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Nonlinear GMV Control for Unstable State Dependent Multivariable Models

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***Abstract* — A *Nonlinear Generalized Minimum Variance* control law is derived for systems represented by an input-output state dependent nonlinear subsystem that may be open-loop unstable. The solution is obtained using a model for the multivariable discrete-time process that includes a state-dependent (nonlinear and possibly unstable) model that links the output and any *unstructured* nonlinear input subsystem. The input subsystem can involve an operator of very general nonlinear form, but this has to be assumed to be stable. This is the first *NGMV* control solution that is suitable for systems containing an unstable *nonlinear* sub-system contained in a state-dependent model.**

I. INTRODUCTION

A control law is proposed based on an extension of the well known *Minimum Variance* (*MV*) controller for nonlinear multivariable systems. The nonlinear (*NL*) system model includes several possible sub-systems that provide alternative ways of modeling the linear/non-linear system. The new innovation is the addition of a *state dependent* model sub-system. The *MV* controller was derived by *Åström* [1], assuming the plant was linear and minimum-phase, and was later derived for processes that could be non-minimum phase. *Hastings-James* [2] and later *Clarke and Hastings-James* [3] modified the first of these control laws by adding a control signal costing term. This was termed a *Generalized Minimum Variance* (*GMV*) control law and enabled non-minimum phase processes to be stabilized.

A family of so-called *Nonlinear Generalized Minimum Variance* (*NGMV*) controllers was derived recently for nonlinear *model based* multivariable systems. The assumption was made that the plant model could be decomposed into a set of delay terms, a very general input nonlinear subsystem that had to be stable and a linear subsystem that could be represented in polynomial matrix or state equation form and include unstable modes. This problem was analyzed by *Grimble* ([4], [5]) and *Grimble and Majecki* ([6], [7]). The major contribution here, over the basic *NGMV* control law in *Grimble* [8] involves the introduction of a more general model structure, where the nonlinearities may be associated with either inputs or outputs and include open-loop unstable elements. The solution of the *NGMV* control problem for a system with linear state-space models has been considered previously [11] but the use of a *state-dependent* model that may be open loop unstable is new.

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II. NONLINEAR OPERATOR AND STATE-DEPENDENT MODELS

The system is shown in Fig. 1, including the *nonlinear* plant models and the linear reference/disturbance models. The output of the unstructured nonlinear system is denoted $u_1(t-k)$ and this acts as an input to the state-dependent block. The first plant sub-system is of a very general nonlinear operator form: $\mathcal{W}_1(.,.)$, where the model without explicit delay terms: $\mathcal{W}_{1k}(.,.)$ and $u_1(t)=(\mathcal{W}_{1k}u)(t)$. The second is of a *state-dependent nonlinear equation form* that may be given the input-output representation: $\mathcal{W}_0(.,.)$, or without delay: $\mathcal{W}_{0k}(.,.)$. The total output of the system $y(t)$ depends upon the state-dependent block. Let the total output including the disturbance model signal $d(t)$ be expressed as: $y(t)=d(t)+(\mathcal{W}_0 u_1)(t)$. The zero-mean white *noise* is denoted $\{v(t)\}$ and it has a covariance matrix R_f. There is no loss of generality in assuming the zero-mean, white noise signals: $\{\omega(t)\}$ and $\{\xi_0(t)\}$ that feed reference and disturbance models have identity covariance matrices and a *Gaussian distribution*. Signals shown in Fig. 1 follow.

Error signal: $$e(t)=r(t)-y(t) \tag{1}$$

Plant output: $$y(t)=d(t)+\left(\mathcal{W}u\right)(t) \tag{2}$$

where the total linear and nonlinear plant input/output model: $\left(\mathcal{W}u\right)(t)=\left(\mathcal{W}_{0k}\mathcal{W}_1 u\right)(t)$.

