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Observer design for a class of nonlinear systems using dynamic scaling with application to adaptive control

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***Abstract*— A constructive algorithm for designing an observer for a class of nonlinear systems is presented. We follow the invariant manifold based approach which allows to shape the dynamics of the estimation error. However, this shaping relies on the solution of a PDE which becomes difficult for multi-output systems. In this paper we remove this restriction by adding to the reduced-order observer an output filter and a single dynamic scaling parameter. We show that this method can be applied to systems with unknown parameters, leading to a new class of adaptive controllers. As an application, we consider two examples: an induction motor with unknown load and a longitudinal controller for an aircraft with unknown aerodynamic properties.**

I. Introduction

A general methodology for designing observers for nonlinear systems, introduced in [1], [2], is based on the construction of a manifold with specific properties, namely invariance and attractivity. This approach was first developed for systems that are affine in the unmeasured states in [3].

Invariant and attractive manifolds in observer design have been introduced in the work of Luenberger [4] for linear systems and have been recently exploited for nonlinear systems in [5], [6], [7] and [8].

In the approach of [1], [2], a parameterised description of a manifold in the extended state space is given and the observer dynamics are selected to render this manifold invariant. The crucial issue is therefore the attractivity of the manifold, which has to be achieved by solving a partial differential equation (PDE). For systems that are linear in the unmeasured states [3] this PDE takes the form

$$\frac{\partial \beta}{\partial y} = B(y), \qquad (1)$$

where $\beta(y)$ is a vector function of dimension equal to the number of unmeasured states, and $B(y)$ is a given matrix that depends on the system. When the dimension of y is larger than one, this PDE is solvable only if $B(y)$ is a Jacobian matrix, which occurs only under certain (restrictive) conditions.

In this paper we remove this restriction by introducing a *dynamic* scaling factor in the estimator dynamics and by adding an output filter. It must be noted that dynamic scaling has been widely used in the framework of high-gain observers, see *e.g.* [9], [10], [11], [12], but is used in the present work in a different context.

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The paper is organised as follows. In Section II we revisit the reduced-order observer design introduced in [3] and in Section III we describe the proposed algorithm. As an example, we design an observer for an induction motor with unknown load. Section IV shows that the proposed approach can be applied to the design of adaptive controllers for linearly parameterised systems and in the case of systems in feedback form it provides an alternative to the classical adaptive backstepping design. As an example, we consider the longitudinal control problem for an aircraft with unknown aerodynamic parameters. Section V concludes the paper with some summarising remarks and suggestions for future work.

II. Reduced-order observer design

Consider a class of nonlinear systems described by equations of the form

$$\begin{aligned} \dot{y} &= f(y,u) + \Phi(y)\eta, \\ \dot{\eta} &= h(y,u) + A(y)\eta, \end{aligned} \qquad (2)$$

where $y \in \mathbb{R}^m$ is the measured part of the state and $\eta \in \mathbb{R}^n$ is the unmeasured part (which may also include unknown parameters, *i.e.* equations of the form $\dot{\eta}_i = 0$).

Following the approach of [3] we define the observer error

$$z = \hat{\eta} - \eta = \xi + \beta(y) - \eta,$$

where $\xi \in \mathbb{R}^n$ is the observer state and $\beta(\cdot) : \mathbb{R}^m \to \mathbb{R}^n$ is a mapping to be determined. Defining the observer as

$$\dot{\xi} = h(y,u) + A(y)\hat{\eta} - \frac{\partial \beta}{\partial y}\left(f(y,u) + \Phi(y)\hat{\eta}\right)$$

yields the error dynamics

$$\dot{z} = \left[A(y) - \frac{\partial \beta}{\partial y}\Phi(y)\right] z.$$

To complete the design it is necessary to assign the function $\beta(y)$ so that the above system has a uniformly (asymptotically, if convergence of the estimation error is required) stable equilibrium at zero.

Note that, if the system (2) is detectable, we expect to be able to find an output injection matrix $B(y)$ such that the system $\dot{z} = [A(y) - B(y)\Phi(y)]\, z$ has a uniformly (asymptotically) stable equilibrium at zero. However, if the dimension of y is larger than one, it may not be possible to find a $\beta(y)$ such that (1) holds, *i.e.* $B(y)$ may not be a Jacobian matrix. (Obviously, if y has dimension one, we can simply select $\beta(y)$ to be the integral of $B(y)$.)

To overcome this obstacle in the following section we propose a dynamic extension to the reduced-order observer which consists of an output filter (of order m) and a single dynamic scaling factor (*i.e.* the proposed observer is of order $n+m+1$). The idea is to employ the output filter to ensure that

$$\frac{\partial \beta}{\partial y} = \Psi(y, \hat{y}), \tag{3}$$

where $\hat{y}$ is the filtered output and $\Psi(\cdot)$ is such that $\Psi(y, y) = B(y)$, and then use dynamic scaling to compensate for the mismatch between y and $\hat{y}$. The obvious gain from this modification is that $\Psi(\cdot)$ can be chosen so that (3), in contrast with (1), is easily solvable.

