-based nonlinear extension of the truncated Kalman filter (PKF) described in [8, 10], which considers inequality-constrained linear systems, to the interval-constrained nonlinear state-estimation problem. TUKF is obtained by appending the PDF truncation procedure to the UKF equations by feedback recursion. TUKF is a three-step algorithm whose forecast step is given by (3.1) and

$$\mathcal{X}_{k-1|k-1} = \hat{x}^{\mathrm{t}}_{k-1|k-1}1_{1\times(2n+1)} + \sqrt{n+\lambda}\left[0_{n\times 1} \quad (P^{xx\mathrm{t}}_{k-1|k-1})^{1/2} \quad -(P^{xx\mathrm{t}}_{k-1|k-1})^{1/2}\right], \quad (5.17)$$

together with (3.3)-(3.10), whose data-assimilation step is given by (3.11)-(3.13), and whose *truncation* step is reviewed in [8, p. 218–222] and [10, p. 101–105]. For brevity, we do not present the truncation equations.

TUKF has two advantages. First, unlike SIUKF, CIUKF, CUKF, and PUKF, it avoids the explicit online solution of a constrained optimization problem at each time step. Second, it assimilates the interval-constraint information in both the state estimate $\hat{x}^{\mathrm{t}}_{k|k}$ and the error covariance $P^{xx\mathrm{t}}_{k|k}$.

F. Truncated Interval Unscented Kalman Filter

We present now the truncated interval unscented Kalman filter (TIUKF) obtained from the combination of IUKF, which uses ICUT, and the PDF truncation approach of TUKF.

TIUKF is a three-step estimator whose forecast step is given by (5.1)-(5.7), where $\hat{x}^{\mathrm{t}}_{k-1|k-1}$ replaces $\hat{x}_{k-1|k-1}$, together with (3.3), (5.8)-(5.9), (3.6)-(3.7), (5.14)-(5.16), whose data-assimilation step is given by (3.11)-(3.13), and whose truncation step is equal to the truncation step of TUKF; see Section V-E. Note that TIUKF enforces (4.1) during both the forecast step (in both $\hat{x}_{k|k-1}$ and $P^{xx}_{k|k-1}$) and truncation step (in both $\hat{x}^{\mathrm{t}}_{k|k}$ and $P^{xx\mathrm{t}}_{k|k}$).

G. Projected Unscented Kalman Filter

The projected unscented Kalman filter (PUKF) is the unscented-based nonlinear extension of the projected Kalman filter (PKF) [8, 9], which considers inequality-constrained linear systems.

Let $W_k \in \mathbb{R}^{n\times n}$ be a positive-definite weighting matrix. PUKF is obtained by appending to UKF the projection equation

$$\hat{x}^{\mathrm{p}}_{k|k} = \underset{\{x_k:\ d_k \ \le\ x_k\ \le\ e_k\}}{\arg\ \min} J_3(x_k), \quad (5.18)$$

where $J_3(x_k) \triangleq (x_k-\hat{x}_{k|k})^{\mathrm{T}} W_k^{-1}(x_k-\hat{x}_{k|k})$. If the estimate $\hat{x}_{k|k}$ given by (3.12) does not satisfy (4.1), then it is projected onto the boundary of (4.1). Similar to [9], we set $W_k = P^{xx}_{k|k}$, where $P^{xx}_{k|k}$ is given by (3.13).

Thus, PUKF is a three-step algorithm whose forecast step is given by (3.1)-(3.10), whose data-assimilation step is given by (3.11)-(3.13), and whose *projection* step is given by (5.18). Note that, unlike the aforementioned algorithms, the constrained estimate $\hat{x}^{\mathrm{p}}_{k|k}$ is not recursively fed back in the forecast step at $k+1$; see (3.2).

H. Algorithms: Summary of Characteristics

In this section, we compare the structure of the UKF, SIUKF, CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF algorithms. Table I lists each algorithm with relation to the specific approaches used for the forecast and data-assimilation steps. Actually, TUKF, TIUKF, and PUKF are three-step algorithms that employ the classical KF update during the data-assimilation step and use a third step (truncation or projection) to enforce the interval constraint.

