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Recursive State Estimation for Linear Systems with Mixed Stochastic and Set-Bounded Disturbances

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Abstract— **Recursive state estimation is considered for discrete time linear systems with mixed process and measurement disturbances that have stochastic and (convex) set-bounded terms. The state estimate is formed as a linear combination of initial guess and measurements, giving an estimation error of the same mixed type (and causing minimal interference between the two kinds of error). An ellipsoidal over-approximation to the set-bounded estimation error term allows to formulate a linear matrix inequality (LMI) for optimization of the filter gain, considering both parts of the estimation error in the objective. With purely stochastic disturbances, the standard Kalman Filter is recovered. The state estimator is shown to work well for an event based estimation example, where measurements are very coarsely quantized.**

I. INTRODUCTION

In many control systems, there exist some disturbances that are best modelled as stochastic, and other disturbances that are better modelled as set-bounded uncertainties. The classical approach to state estimation in such cases is to approximate the set-bounded uncertainties by stochastic ones, allowing to use a standard Kalman Filter. Another approach is to approximate the stochastic disturbances by set-bounded ones, and use a state estimator for set-bounded uncertainty.

It is, however, not straightforward to translate between stochastic and set-bounded disturbances, since they do not combine in the same way. Two measurements of the same variable with independent identically distributed (I.I.D.) stochastic noise combine to form an estimate with only half the error variance. Two measurements with set-bounded uncertainty $y_i = x + z_i, |z_i| \leq 1$ may on the other hand be little better than just one if $y_1 \approx y_2$, not uncommon of situations where this kind of disturbance model is applied.

Thus, it is useful to be able to deal with both kinds of disturbances at the same time. The contribution of this paper is the formulation of an estimator that can deal with general state estimation problems with mixed disturbances. The optimization of the filter gain required in each step is expressed as an LMI. Since the basic structure is that of a Kalman Filter, the estimator reduces to a Kalman Filter in the case of purely stochastic disturbances.

There is much previous work for the cases of only stochastic or only set-bounded disturbances, and also some variations on mixing the two. With only stochastic disturbances, the optimal solution is the classical Kalman Filter (see [6], [7]). State estimation with set bounded disturbances is considered in [1] and [3]. Kalman Filtering with a set-bounded initial expectation in the prior is treated in [8]. For a different approach to mixed disturbance estimation, see [4] and references therein.

When dealing with set-bounded disturbances, there is the issue of how to represent the uncertainty sets that arise as data is combined. Unlike Gaussian noise, there is no general exact closed form representation of limited complexity. We first present the general equations, which can be used with polytopic uncertainty sets. These will however grow quickly in complexity. We will thus focus on the ellipsoidal approximation of uncertainty sets; together with a recursive formulation of the estimator this gives a fixed complexity for the estimator operations.

The rest of the paper is laid out as follows. The mixed state estimation problem to be solved is stated in section II, including the basic estimator structure. Section III covers some preliminaries used in the solution. The first step of the solution is taken in section IV, which shows how to decompose the problem into the stochastic part, treated in section V, and the set-bounded part, treated in section VI. The latter section contains the central theorem to express the set-bounded part of the filter's optimization criterion for a combination of polytopic and ellipsoidal uncertainties, which is proved in the appendix. Section VII compares the proposed estimator with a grid based Bayesian estimator and a Kalman Filter for an example problem. Conclusions are given in section VIII.

II. PROBLEM FORMULATION

The objective is to perform recursive state estimation for discrete time dynamic systems modelled by

$$x_k = Ax_{k-1} + u_{k-1} + e_{k-1}^{\text{proc.}} \tag{1}$$
$$y_k = Cx_k + e_k^{\text{meas.}} \tag{2}$$

where A and C are the dynamics and measurements matrices, and the state x_k, the known control input u_k, the measurements y_k, the process disturbance $e_k^{\text{proc.}}$, and the measurement disturbance $e_k^{\text{meas.}}$ are vectors. Also A and C may be time dependent.