III. Main result

We now present the proposed algorithm for constructing an observer (of order $n+m+1$) for the class of systems (2), under the following detectability-like assumption.

Assumption 1: There exist a continuously differentiable $n \times m$ matrix function $B(\cdot)$, a scalar function $\rho(\cdot) \geq 0$ and a constant $\gamma > 0$ such that

$$\tfrac{1}{2}\left([A(y) - B(y)\Phi(y)]^\top + [A(y) - B(y)\Phi(y)] \right) \leq -\rho(y) I - \gamma \Phi(y)^\top \Phi(y).$$

Remark 1: The foregoing assumption simply implies that the system $\dot{z} = [A(y) - B(y)\Phi(y)]\, z$ has a uniformly globally stable equilibrium at zero and $\Phi(y(t))z(t)$ is square-integrable. If, in addition, $\rho(\cdot)$ is strictly positive, then the observer error z converges to zero. (However, this last property is often not required.)

Note that, when $A(y) \equiv 0$, Assumption 1 is trivially satisfied (with $\rho(y) \equiv 0$) by selecting $B(y) = \gamma \Phi(y)^\top$. This is the case, for instance, in adaptive control, where η is a vector of unknown parameters (see Section IV for details).

Consider the scaled observer error

$$z = \frac{\hat{\eta} - \eta}{r} = \frac{\xi + \beta(y, \hat{y}) - \eta}{r},$$

where $\beta(y, \hat{y}) = [\beta_1(y, \hat{y}), \ldots, \beta_n(y, \hat{y})]^\top$ are functions to be specified and the auxiliary state $\hat{y}$ is obtained from the filter

$$\dot{\hat{y}} = f(y, u) + \Phi(y)\hat{\eta} - K(y, r, \hat{y} - y)(\hat{y} - y), \tag{4}$$

where $K(\cdot)$ is a positive-definite matrix function. Defining the observer as

$$\dot{\xi} = h(y, u) + A(y)\hat{\eta} - \frac{\partial \beta}{\partial y}\left(f(y, u) + \Phi(y)\hat{\eta}\right) - \frac{\partial \beta}{\partial \hat{y}}\dot{\hat{y}} \tag{5}$$

yields the error dynamics

$$\dot{z} = \left[A(y) - \frac{\partial \beta}{\partial y}\Phi(y) \right] z - \frac{\dot{r}}{r} z. \tag{6}$$

The observer design problem is now reduced to the problem of finding a function $\beta(y, \hat{y})$ and a dynamic scaling $\dot{r}$ such that the system (6) has a uniformly globally (asymptotically) stable equilibrium at the origin.

Let the desired output injection matrix, satisfying Assumption 1, be given by

$$B(y) = \left[\, B_1(y) \ \cdots \ B_m(y) \,\right] = \begin{bmatrix} b_{11}(y) & \cdots & b_{1m}(y) \\ \vdots & & \vdots \\ b_{n1}(y) & \cdots & b_{nm}(y) \end{bmatrix}$$

and consider the function

$$\beta(y, \hat{y}) = \int B_1(y_1, \hat{y}_2, \ldots, \hat{y}_m)\mathrm{d}y_1 + \cdots + \int B_m(\hat{y}_1, \ldots, \hat{y}_{m-1}, y_m)\mathrm{d}y_m, \tag{7}$$

which is such that

$$\frac{\partial \beta}{\partial y} = \left[\, B_1(y_1, \hat{y}_2, \ldots, \hat{y}_m) \ \cdots \ B_m(\hat{y}_1, \ldots, \hat{y}_{m-1}, y_m) \,\right].$$

Let $e = \hat{y} - y$ and note that, since $B(\cdot)$ is continuously differentiable, we can write

$$\begin{aligned} B_1(y_1, \hat{y}_2, \ldots, \hat{y}_m) &= B_1(y) - \sum_{j=1}^m e_j \delta_{1j}(y, e), \\ &\ \vdots \\ B_m(\hat{y}_1, \ldots, \hat{y}_{m-1}, y_m) &= B_m(y) - \sum_{j=1}^m e_j \delta_{mj}(y, e), \end{aligned}$$

for some functions $\delta_{ij}(\cdot) \in \mathbb{R}^n$ with $\delta_{ii}(y, e) \equiv 0$. Substituting the above equations into (6) yields

$$\dot{z} = [A(y) - B(y)\Phi(y)]\, z + \sum_{j=1}^m e_j \Delta_j(y, e)\Phi(y) z - \frac{\dot{r}}{r} z, \tag{8}$$

where $\Delta_j(y, e) = [\delta_{1j}(y, e), \ldots, \delta_{mj}(y, e)]$, while the dynamics of e are given by

$$\dot{e} = -K(y, r, e)e + r\Phi(y)z. \tag{9}$$

The system (8)-(9) has an equilibrium at zero and this can be rendered uniformly globally stable by selecting the dynamics of the scaling factor r and the matrix $K(\cdot)$ appropriately, as described in the following lemma.