Reference signal: $$r(t)=W_r\omega(t) \tag{3}$$

Observations signal: $$z(t)=y(t)+v(t) \tag{4}$$

Noisy error signal:
$$e_0(t)=r(t)-z(t)=r(t)-d(t)-v(t)-\left(\mathcal{W}u\right)(t) \tag{5}$$

Linear Sub-System Models: The linear state-space system models may now be introduced. The reference and the error weighting models are assumed to be linear. The state-space system models, for the ($r\times m$) multivariable system, shown in Fig. 1, may be listed as follows:

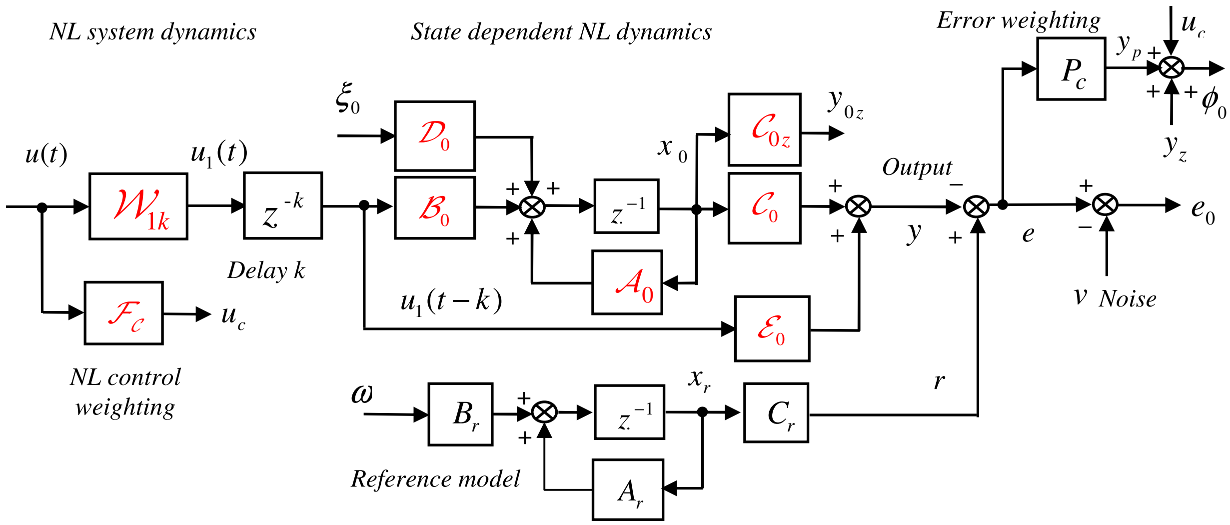

Fig. 1: Nonlinear Unstructured and State-Dependent Plant Model Including Disturbance and Reference

Reference model:

$$x_r(t+1) = A_r x_r(t) + D_r \omega(t), \qquad x_r(t) \in R^{n_r} \tag{6}$$

$$r(t) = C_r x_r(t) \text{ and } W_r(z^{-1}) = C_r(zI - A_r)^{-1} D_r \tag{7}$$

Also introduce the *dynamic* error weighting that is used in the cost-index $y_p(t) = P_c(z^{-1})\big(r(t) - y(t)\big)$ defined later.

This has the following state-space representation:

$$x_p(t+1) = A_p x_p(t) + B_p\big(r(t) - y(t)\big), \qquad x_p(t) \in R^{n_p} \tag{8}$$

$$y_p(t) = C_p x_p(t) + E_p\big(r(t) - y(t)\big) \tag{9}$$

A. Nonlinear Sub-Systems

The non-linear system models, defining the system output: $y(t) = d(t) + (\mathcal{W}u)(t)$ and the input non-linear operator model are explained further below. Let

$$(\mathcal{W}u)(t) = (\mathcal{W}_k z^{-k} u)(t) \tag{10}$$

where k denotes the magnitude of the common delay elements in the output signal paths. The delay free model may be written: $(\mathcal{W}_k u)(t) = (\mathcal{W}_{0k}\mathcal{W}_{1k} u)(t)$ and the total forward path *NL* plant model:

$$(\mathcal{W}u)(t) = (\mathcal{W}_{0k}\mathcal{W}_1 u)(t) = (\mathcal{W}_{0k} z^{-k}\mathcal{W}_{1k} u)(t) \tag{11}$$

The signal input to the linear and *NL* state-dependent dynamics is denoted: $u_1(t) = \mathcal{W}_{1k}(.,.)u(t)$. The subsystem: $\mathcal{W}_{1k}$ is assumed *finite gain stable* but the *NL state-dependent model* may be unstable.