All error terms e^i are the sum of a stochastic term w^i and a set-bounded term δ^i,

$$e^i = w^i + \delta^i$$
$$\mathrm{E}(w^i) = 0, \qquad \mathrm{E}(w^i(w^i)^T) = R^i$$
$$\delta^i \in \Delta^i$$

for some positive semidefinite covariance matrix R^i and convex uncertainty set Δ^i. The stochastic terms of the

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process and measurement disturbance $w_k^{\text{proc.}}$ and $w_k^{\text{meas.}}$ for all times are assumed mutually uncorrelated.

Given the system above and an initial state estimate $\hat{x}_0$ with mixed error

$$e_0 = x_0 - \hat{x}_0$$

we want to form a running state estimate as a linear combination of the initial state and the measurements. The dynamics (1) are used to form the *predicted* estimate $\hat{x}_{k|k-1}$ from the previous *filtered* estimate $\hat{x}_{k-1|k-1}$:

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + u_{k-1}. \tag{3}$$

The measurement y_k is then used to form the current filtered estimate

$$\begin{aligned} \hat{x}_{k|k} &= \hat{x}_{k|k-1} + L_k\Big(y_k - C\hat{x}_{k|k-1}\Big) \\ &= \underbrace{\begin{pmatrix} I - L_kC & L_k \end{pmatrix}}_{X_k} \begin{pmatrix} \hat{x}_{k|k-1} \\ y_k \end{pmatrix} \end{aligned} \tag{4}$$

using some suitable filter gain L_k. We wish to choose L_k to minimize the estimation error in some appropriate sense. The matrix X_k specifies how to weigh together the predicted state estimate and the current measurement, and represents the action of the filtering step.

III. NOTATION AND PRELIMINARIES

The Minkowski sum of two sets X_k and Y is defined as

$$X + Y = \{x + y; x \in X, y \in Y\}.$$

Similarly, we will let the sum $X + y$ of a set X and a vector y be the translation $X + \{y\}$. The product of a set X and a matrix A will be interpreted as the element-wise product

$$AX = \{Ax; x \in X\}.$$

We will also use the product of two sets X, Y as the stacked Cartesian product

$$X \times Y = \left\{ \begin{pmatrix} x \\ y \end{pmatrix}; x \in X, y \in Y \right\}.$$

For a matrix A, we denote by $A > 0$ ($A \geq 0$) that A is positive (semi-)definite. For a block matrix

$$M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

with $D > 0$, the conditions that $M \geq 0$ and that the Schur Complement (see [2, ch. 2.1, pp. 7-8]) of D in M

$$\Delta = A - BD^{-1}B^T$$

is positive semidefinite, $\Delta \geq 0$, are equivalent.

IV. PROBLEM DECOMPOSITION

We begin by decomposing the problem into a stochastic and a set-bounded part. The dynamics (1) combined with the prediction (3) gives the next prediction error

$$e_{k|k-1} = Ae_{k-1|k-1} + e_{k-1}^{\text{proc.}} \tag{5}$$

while the measurement equation (2) combined with the filtering step (4) gives the next filtered error

$$e_{k|k} = X_k \begin{pmatrix} e_{k|k-1} \\ e_k^{\text{meas.}} \end{pmatrix}. \tag{6}$$

The minimization of the expected/worst-case estimation error will guide the selection of the filter gain L_k, which will then be used to update the point estimate according to (4). L_k can be optimized online, or, since it is independent of the point estimate, it can be calculated ahead of time if the disturbance characteristics are known, e.g. if they are periodic or stationary.

The estimation errors $e_{k|k-1}$ and $e_{k|k}$ are composed of a stochastic and a set-bounded part, and are formed by forming each part separately. The two parts will be coupled only in the search for the optimal filter gain L_k in the filtering step, which we find by minimizing the cost function

$$V(L) = \text{tr}\; W\big(R_{k|k}(L) + \alpha r(L)^2 P(L)\big) \tag{7}$$

where $W > 0$ is a weight on the estimation error for different states, $\alpha > 0$ is the relative penalty on set-bounded error, $R_{k|k}(L)$ is the filtered error covariance, and $P_k(L)$ and $r(L)$ bound the set-bounded error after filtering $\delta_{k|k} \in \Delta_{k|k}(L)$ inside an ellipsoid:

$$\delta_{k|k}^T P(L)^{-1} \delta_{k|k} \leq r(L)^2 \qquad \forall\, \delta_{k|k} \in \Delta_{k|k}(L). \tag{8}$$

Either P or r can be fixed for the optimization step, depending on whether we want to prespecify the shape of the ellipsoid circumscribed around $\Delta_{k|k}(L)$.