Lemma 1: Consider the system (2) and suppose that Assumption 1 holds. Let

$$\dot{r} = -\frac{\rho(y)}{2}(r-1) + cr \sum_{j=1}^m e_j^2 \|\Delta_j(y, e)\|^2, \quad r(0) = 1, \tag{10}$$

with $c \geq m/(2\gamma)$, where $\|\cdot\|$ denotes the induced 2-norm, and

$$K(y, r, e) = kr^2 I + \epsilon c r^2 \mathrm{diag}(\|\Delta_j(y, e)\|^2), \tag{11}$$

with $k > 0$ and $\epsilon > 0$ constants. Then the system (8)-(9)-(10) has a uniformly globally stable manifold of equilibria defined by $\Omega = \{\, (z, r, e) \mid z = e = 0 \,\}$. Moreover, $z(t), r(t), e(t) \in \mathcal{L}_\infty$ and $e(t), \Phi(y(t))z(t) \in \mathcal{L}_2$. If, in addition, $\rho(\cdot) > 0$, then $z(t)$ converges to zero.

Proof: Consider the positive-definite and proper function $V(z) = \frac{1}{2}|z|^2$, whose time-derivative along the trajectories of (8) satisfies

$$\begin{aligned} \dot{V} \leq & -\frac{\rho(y)}{2}|z|^2 - \gamma|\Phi(y)z|^2 \\ & + \frac{1}{2}\sum_{j=1}^m e_j z^\top \left[\Delta_j(y, e)\Phi(y) + \Phi(y)^\top \Delta_j(y, e)^\top \right] z \\ & - c\sum_{j=1}^m e_j^2 \|\Delta_j(y, e)\|^2 |z|^2 \leq -\frac{\rho(y)}{2}|z|^2 - \frac{\gamma}{2}|\Phi(y)z|^2. \end{aligned}$$

As a result, the system (8) has a uniformly globally stable equilibrium at the origin, $z(t) \in \mathcal{L}_\infty$ and $\Phi(y(t))z(t) \in \mathcal{L}_2$ (the latter is obtained by integrating the last inequality). Moreover, the above holds true independently of the behaviour of $y(t)$ and $e(t)$.

Consider now the function $W(e,z) = \frac{1}{2}|e|^2 + \frac{1}{k\gamma}V(z)$, whose time-derivative along the trajectories of (8)-(9) satisfies $\dot{W} \leq -kr^2|e|^2 + re^\top\Phi(y)z - \frac{1}{2k}|\Phi(y)z|^2 \leq -\frac{k}{2}r^2|e|^2$, from which we conclude that the system (8)-(9) has a uniformly globally stable equilibrium at $(z,e) = (0,0)$, hence the manifold Ω is uniformly globally stable, and $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

It remains to show that r is bounded. To this end, consider the combined Lyapunov function $U(e,z,r) = W(e,z) + \frac{\epsilon}{2}r^2$, whose time-derivative along the trajectories of (8)-(9)-(10) satisfies $\dot{U} \leq -\frac{k}{2}r^2|e|^2 - \epsilon c r^2 e^\top \mathrm{diag}(\|\Delta_j(y,e)\|^2)e + \epsilon c r^2 \sum_{j=1}^m e_j^2 \|\Delta_j(y,e)\|^2$. Note that the last two terms cancel out, hence $r(t) \in \mathcal{L}_\infty$ and $\lim_{t\to\infty} e(t) = 0$, which concludes the proof. ∎

Remark 2: The term $-\frac{\rho(y)}{2}(r-1)$ appearing in (10) is not needed to prove stability (note that $\rho(y)$ may be zero), but it has been introduced to ensure that, when $\rho(y) > 0$, r stays bounded in the presence of noise (and eventually converges to its initial value). Note also that, from (10), we have that $r(t) \geq 1$, for all $t \geq 0$.

Summarising, the proposed observer is described by the equations (4), (5), (7), (10) and (11) with $\hat{\eta} = \xi + \beta(y, \hat{y})$.