Moreover, Table II indicates, for each algorithm, whether or not the state estimates, as well as the pseudo error covariance, are affected by the interval constraint in each step of the estimator. Also, we account for the number of constrained optimization problems that must be explicitly solved at each time step to enforce the (4.1).

It is important to mention that SIUKF, CUKF, and CIUKF can handle nonlinear inequality and/or equality constraints

Alg.	Sec.	forecast $\hat{x}_{k\|k-1}$	forecast $P^{xx}_{k\|k-1}$	DA $\hat{x}_{k\|k}$	DA $P^{xx}_{k\|k}$	truncation/projection $\hat{x}^{\mathrm{t}}_{k\|k}$ or $\hat{x}^{\mathrm{p}}_{k\|k}$	truncation/projection $P^{xx\mathrm{t}}_{k\|k}$ or $P^{xx\mathrm{p}}_{k\|k}$	# COP
UKF [3]	III	no	no	no	no	–	–	0
SIUKF [14]	V-A	yes	yes	yes	yes	–	–	$2n+1$
CUKF	V-B	no	no	yes	no	–	–	1
CIUKF	V-C	yes	yes	yes	no	–	–	1
IUKF	V-D	yes	yes	no	no	–	–	0
TUKF	V-E	no	no	no	no	yes	yes	0
TIUKF	V-F	yes	yes	no	no	yes	yes	0
PUKF	V-G	no	no	no	no	yes	–	1

II. Summary of characteristics of state-estimation algorithms (Alg.) for interval-constrained nonlinear systems. It is shown whether or not the state estimates and the pseudo error covariance assimilate the interval constraint information in each step of the estimator. Also, it is indicated how many constrained optimization problems (COP) are solved at each time step.

in addition to (4.1) during the data-assimilation step. Likewise, PUKF is able to enforce such constraints during the projection step, while TUKF and TIUKF can enforce linear equality constraints during the truncation step.

VI. Numerical Example: Continuously Stirred Tank Reactor [1]

We consider the gas-phase, reversible reactions

$$\mathrm{A} \underset{k_2}{\overset{k_1}{\rightleftharpoons}} \mathrm{B}+\mathrm{C}, \tag{6.1}$$

$$2\mathrm{B} \underset{k_4}{\overset{k_3}{\rightleftharpoons}} \mathrm{B}+\mathrm{C}, \tag{6.2}$$

with reaction-rate proportions $k_1 = 0.5$, $k_2 = 0.05$, $k_3 = 0.2$, $k_4 = 0.01$, stoichiometric matrix $S = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}$, and reaction rates $r(t) = \begin{bmatrix} k_1x_1(t) - k_2x_2(t)x_3(t) \\ k_3x_2^2(t) - k_4x_3(t) \end{bmatrix}$. Let the state vector $x(t) \in \mathbb{R}^3_+$ be given by the concentrations of A, B, and C in mol/l. We assume that these reactions take place in a well-mixed, isothermal continuously stirred tank reactor (CSTR), whose dynamics are given by

$$\dot{x}(t) = \frac{1}{3}\left(S^{\mathrm{T}} r(t) + \frac{1}{V_R}\left(\begin{bmatrix} c_f & -x(t) \end{bmatrix} u(t)\right)\right), \tag{6.3}$$

where $V_R = 100\mathrm{l}$ is the reactor volume, $c_f = \begin{bmatrix} 0.5 & 0.05 & 0 \end{bmatrix}^{\mathrm{T}}$ mol/l denotes inlet concentrations, $u(t) = \begin{bmatrix} q_f & q_o \end{bmatrix}^{\mathrm{T}}$ is the input vector, and $q_f \geq 0$ and $q_o \geq 0$ are the volumetric inlet and effluent flow rates. We set $x(0) = \begin{bmatrix} 0.5 & 0.05 & 0 \end{bmatrix}^{\mathrm{T}}$ and $q_f = q_o = 1$