To carry out the minimization, we take the following steps:

- Form LMI conditions linear in L for
  - the stochastic part: $R \geq R_{k|k}(L)$
  - the set-bounded part: (P, r) satisfying (8)
- Minimize
$$\bar{V} = \text{tr}\; W(R + \alpha r^2 P)$$
  under these LMI conditions.

When we introduce ellipsoidal approximation of the set-bounded error $\Delta_{k|k}$, we will merge the prediction and filtering steps for this part to reduce conservatism.

V. STOCHASTIC PART

We consider the update and optimization of the stochastic estimation error terms. The prediction and filtering steps (5) and (6) give the stochastic error covariances

$$R_{k|k-1} = AR_{k-1|k-1}A^T + R_{k-1}^{\text{proc.}} \tag{9}$$

$$R_{k|k} = X_k \underbrace{\begin{pmatrix} R_{k|k-1} & 0 \\ 0 & R_k^{\text{meas.}} \end{pmatrix}}_{R_k^{\text{pm}}} X_k^T \tag{10}$$

for $w_{k|k-1}$ and $w_{k|k}$ respectively, since if $\mathrm{E}(ww^T) = R$,

$$\mathrm{E}\Big((Aw)(Aw)^T\Big) = A\,\mathrm{E}(ww^T)A^T = ARA^T.$$

The prediction step (9) is straightforward. To form an LMI for the filtering step (10), we first factor R_k^{pm} as

$$R_k^{\text{pm}} = SR^{\text{pm}0}S^T, \qquad R^{\text{pm}0} > 0.$$

By the Schur Complement, the condition $R \geq R_{k|k}$ or

$$R - X_k S R^{\text{pm}0} S^T X_k^T \geq 0$$

is then equivalent (since $R^{\text{pm}0} > 0$) to the LMI

$$\begin{pmatrix} R & X_k S \\ S^T X_k^T & (R^{\text{pm}0})^{-1} \end{pmatrix} \geq 0,$$

which is linear in L and R.

VI. SET-BOUNDED PART

We now consider the update and optimization of the set-bounded estimation error terms. The operations are first formulated for general uncertainty sets, and then the case of ellipsoidal over-approximation is treated.

A. General Uncertainty Sets

From the prediction step (5), we must have $\delta_{k|k-1} \in \Delta_{k|k-1}$,

$$\Delta_{k|k-1} = A\Delta_{k-1|k-1} + \Delta_{k-1}^{\text{proc.}}.$$

If $\Delta_{k-1|k-1}$ and $\Delta_{k-1}^{\text{proc.}}$ are polytopes, so is $\Delta_{k|k-1}$.

For the filtering step, we have

$$\delta_{k|k} = X_k \underbrace{\begin{pmatrix} \delta_{k|k-1} \\ \delta_k^{\text{meas.}} \end{pmatrix}}_{\delta_k^{\text{pm}}}.$$

The constraint (8) can be expressed for any $\delta_k^{\text{pm}} \in \Delta_k^{\text{pm}} = \Delta_{k|k-1} \times \Delta_k^{\text{meas.}}$ as a second order cone constraint when P is fixed:

$$r \geq ||P^{-\frac{1}{2}}\delta_{k|k}|| = ||P^{-\frac{1}{2}}X_k\delta_k^{\text{pm}}||$$

or in general by the Schur Complement (since $P > 0$) as an LMI

$$\begin{aligned} & r^2 - (\delta_k^{\text{pm}})^T X_k^T P^{-1} X_k \delta_k^{\text{pm}} \geq 0 \\ \Longleftrightarrow & \begin{pmatrix} P & X_k\delta_k^{\text{pm}} \\ (\delta_k^{\text{pm}})^T X_k^T & r^2 \end{pmatrix} \geq 0. \end{aligned}$$

If Δ_k^{pm} is a polytope, it is enough to consider the constraint at the vertices, since an ellipsoid contains a set of vertices iff it contains the convex hull of those vertices (the polytope).