Example 1 (Induction motor): The two-phase equivalent model of an induction motor in the stator reference frame is described by the equations [13]

$$\begin{aligned} \dot{y} &= \begin{bmatrix} -a_0 y_1 + a_2 u_1 \\ -a_0 y_2 + a_2 u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} a_1\mu & a_1 y_3 & 0 \\ -a_1 y_3 & a_1\mu & 0 \\ a_3 y_2 & -a_3 y_1 & -1 \end{bmatrix} \eta \\ \dot{\eta} &= \begin{bmatrix} a_4 y_1 \\ a_4 y_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -\mu & -y_3 & 0 \\ y_3 & -\mu & 0 \\ 0 & 0 & 0 \end{bmatrix} \eta, \end{aligned} \tag{12}$$

where $y = [i_a, i_b, n_p\omega]^\top$, $\eta = [\lambda_a, \lambda_b, n_p\tau_L/J_m]^\top$, i_a, i_b are the stator currents, λ_a, λ_b are the rotor fluxes, ω is the rotor speed, u_1, u_2 are the stator voltages, n_p is the number of pole pairs, J_m is the rotor moment of inertia, τ_L is the unknown load torque, and a_0, a_1, a_2, a_3, a_4 and μ are positive constants, see [13] for details.

The system (12) is of the form (2) with

$$\Phi(y) = \begin{bmatrix} a_1\mu & a_1 y_3 & 0 \\ -a_1 y_3 & a_1\mu & 0 \\ a_3 y_2 & -a_3 y_1 & -1 \end{bmatrix}, \quad A(y) = \begin{bmatrix} -\mu & -y_3 & 0 \\ y_3 & -\mu & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assumption 1 is satisfied (with $\rho(y) \equiv 0$) by selecting the output injection matrix

$$B(y) = \gamma\Phi(y)^\top = \gamma \begin{bmatrix} a_1\mu & -a_1 y_3 & a_3 y_2 \\ a_1 y_3 & a_1\mu & -a_3 y_1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Consider now the function

$$\beta(y) = \gamma \begin{bmatrix} a_1\mu y_1 - a_1\hat{y}_3 y_2 + a_3\hat{y}_2 y_3 \\ a_1\hat{y}_3 y_1 + a_1\mu y_2 - a_3\hat{y}_1 y_3 \\ -y_3 \end{bmatrix},$$

which is such that

$$\frac{\partial\beta}{\partial y} = B(y) - \begin{bmatrix} 0 & a_1 e_3 & -a_3 e_2 \\ -a_1 e_3 & 0 & a_3 e_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the last matrix we have

$$\Delta_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0 & 0 & -a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \Delta_3 = \begin{bmatrix} 0 & a_1 & 0 \\ -a_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

hence from (10) and (11) the dynamic scaling and the gain matrix $K(\cdot)$ are selected, respectively, as

$$\dot{r} = \frac{3}{2\gamma} r \left(a_3^2 e_1^2 + a_3^2 e_2^2 + a_1^2 e_3^2\right)$$

and $K(r) = kr^2 I + \frac{3\epsilon}{2\gamma} r^2 \mathrm{diag}(a_3^2, a_3^2, a_1^2)$.

Remark 3: Assuming that y_1 and y_2 are bounded, the system $\dot{z} = [A(y) - B(y)\Phi(y)]\, z$ can also be stabilised (uniformly in y) by selecting the output injection matrix

$$B(y) = \gamma \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

which is the Jacobian of $\beta(y) = [0, 0, -\gamma y_3]$, thus obviating the need for the output filter and the scaling. However, this solution has two significant drawbacks. First, the convergence rate of z_1, z_2 depends only on μ and cannot be assigned. This is due to the fact that the observer for η_1 and η_2 (the rotor fluxes) is open-loop (and hence non-robust). Second, the dynamics of z_3 are strongly coupled with y_1 and y_2, and this may adversely affect performance.

IV. ADAPTIVE CONTROL DESIGN

In this section we consider a special case of the class of systems (2), where the vector η consists solely of unknown constant parameters, and we provide conditions for constructing an adaptive control law using the observer of Section III. The algorithm is then applied to the longitudinal control problem for an aircraft with unknown aerodynamic coefficients.

We consider linearly parameterised systems of the form

$$\dot{x} = f(x, u) + \Phi(x)\theta, \tag{13}$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$, where $\theta \in \mathbb{R}^p$ is an unknown constant vector and each element of the vector $\Phi(x)\theta$ has the form $\varphi_i(x)^\top\theta_i$, with $\theta_i \in \mathbb{R}^{p_i}$. The control problem is to find a continuous adaptive state feedback control law such that all trajectories of the closed-loop system are bounded and

$$\lim_{t\to\infty} x(t) = x^*, \tag{14}$$

where x^* is a desired set point (the extension to tracking is trivial and is not considered here).

Remark 4: A special subclass of (13) is the so-called parametric strict feedback form which is given by

$$\dot{x}_i = x_{i+1} + \varphi_i(x_1, \ldots, x_i)^\top \theta_i, \tag{15}$$

with states $x_i \in \mathbb{R}$, $i = 1, \ldots, n$ and control input $u \triangleq x_{n+1}$. This class of systems can be stabilised using adaptive backstepping, see [14] and related works. The drawback of this approach is that the resulting closed-loop system dynamics depend strongly on the estimation error which is only guaranteed to be bounded. This can have a detrimental effect on performance. To counteract this problem, an alternative method has been developed in [15], see also [16], which is based on the reduced-order observer design of Section II and which effectively recovers the performance of the known-parameters controller by imposing a closed-loop cascaded structure. However, the application of this method relies on a rather restrictive structural assumption (see Assumption 1 in [15]). The result in this section can therefore be used to remove this assumption.