To perform state estimation using UKF, SIUKF, CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF, we integrate the process model (6.3) with $T_{\mathrm{s}} = 0.25\,\mathrm{s}$ using the 4th-order Runge-Kutta algorithm such that $x_k \triangleq x(kT_{\mathrm{s}})$. To help convergence using UKF with $\hat{x}_{0|0} = x_0$, we set $Q_{k-1} = 10^{-6}\, I_{3\times3}$. For uniformity, this value is used in the remaining cases. Also, we assume that we measure the total pressure

$$y_k = \begin{bmatrix} R & R & R \end{bmatrix} x_k + v_k, \tag{6.4}$$

where $R = 32.84$ atm×l/mol is a constant, and $R_k = 0.25^2$ is the variance of $v_k \in \mathbb{R}$. We want to enforce the interval constraint (4.1), where $d_k = 0_{3\times1}$ and $e_k = \infty_{3\times1}$. First, we set a poor initialization given by $\hat{x}_{0|0} = \begin{bmatrix} 0 & 0 & 3.5 \end{bmatrix}^{\mathrm{T}}$ and $P^{xx}_{0|0} = 4\, I_{3\times3}$ and we refer to it as case 1. Case 1 is investigated in [1]. We also investigate a second case (case 2) with good initialization given by $\hat{x}_{0|0} = \begin{bmatrix} 0.6 & 0.1 & 0.05 \end{bmatrix}^{\mathrm{T}}$ and $P^{xx}_{0|0} = 0.5\, I_{3\times3}$. Whenever a constrained optimization problem is solved, since the measurement model is linear, we use the function `quadprog` of Matlab, which implements a subspace trust region optimization method for quadratic programming.

Table III presents a performance comparison among the aforementioned algorithms for a 100-run Monte Carlo simulation, regarding the root-mean-square error of the each state $\mathrm{RMSE}_j = \sqrt{\frac{1}{N}\sum_{k=1}^{N}(x_{j,k} - \hat{x}_{j,k|k})^2}$, $j = 1,\ldots,n$, where N is the final time, and the mean CPU processing time per time step. When applicable, we replace $\hat{x}_{k|k}$ by either $\hat{x}^{\mathrm{t}}_{k|k}$ or $\hat{x}^{\mathrm{p}}_{k|k}$. Figure 3 shows the state estimate for $x_{3,k}$ for a given simulation. For case 1, UKF does not converge for $kT_{\mathrm{s}} < 180$ s and yields estimates violating (4.1) for $kT_{\mathrm{s}} < 80$ s; see Figure 3a. Note also that, although CUKF and PUKF slightly improve convergence, they still result in large error. On the other hand, SIUKF and the truncation-based algorithms TUKF and TIUKF yield the smallest RMSE indices; see Table III. CIUKF and IUKF also provide a good performance, although slightly inferior compared to SIUKF, TUKF, and TIUKF.

For case 2, all constrained algorithms yield more accurate estimates for x_2 and x_3 than UKF. However, among the interval-constrained methods, CUKF and PUKF present the worst results. In sum, we observe that methods that assimilate the interval-constraint information in both the state estimate and pseudo error covariance, namely, CIUKF, SIUKF, TUKF, TIUKF, and IUKF, provide an improved performance compared to UKF. Similar results are presented in [10] for a batch reactor [14].

Regarding computational cost, IUKF, TIUKF, and TUKF are competitive with UKF. Furthermore, CUKF, CIUKF, and PUKF are three to five times slower than UKF, whereas SIUKF is thirteen to twenty times slower than UKF.

VII. Concluding Remarks

We have addressed the interval-constrained state-estimation problem for nonlinear systems. We have investigated how combinations of one of two candidate unscented approaches for forecast and one of five candidate methods for data assimilation (see Table I) can be used. In doing so, we reviewed UKF and SIUKF and introduced CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF. These methods were compared with relation to whether or not the