B. Ellipsoidal Uncertainty Sets

Now suppose that the filtered set-bounded error from the previous step $\Delta_{k-1|k-1}$, and possibly the process or measurement disturbance parts $\Delta_{k-1}^{\text{proc.}}$ and $\Delta_k^{\text{meas.}}$, are described by ellipsoids. In this case we can use the ellipsoid (8) to find an ellipsoidal over-approximation for $\Delta_{k|k}$ to use in the next step. To formulate (8) as an LMI in this case, we need the following theorem.

Theorem 1 (Ellipsoid Bounding Weighted Ellipsoid Sum): Given a number of ellipsoids $\mathcal{E}_i, i = 1 \ldots n$:

$$z_i \in \mathcal{E}_i \Longleftrightarrow \begin{cases} z_i = G_i x_i + b_i \\ x_i^T Q_i x_i \leq r_i^2 \end{cases}$$

the weighted Minkowski sum

$$\mathcal{A} = X\sum_i \mathcal{E}_i = \left\{ x = Xz; z = \sum_i z_i, z_i \in \mathcal{E}_i\, \forall i \right\}$$

can be proved by the S-procedure (see [2, ch. 2.6.3, pp. 23-24]) to be contained in the centered target ellipsoid $\mathcal{E}$,

$$x \in \mathcal{E} \Longleftrightarrow x^T P^{-1} x \leq r^2 \tag{11}$$

iff the LMI condition

$$\begin{pmatrix} P & XG & Xb \\ G^T X^T & Q\tau & \\ b^T X^T & & r^2 - \sum_i \tau_i r_i^2 \end{pmatrix} \geq 0 \tag{12}$$

is satisfied for some scalars $\tau_i \geq 0$, where $b = \sum_i b_i$, and

$$G = \begin{pmatrix} G_1 & G_2 & \ldots & G_n \end{pmatrix}, \qquad Q_\tau = \operatorname{diag}\big(\{\tau_i Q_i\}_i\big).$$

If $n = 1$ and $r_1 > 0$, the condition (12) is also necessary for $\mathcal{A} \subseteq \mathcal{E}$.

Proof: See the appendix.

1) Using the theorem: We let $P = P$ and $z = \delta_k^{\text{pm}}$, where Δ_k^{pm} is a sum of ellipsoids. With one centered ellipsoid ($b_i = 0$) containing each of the previous filtered error, the process and measurement disturbances:

$$\Delta_{k-1|k-1} \subseteq \mathcal{E}_1, \qquad \Delta_{k-1}^{\text{proc.}} \subseteq \mathcal{E}_2, \qquad \Delta_k^{\text{meas.}} \subseteq \mathcal{E}_3$$

the set-bounded part gets the prediction step $\Delta_{k|k-1} \subseteq A\mathcal{E}_1 + \mathcal{E}_2$ and the filtering step

$$\Delta_{k|k} \subseteq X_k(\Delta_{k|k-1} \times \mathcal{E}_3) \subseteq X_k\big((A\mathcal{E}_1 + \mathcal{E}_2) \times \mathcal{E}_3\big).$$

The ellipsoid sum for $\Delta_{k|k}$ can thus be expressed with the theorem, plugging in the ellipsoids $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, and

$$G_1 = \begin{pmatrix} A \\ 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

Thus we can use the LMI condition (12) to circumscribe an ellipsoid around $\Delta_{k|k}$.

2) Variations: We can use more or fewer ellipsoidal terms for the uncertainty sets Δ_i, and also polytopic terms. For polytopic terms, the sum $\mathcal{P}$ of all such terms is first formed. As in the case with only polytopic terms, the LMI must be written once for each vertex of $\mathcal{P}$. If $\mathcal{P}$ is symmetric, we need only write half as many LMI:s since the centered target ellipsoid $\mathcal{E}$ sees no difference between the vertices v and $-v$. A polytope vertex can be represented by a zero-dimensional ellipsoid with $b_i \neq 0$.