Remark 5: The result in Lemma 1 is directly applicable to (13). However, in this section we design a separate observer for each vector θ_i to facilitate the control design (for example, to enable a step-by-step construction of the control law, which is necessary when dealing with systems in feedback form). See [17], [18] for an alternative approach utilising scaling, which is not based on parameter estimation.

A. Estimator design

We first show how an estimator for θ_i can be constructed using the observer design of Section III. We then give conditions under which the estimator can be combined with a certainty equivalence control law to obtain an adaptive controller.

Consider the system (13) and let

$$z_i = \frac{\hat{\theta}_i - \theta_i}{r_i} = \frac{\xi_i + \beta_i(x_i, \hat{x}) - \theta_i}{r_i},$$

for $i = 1, \ldots, n$, where ξ_i are the estimator states, r_i are scaling factors, $\beta_i(\cdot)$ are functions to be specified, and the auxiliary states $\hat{x}_i$ are obtained from the filters

$$\dot{\hat{x}}_i = f_i(x, u) + \varphi_i(x)^\top \hat{\theta}_i - k_i(x, r, \hat{x} - x)(\hat{x}_i - x_i), \tag{16}$$

for $i = 1, \ldots, n$, where $k_i(\cdot)$ are positive functions. Using the above definitions and the update laws

$$\dot{\xi}_i = -\frac{\partial \beta_i}{\partial x_i}\big(f_i(x, u) + \varphi_i(x)^\top \hat{\theta}_i\big) - \sum_{j=1}^{n} \frac{\partial \beta_i}{\partial \hat{x}_j} \dot{\hat{x}}_j \tag{17}$$

yields the error dynamics

$$\dot{z}_i = -\frac{\partial \beta_i}{\partial x_i} \varphi_i(x)^\top z_i - \frac{\dot{r}_i}{r_i} z_i. \tag{18}$$

Note that the system (18), for $i = 1, \ldots, n$, can be regarded as a linear time-varying system with a block diagonal dynamic matrix. In order to render the diagonal blocks negative-semidefinite, we select the functions $\beta_i(\cdot)$ as

$$\beta_i(x_i, \hat{x}) = \gamma_i \int \varphi_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, x_i, \hat{x}_{i+1}, \ldots, \hat{x}_n) \mathrm{d}x_i, \tag{19}$$

where γ_i are positive constants.

Let $e_i = \hat{x}_i - x_i$ and note that, since $\varphi_i(\cdot)$ is continuously differentiable, we can write

$$\varphi_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, x_i, \hat{x}_{i+1}, \ldots, \hat{x}_n) = \varphi_i(x) - \sum_{j=1}^{n} e_j \delta_{ij}(x, e),$$

for some functions $\delta_{ij}(\cdot)$, with $\delta_{ii}(x, e) \equiv 0$. Using the above equation and substituting (19) into (18) yields the error dynamics

$$\dot{z}_i = -\gamma_i \varphi_i(x) \varphi_i(x)^\top z_i + \gamma_i \sum_{j=1}^{n} e_j \delta_{ij}(x, e) \varphi_i(x)^\top z_i - \frac{\dot{r}_i}{r_i} z_i, \tag{20}$$

while, from (13) and (16), the dynamics of $e_i = \hat{x}_i - x_i$ are given by

$$\dot{e}_i = -k_i(x, r, e) e_i + r_i \varphi_i(x)^\top z_i. \tag{21}$$

The system (20)-(21) has an equilibrium at zero and this can be rendered uniformly globally stable by selecting the dynamics of the scaling factors r_i and the functions $k_i(\cdot)$ as described in the following lemma.

Lemma 2: Consider the system (13) and let

$$\dot{r}_i = c_i r_i \sum_{j=1}^{n} e_j^2 |\delta_{ij}(x, e)|^2, \qquad r_i(0) = 1, \tag{22}$$

for $i = 1, \ldots, n$, with $c_i \geq \gamma_i n/2$, where $|\cdot|$ denotes the 2-norm, and

$$k_i(x, r, e) = \lambda_i r_i^2 + \epsilon \sum_{\ell=1}^{n} c_\ell r_\ell^2 |\delta_{\ell i}(x, e)|^2, \tag{23}$$

where $\lambda_i > 0$ and $\epsilon > 0$ are constants. Then the system (20)-(21)-(22) has a uniformly globally stable manifold of equilibria defined by $\Omega = \{(z, r, e) \mid z = e = 0\}$. Moreover, $z_i(t) \in \mathcal{L}_\infty$, $r_i(t) \in \mathcal{L}_\infty$, $e_i(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and $\varphi_i(x(t))^\top z_i(t) \in \mathcal{L}_2$, for all $i = 1, \ldots, n$. If, in addition, $\varphi_i(x(t))$ and its time-derivative are bounded, then the signals $\varphi_i(x(t))^\top z_i(t)$ converge to zero.