	UKF	IUKF	SIUKF	TUKF	TIUKF	CUKF	CIUKF	PUKF
	Average of RMSE (mol/l)							
$x_{1,k}$	0.131 (8)	0.050 (4)	0.027 (1)	0.048 (3)	0.045 (2)	0.082 (6)	0.058 (5)	0.083 (7)
	0.014 (6)	*0.008* (1)	*0.012* (5)	*0.018* (8)	*0.018* (7)	*0.011* (3)	*0.009* (2)	*0.011* (4)
$x_{2,k}$	0.554 (8)	0.044 (4)	0.012 (2)	0.010 (1)	0.020 (3)	0.204 (6)	0.059 (5)	0.206 (7)
	0.079 (8)	*0.005* (1)	*0.010* (4)	*0.009* (3)	*0.010* (5)	*0.055* (6)	*0.007* (2)	*0.055* (7)
$x_{3,k}$	0.628 (8)	0.152 (4)	0.132 (1)	0.136 (2)	0.137 (3)	0.263 (6)	0.163 (5)	0.267 (7)
	0.081 (8)	*0.006* (1)	*0.013* (3)	*0.015* (4)	*0.016* (5)	*0.056* (6)	*0.008* (2)	*0.056* (7)
	CPU Processing Time per Iteration (ms)							
	9.6 (3)	6.0 (1)	189.6 (8)	11.1 (4)	8.8 (2)	38.0 (6)	33.4 (5)	38.5 (7)
	6.3 (2)	*5.7* (1)	*192.5* (8)	*9.1* (4)	*8.8* (3)	*33.2* (5)	*33.5* (6)	*33.7* (7)

III. Average of RMSE and mean CPU processing time per iteration for a 100-run Monte Carlo simulation for the CSTR system using UKF, SIUKF, CUKF, CIUKF, IUKF, TUKF, TIUKF, and PUKF. Case-2 results are in italic. In each row, the numbers inside parentheses sort the performance indices in increasing order.

(a) (b)

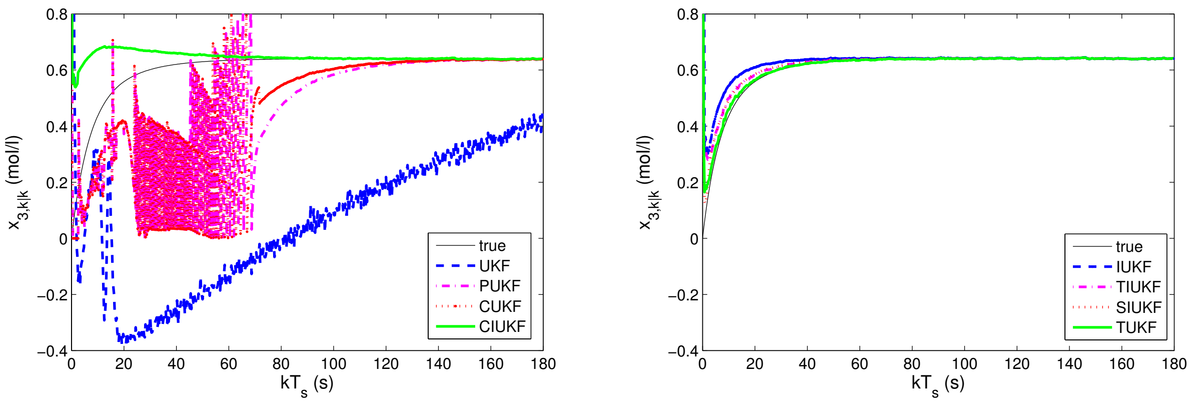


Fig. 3. Estimate of $x_{3,k}$ for the poor initialization case using (a) UKF, PUKF, CUKF, and CIUKF, and (b) SIUKF, IUKF, TUKF, and TIUKF.

state estimates, as well as the pseudo error covariance, are affected by the interval constraint in each step; see Table II.

We discussed an illustrative example, whose state estimates are constrained to be nonnegative. Whenever an interval-constrained algorithm was used, more accurate state estimates were obtained compared to UKF. We observed that, for a good initialization, the performance indices of all constrained algorithms were competitive, except for CUKF and PUKF. However, when a poor initialization was set, the UKF estimates violated the interval constraint. In this case, only a slight improvement over UKF was observed using CUKF and PUKF. On the other hand, SIUKF and the truncation-based algorithms TUKF and TIUKF provided the best performance. CIUKF and IUKF also provided a good performance, although slightly inferior compared to SIUKF, TUKF, and TIUKF.

Since the methods investigated in this study are approximate, it is not clear to point out which one is the best method. Instead, it seems that the choice of a method depends on the application. However, for the example investigated, considering the tradeoff between accuracy and computational cost, TIUKF and TUKF seem to be promising algorithms to enforce interval constraints in nonlinear systems.

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