A polytope that is the sum of one-dimensional polytopes (line segments) may expressed more economically as a sum of one-dimensional ellipsoids. However, the result may be more conservative since forming the sum of ellipsoids relies on the S-procedure.

The use of both P and r as variables in the condition (11) for the target ellipsoid may seem redundant, but it allows to state a possibly simpler optimization problem if the shape of the target ellipsoid is fixed. (I.e. to some shape desired in a stationary situation.) It is of course possible to constrain P to other spaces than to be fully free or with a prespecified shape. Another use for r could be to improve the numerical conditioning of the optimization problem by guessing the size of the resulting ellipsoid before optimizing for P.

VII. SIMULATIONS

A. Example System

Consider a double integrator process with dynamics

$$x_{k+1} = \underbrace{\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}}_{A} x_k + \underbrace{\begin{pmatrix} \frac{1}{2}h^2 \\ h \end{pmatrix}}_{B} u_k + w_k^{\text{proc.}}$$

$$\mathrm{E}\big(w_k^{\text{proc.}}\big) = 0, \quad \mathrm{E}\Big(\big(w_k^{\text{proc.}} w_k^{\text{proc.}}\big)^T\Big) = \underbrace{\frac{1}{4}\begin{pmatrix} \frac{1}{3}h^3 & \frac{1}{2}h^2 \\ \frac{1}{2}h^2 & h \end{pmatrix}}_{R_k^{\text{proc.}}}$$

where $h = 0.1$ is the sample time, $(x_k)_1$ is the position and $(x_k)_2$ the velocity. White process noise enters along with the control acceleration u_k.

The measurements are coarsely quantized:

$$y_k = \text{round}\big(Cx_k\big), \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

where round(x) rounds x to the nearest integer. Using the current framework, we can model the measurement by

$$y_k = Cx_k + \delta_k^{\text{meas.}}, \quad \delta_k^{\text{meas.}} \in \Delta_k^{\text{meas.}} = [-\tfrac{1}{2}, \tfrac{1}{2}].$$

With the sampling time h small enough, we may consider $(x_k)_1$ to be almost completely known at all *events*, when y_k changes value. This measurement may be modelled as

$$\begin{aligned} \frac{1}{2}\big(y_k + y_{k-1}\big) &= Cx_k + w_k^{\text{meas.}}, \\ \mathrm{E}\big(w_k^{\text{meas.}}\big) = 0, \quad \mathrm{E}\Big(\big(w_k^{\text{meas.}} w_k^{\text{meas.}}\big)^T\Big) &= R_k^{\text{meas.}}, \end{aligned} \tag{13}$$

where $R_k^{\text{meas.}}$ gives a suitable approximation of the error in the guess $Cx_k \approx \frac{1}{2}\big(y_k + y_{k-1}\big)$. We take $R_k^{\text{meas.}} = (R_k^{\text{proc.}})_{11}$.

Since the system is unstable, we stabilize it with the control law

$$u_k = -\begin{pmatrix} 1 & 2 \end{pmatrix} \hat{x}_k,$$

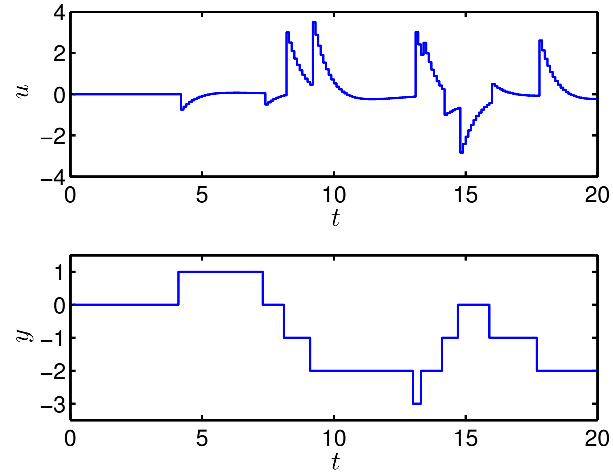

Fig. 1. Test sequence for the observers

E_{Mixed}	E_{Grid}	E_{Kalman}
$\begin{pmatrix} 0.054 & 0.063 \\ 0.063 & 0.180 \end{pmatrix}$	$\begin{pmatrix} 0.045 & 0.053 \\ 0.053 & 0.157 \end{pmatrix}$	$\begin{pmatrix} 0.444 & 0.223 \\ 0.223 & 0.285 \end{pmatrix}$