Proof: Consider the positive-definite and proper function $V_i(z_i) = \frac{1}{2\gamma_i}|z_i|^2$, whose time-derivative along the trajectories of (20)-(22) satisfies

$$\dot{V}_i \leq -\big(\varphi_i(x)^\top z_i\big)^2 + \sum_{j=1}^{n} \left[\frac{1}{2n}\big(\varphi_i(x)^\top z_i\big)^2 + \frac{n}{2} e_j^2 \big(\delta_{ij}^\top z_i\big)^2\right] - \frac{\dot{r}_i}{\gamma_i r_i}|z_i|^2 \leq -\frac{1}{2}\big(\varphi_i(x)^\top z_i\big)^2.$$

As a result, the system (20) has a uniformly globally stable equilibrium at the origin, $z_i(t) \in \mathcal{L}_\infty$ and $\varphi_i(x(t))^\top z_i(t) \in \mathcal{L}_2$, for all $i = 1, \ldots, n$.

Consider now the function $W_i(e_i, z_i) = \frac{1}{2}|e_i|^2 + \frac{1}{\lambda_i} V_i(z_i)$, whose time-derivative along the trajectories of (20)-(21)

satisfies $\dot{W}_i \le -k_i(x,r,e)e_i^2 + \frac{\lambda_i}{2}r_i^2e_i^2 \le -\frac{\lambda_i}{2}r_i^2e_i^2$, for any $\lambda_i > 0$, from which we conclude that the system (20)-(21) has a uniformly globally stable equilibrium at $(z_i, e_i) = (0,0)$, hence the manifold Ω is uniformly globally stable, and $e_i(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

To show that the r_i's are bounded consider the combined Lyapunov function $U(e,z,r) = \sum_{i=1}^n \left[W_i(e_i,z_i) + \frac{\epsilon}{2}r_i^2\right]$, whose time-derivative along the trajectories of (20)-(22)-(21) satisfies $\dot{U} \le -\sum_{i=1}^n \left(k_i(x,r,e) - \frac{\lambda_i}{2}r_i^2\right)e_i^2 + \epsilon\sum_{i=1}^n \left[c_i r_i^2 \sum_{j=1}^n e_j^2|\delta_{ij}(x,e)|^2\right]$. Note now that the last term is equal to $\epsilon\sum_{i=1}^n\sum_{\ell=1}^n c_\ell r_\ell^2|\delta_{\ell i}(x,e)|^2e_i^2$, hence selecting $k_i(\cdot)$ from (23) ensures $\dot{U} \le -\sum_{i=1}^n \frac{\lambda_i}{2}e_i^2$, which proves that $r_i(t) \in \mathcal{L}_\infty$ and $\lim_{t\to\infty} e_i(t) = 0$. Finally, when $\varphi_i(x(t))$ and its time-derivative are bounded, it follows from Barbalat's Lemma that $\varphi_i(x(t))^\top z_i(t)$ converge to zero. ■

Remark 6: In the special case in which $\varphi_i(\cdot)$ is a function of x_i only, the auxiliary states (16) are not used in the adaptive law and $\delta_{ij}(x,e) \equiv 0$ which implies that $\dot{r}_i = 0$, hence we can simply fix the scaling factors r_i to be equal to one. The same simplification occurs when only one of the functions $\varphi_i(\cdot)$ in nonzero.

B. Certainty equivalence control

A possible way of exploiting the estimator properties given in Lemma 2 to design an adaptive control law based on certainty equivalence is given in the following theorem.

Theorem 1: Consider the system (13) and suppose that there exists a control law $u = v(x, \theta + z)$ such that, for all trajectories of the closed-loop system,

(a) $\varphi_i(x(t))^\top z_i(t) \in \mathcal{L}_2 \Longrightarrow x(t) \in \mathcal{L}_\infty$, and

(b) $\lim_{t\to\infty}\varphi_i(x(t))^\top z_i(t) = 0 \Longrightarrow \lim_{t\to\infty} x(t) = x^*$.

Then there exists an adaptive state feedback control law such that all closed-loop signals are bounded and (14) holds.

The proof follows directly from Lemma 2 and conditions (a) and (b), hence it is omitted.

Finally, it is possible to derive a counterpart of the result in [15] with the dynamic scaling-based estimator replacing the estimator in [15], as the following corollary shows.