TABLE I

MEAN QUADRATIC ERRORS OVER A 10^5 TIME STEP TEST SEQUENCE.

which places the poles in approximately $z = e^{-h}$. The state estimate $\hat{x}_k$ is taken from a simple heuristic state estimator that:

- runs in open loop between events
- updates at events:

$$\begin{aligned} (\hat{x}_k)_1 &= \frac{1}{2}\big(y_k + y_{k-1}\big) \\ (\hat{x}_k)_2 &= \frac{(\hat{x}_k)_1 - (\hat{x}_{k_{\text{last}}})_1}{h(k - k_{\text{last}})} \end{aligned}$$

where k_{last} is the time index of the last event or known initial state.

The process was simulated with the heuristic controller to produce the test sequence u_k, y_k in Fig. 1. The corresponding state sequence x_k can seen in Fig. 2. (together with state estimates from different estimators)

B. Estimator Implementation For The Example

In this example, the process noise is purely stochastic, and the set-bounded measurement error $\Delta_k^{\text{meas.}}$ can be represented as an interval symmetric around the origin, so the target ellipsoid $\mathcal{E} \supseteq \Delta_{k|k}$ should enclose the sum of an ellipsoid for $\Delta_{k|k-1}$ and the polytope for $\Delta_k^{\text{meas.}}$. Since we have only one ellipsoid in the sum, (12) is both necessary and sufficient for the target ellipsoid $\mathcal{E}$ to enclose it. Since the polytope $\Delta_k^{\text{meas.}}$ is symmetric with two vertices, we need only one instance of the LMI condition (12).

C. Performance Comparison

Three filters were compared on the test sequence:

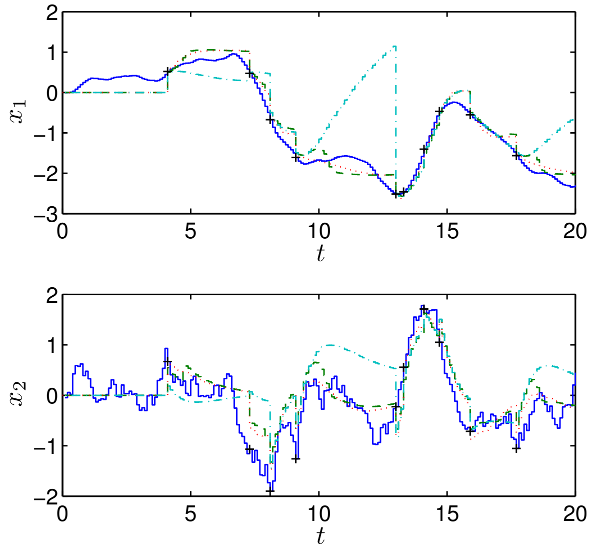

Fig. 2. Actual states and state estimates generated by the observers. Actual states (solid), Mixed Estimator (dashed), Grid Filter (dotted), Kalman Filter (dash-dotted). Events are marked with a $+$ sign.

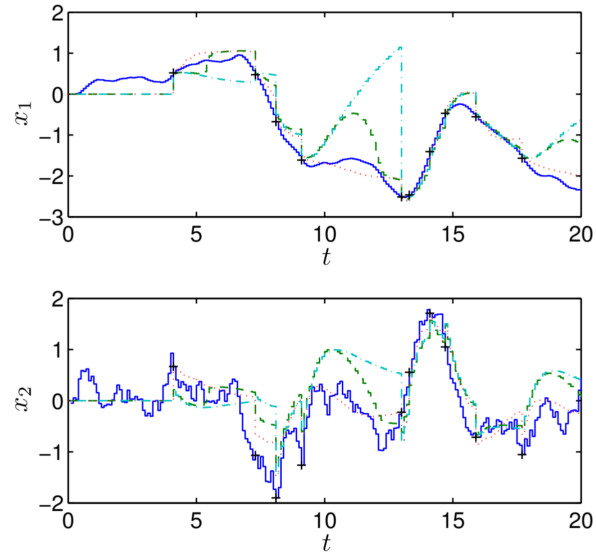

Fig. 3. Actual states and state estimates, with $\alpha = 10$ for the Mixed Estimator, which makes it follow the Kalman Filter for too long.

- The Mixed Estimator proposed in this paper using ellipsoidal over-bounding of $\Delta_{k|k}$ in each step, with
$$\alpha = 1, \qquad W = \begin{pmatrix} 1 & -0.3 \\ -0.3 & 0.4 \end{pmatrix}.$$
The weight matrix W was chosen by letting W^{-1} be roughly proportional to the error covariance of the Grid Filter (see below) a long time after an event.
- A *Grid Filter*; a discretization of the Bayesian Estimator for the system (with approximately 32 000 states). See [5] for more about the Bayesian Estimator for this system.
- A Kalman Filter that uses only the measurements (13) at events, and runs in open loop in between.

Table I shows the average estimation error of the filters over a test sequence of 10^5 time steps, evaluated as

$$E = \frac{1}{N}\sum_{k=1}^{N}(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T.$$

The Mixed Estimator is seen to come quite close to the Grid Filter performance, but the Kalman Filter is far behind. Fig. 2 shows actual state trajectories together with the estimates. Events are marked with $+$ signs. When events are frequent, all estimators seem to follow the state trajectories reasonably well, especially for the position x_1. When there is longer time between events, the Kalman Filter seems to lose track. The Mixed Filter is much better at following the Bayesian estimate. The strategy it uses seems to be something like:

- At an event, update the state estimate.
- Continue by open loop predictions some time after each event, while the prediction error is small.
- When the prediction error becomes too large, start to incorporate the imprecise measurements available.

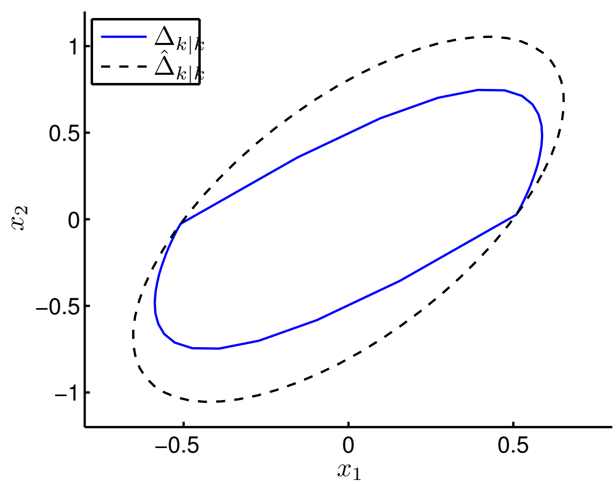

Fig. 4. Actual set bounded error and ellipsoidal approximation used by the Mixed Filter at $t = 12.9$, just before an event.

Fig. 3 shows the same simulation with $\alpha = 10$ for the Mixed Estimator. The weight α adjusts the tradeoff between stochastic and set-bounded estimation error. With higher α it is seen that the Mixed Filter waits longer to incorporate the uncertain measurements after each events. The value $\alpha = 1$ used in Fig. 2 seems to give a more reasonable tradeoff.

The uncertainty set $\Delta_{k|k}$ (a polytope in this example) and the recursive ellipsoidal over-approximation $\hat{\Delta}_{k|k}$ used by the mixed filter can be seen in Fig. 4, just prior to the event at $t = 13$. The actual set takes up perhaps $\frac{2}{3}$ of the ellipsoid's volume, and that they more or less touch at the sharpest corners of the polytope.

VIII. CONCLUSION

This paper describes the design of a state estimator for linear systems with process and measurement disturbances containing both stochastic and set bounded terms. The estimator structure that is borrowed from the Kalman Filter is optimal for purely stochastic disturbances, and allows the two parts of the estimation error to be treated efficiently and almost independently. The filter gain is optimized by solving a Linear Matrix Inequality (LMI) problem.