Corollary 1: Consider the system (15) with input $u \triangleq x_{n+1}$, where $\phi_i : \mathbb{R}^i \to \mathbb{R}^{p_i}$ are $\mathcal{C}^{n-i}$ mappings. Then there exists an adaptive state feedback control law such that all trajectories of the closed-loop system are bounded and $\lim_{t\to\infty}[x_1(t) - x_1^*(t)] = 0$, where $x_1^*(t)$ is a bounded $\mathcal{C}^n$ reference signal.

Example 2 (Aircraft longitudinal control): The longitudinal motion of an aircraft can be described by the equations

$$\begin{aligned}\dot{V} &= -g\sin(\vartheta - \alpha) + \frac{T_x}{m}\cos(\alpha) - \frac{D}{m},\\ \dot{\alpha} &= q + \frac{1}{V}\Big(g\cos(\alpha - \vartheta) - \frac{T_x}{m}\sin(\alpha) - \frac{L}{m}\Big)\\ \dot{\vartheta} &= q, \qquad \dot{q} = \frac{M}{I_y},\end{aligned}$$

where V is the total airspeed, α is the incidence angle, ϑ is the pitch angle, q is the pitch rate, T_x is the thrust (along the x body axis), m is the aircraft mass, g is the gravitational acceleration, I_y is the moment of inertia, M is the pitching moment, and D and L are the aerodynamic forces corresponding to drag and lift, respectively, see [19] for more details.

For the purposes of this example we consider a simple model for the drag and lift given by the parameterised functions

$$D = \frac{1}{2}\rho V^2 S\left(C_{D0} + C_{D\alpha}\alpha^2\right), \quad L = \frac{1}{2}\rho V^2 S C_{L\alpha}\alpha,$$

where ρ is the air density, S is the wing area and $C_{D0}, C_{D\alpha}$, $C_{L\alpha}$ are constant coefficients. To simplify the control design we also assume that the pitch rate can be directly controlled and concentrate on the first two equations. (Note, however, that the proposed control law can be modified to take into account the dynamics of the pitch rate, including the uncertainty in the pitching moment.)

Define the states $x_1 = V$, $x_2 = \alpha$, the control inputs $u_1 = T_x/m$, $u_2 = q$, and the unknown parameters

$$\theta_1 = -\frac{\rho S}{2m}\begin{bmatrix} C_{D0} \\ C_{D\alpha}\end{bmatrix}, \quad \theta_2 = -\frac{\rho S}{2m}C_{L\alpha},$$

and note that the system can be rewritten in the form (13), namely

$$\begin{aligned}\dot{x}_1 &= -g\sin(\vartheta - x_2) + u_1\cos(x_2) + \varphi_1(x)^\top\theta_1,\\ \dot{x}_2 &= u_2 + \frac{g}{x_1}\cos(x_2 - \vartheta) - \frac{u_1}{x_1}\sin(x_2) + \varphi_2(x)^\top\theta_2,\end{aligned} \tag{24}$$

where $\varphi_1(x) = \left[\, x_1^2,\ x_1^2x_2^2 \,\right]^\top$ and $\varphi_2(x) = x_1x_2$.

Note that due to the physical constraints $x_1 \ge V_{\min} > 0$ and $|x_2| \le \alpha_{\max} < \pi/2$, the system (24) is well-defined and controllable. The control objective is to drive the airspeed x_1 and incidence angle x_2 to their respective set-points x_1^* and x_2^*, despite the lack of information on the drag and lift coefficients.

Following the construction of Section IV-A, the output filter is defined as

$$\begin{aligned}\dot{\hat{x}}_1 &= -g\sin(\vartheta - x_2) + u_1\cos(x_2) + \varphi_1(x)^\top\hat{\theta}_1 - k_1e_1,\\ \dot{\hat{x}}_2 &= u_2 + \frac{g}{x_1}\cos(x_2 - \vartheta) - \frac{u_1}{x_1}\sin(x_2) + \varphi_2(x)^\top\hat{\theta}_2\\ &\quad -k_2e_2,\end{aligned}$$

and the estimator dynamics are given by the equations

$$\begin{aligned}\dot{\xi}_1 &= -\frac{\partial\beta_1}{\partial x_1}\big(\dot{\hat{x}}_1 + k_1e_1\big) - \frac{\partial\beta_1}{\partial\hat{x}_2}\dot{\hat{x}}_2,\\ \dot{\xi}_2 &= -\frac{\partial\beta_2}{\partial x_2}\big(\dot{\hat{x}}_2 + k_2e_2\big) - \frac{\partial\beta_2}{\partial\hat{x}_1}\dot{\hat{x}}_1,\end{aligned}$$

where the functions $\beta_i(\cdot)$ are obtained from (19), namely

$$\beta_1(x_1,\hat{x}_2) = \gamma_1\frac{x_1^3}{3}\begin{bmatrix}1\\ \hat{x}_2^2\end{bmatrix}, \quad \beta_2(x_2,\hat{x}_1) = \gamma_2\hat{x}_1\frac{x_2^2}{2},$$

with $\gamma_1 > 0$ and $\gamma_2 > 0$. Note now that

$$\begin{aligned}\frac{\partial\beta_1}{\partial x_1} &= \gamma_1\varphi_1(x) - \gamma_1\begin{bmatrix}0\\ -x_1^2(2x_2 + e_2)\end{bmatrix}e_2,\\ \frac{\partial\beta_2}{\partial x_2} &= \gamma_2\varphi_2(x) - \gamma_2\left[-x_2\right]e_1,\end{aligned}$$