The estimator can value the usefulness of measurements corrupted by different amounts of stochastic and set bounded disturbances, with a parameter α that can be used to tune the tradeoff between the two kinds of error. An example shows that the estimator performs quite close to an optimal Bayesian Estimator, and that α can be used to adjust how long to wait after receiving a good measurement before incorporating measurements with interval uncertainty.

The estimator reproduces the behavior of the Kalman Filter with set-bounded initial expectation in [8] under the circumstances assumed in that work, when the weight α goes to zero. When α is nonzero, the estimator applies a higher filter gain to eliminate the set-bounded uncertainty faster.

An open issue is how to choose the state weighting matrix W in a systematic fashion.

IX. ACKNOWLEDGMENTS

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APPENDIX

Proof of Theorem 1: This development is based on [2, ch. 3.7.4, pp. 46-47]. The construction is extended to be linear in the transformation X, to handle ellipsoids that are flat in some dimensions, and to specify the centers b_i separately, but is reduced in that we are only interested in centered target ellipsoids $\mathcal{E}$.

To handle the Minkowski sum of ellipsoids, we need a condition for when one ellipsoid contains the intersection of a number of ellipsoids. Given a set of quadratic functions $\{f_i(x)\}_i, i = 1 \dots n$, one sufficient condition to verify that a quadratic function $f(x) \geq 0$ whenever all $f_i(x) \geq 0$ is given by the S-procedure:

$$\exists\, \tau_i \geq 0, i = 1 \dots n: \quad f(x) \geq \sum_i \tau_i f_i(x) \quad \forall x.$$

The condition is also necessary e.g. when $n = 1$ and $f_1(x) > 0$ for some x, see [2, ch. 2.6.3, pp. 23-24].

The condition (12) which we seek to derive is formed by first constructing an extended space where each term of the ellipsoid sum has its own coordinates, and forming the set where all coordinates are within their respective ellipsoids, which is the intersection of ellipsoidal cylinders. We then used the S-procedure to circumscribe an ellipsoidal cylinder parametrized in the sum coordinates.

Let

$$x^T = \begin{pmatrix} x_1^T & x_2^T & \dots & x_n^T \end{pmatrix}, \qquad z = \sum_i z_i.$$

Then, according to the definitions in the theorem,

$$z = Gx + b = \underbrace{\begin{pmatrix} G & b \end{pmatrix}}_{G_e} \underbrace{\begin{pmatrix} x \\ 1 \end{pmatrix}}_{x_e} = G_e x_e.$$

We take the first step of the S-procedure (using $\tau_i \geq 0\, \forall i$) by forming the condition

$$\sum_i \tau_i (r_i^2 - x_i^T Q_i x_i) = \left(\sum_i \tau_i r_i^2 \right) - x^T Q_\tau x \geq 0 \quad (14)$$

which will always be fulfilled when $z_i \in \mathcal{E}_i\, \forall i$.

The condition for the target ellipsoid, $x \in \mathcal{E}$, $x = Xz = XG_e x_e$ is equivalent to

$$r^2 - x_e^T G_e^T X^T P^{-1} X G x_e \geq 0. \quad (15)$$

Subtracting (14) from (15), we form our S-procedure condition, which can clearly only be fulfilled for all x if (15) is fulfilled whenever (14) is:

$$x_e^T \left(\underbrace{\begin{pmatrix} Q_\tau & \\ & r^2 - \sum_i \tau_i r_i^2 \end{pmatrix}}_{Q_e} - G_e^T X^T P^{-1} X G_e \right) x_e \geq 0.$$

As we assume x to be arbitrary, we might as well assume x_e to be arbitrary since scaling of x_e with a nonzero constant does not affect whether the condition holds. The case when the last entry of x_e is zero is approached when $||x|| \to \infty$. Thus we can equivalently consider positive semidefiniteness of the matrix that stands between x_e^T and x_e above.

By the Schur Complement, since $P^{-1} > 0$, this condition is equivalent to

$$\begin{pmatrix} P & XG_e \\ G_e^T X^T & Q_e \end{pmatrix} \geq 0,$$

which is exactly (12).