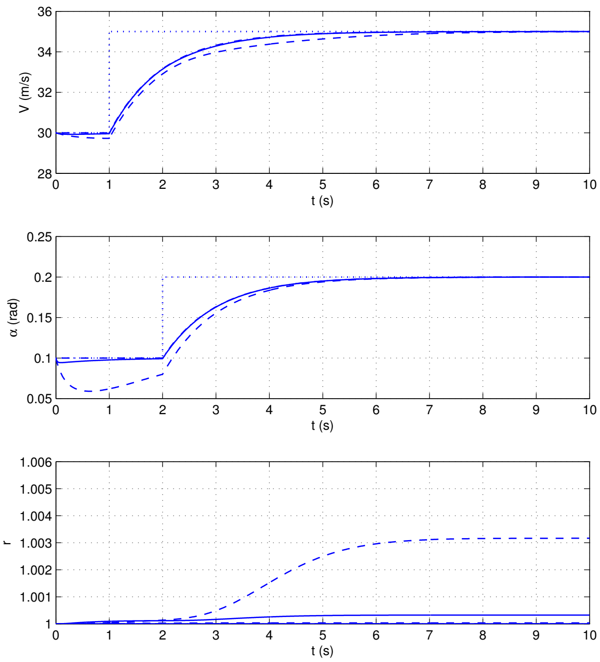

Fig. 1. Responses during set-point changes. Dash-dotted line: Ideal controller. Dashed line: Adaptive controller with $\gamma_1 = 10^{-6}$, $\gamma_2 = 0.5$. Solid line: Adaptive controller with $\gamma_1 = 10^{-5}$, $\gamma_2 = 5$.

where the terms in brackets correspond to the functions $\delta_{12}(\cdot)$ and $\delta_{21}(\cdot)$. Hence, from (22) and (23), the dynamic scaling parameters r_i and the gains k_i are given by

$$\dot{r}_1 = \frac{\gamma_1}{2} x_1^4 (2x_2 + e_2)^2 e_2^2, \qquad \dot{r}_2 = \frac{\gamma_2}{2} x_2^2 e_1^2,$$

and

$$k_1 = \lambda_1 r_1^2 + \epsilon \frac{\gamma_2}{2} r_2^2 x_2^2, \quad k_2 = \lambda_2 r_2^2 + \epsilon \frac{\gamma_1}{2} r_1^2 x_1^4 (2x_2 + e_2)^2,$$

respectively, where $\lambda_1 > 0$, $\lambda_2 > 0$ and $\epsilon > 0$ are constants.

Finally, a control law for the system (24) that satisfies the conditions of Theorem 1 is given by

$$u_1 = \frac{1}{\cos(x_2)} \left(g \sin(\vartheta - x_2) - \varphi_1(x)^\top \hat{\theta}_1 - \mu_1 (x_1 - x_1^*) \right),$$
$$u_2 = -\frac{g}{x_1} \cos(x_2 - \vartheta) + \frac{u_1}{x_1} \sin(x_2) - \varphi_2(x)^\top \hat{\theta}_2 - \mu_2 (x_2 - x_2^*),$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are constants.

The closed-loop system has been simulated using the parameters $\theta_1 = [-0.00063, -0.0358]^\top$ and $\theta_2 = -0.092$, which correspond to an Eclipse-type unmanned aerial vehicle. We consider two set-point step changes: at time $t = 1$ s, the airspeed reference x_1^* changes from 30 m/s to 35 m/s, and at time $t = 2$ s, the incidence angle reference x_2^* changes from 0.1 rad to 0.2 rad. Figure 1 shows the responses for the ideal (known-parameters) controller, and for the proposed adaptive controller for two different sets of gains γ_1, γ_2. We see that by increasing these gains we recover the performance of the ideal controller. Moreover, the scaling factors r_1, r_2 remain relatively small.

V. Conclusions

We have presented a constructive algorithm for designing an observer for a class of nonlinear systems that are affine in the unmeasured states. We have shown that this algorithm can also be applied to systems with unknown parameters, leading to a new class of adaptive controllers, and have provided two illustrative examples. Although we follow the invariant manifold based approach introduced in [1], [3], the contribution of the paper is to remove the restrictions deriving from solvability of PDEs by using a dynamic extension that consists of an output filter and a single dynamic scaling parameter. An area for future work is to extend these tools to the wider class of nonlinear systems considered in [1], [2].

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