B. Total Linear Sub-System State Equation Model

Combining the linear equations for the sub-systems in §2.2 obtain the augmented state equations for the total *linear* part of the system, by augmenting the state vector as: $x_1(t) = \begin{bmatrix} x_r^T(t) & x_p^T(t)\end{bmatrix}^T$. Noting equations (6) and (8), the augmented state, output and observations:

$$x_1(t+1) = A_1 x_1(t) + B_1 y(t) + D_1 \xi_1(t) \tag{12}$$

$$\text{where } A_1 = \begin{bmatrix} A_r & 0 \\ B_p C_r & A_p \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -B_p \end{bmatrix}, \quad D_1 = \begin{bmatrix} D_r \\ 0 \end{bmatrix} \tag{13}$$

Note the *resolvent:* $\Phi_1(z^{-1}) = (zI - A_1)^{-1}$.

C. State-Dependent Dynamics

The second NL system model is represented in the so-called *linear state-dependent (LSD)* state-space form. This has been used for state-dependent Riccati equation optimal control solutions [9] and involves matrices that are time-varying since they are allowed to depend upon the system states. A slight extension of this idea is to allow these matrices to be functions of the model input at time *t - k*. The *NL* model is therefore assumed to have the following state-dependent form ([10], [11]):

$$x_0(t+1) = \mathcal{A}_0(x_0,u_1)\, x_0(t) + \mathcal{B}_0(x_0,u_1)\, u_1(t-k) + \mathcal{D}_0(x_0,u_1)\, \xi_0(t) \tag{14}$$

$$y(t) = \mathcal{C}_0(x_0,u_1)\, x_0(t) + \mathcal{E}_0(x_0,u_1)\, u_1(t-k) \tag{15}$$

where $x_0(t)$ is a vector of *sub-system* states, $u_1(t)$ is a vector of the *LSD* sub-system inputs and $y(t)$ is a vector of output signals. To simplify the notation in (14), (15) write $\mathcal{A}_0(t) = \mathcal{A}_0(x_0(t), u_1(t-k))$ and similarly for the matrices: $\mathcal{B}_0$, $\mathcal{C}_0$ and $\mathcal{E}_0$. The error weighting term $y_p(t) = P_c e(t)$ that is needed in the cost expression may therefore be written in a more concise form, using (12) and (15). Thence, substituting from (15) into (12):

$$x_1(t+1) = A_1 x_1(t) + B_1\mathcal{C}_0 x_0(t) + B_1\mathcal{E}_0 u_1(t-k) + D_1\xi_1(t) \quad (16)$$

The *weighted output* in equation (9) may now be written:

$$\begin{aligned} y_p(t) &= C_p x_p(t) + E_p\left(C_r x_r(t) - y(t)\right) \\ &= \mathcal{C}_{0p} x_0(t) + C_{1p} x_1(t) + (\mathcal{E}_\phi u_1)(t-k) \end{aligned} \quad (17)$$

where $\mathcal{C}_{0p} = -E_p\mathcal{C}_0$ and $C_{1p} = \begin{bmatrix} E_p C_r & C_p \end{bmatrix}$ and the through term $\mathcal{E}_\phi = -E_p\mathcal{E}_0$. (18)

D. Combined Linear and Nonlinear Models

Let the total combined vector of linear and state-dependent sub-system model states be defined to have the form: $x(t) = [x_0(t)^T \quad x_1(t)^T]^T$. Thence, the *combined state vector* for the linear and nonlinear sub-systems and the related disturbance vector inputs become:

$$x(t+1) = \mathcal{A}x(t) + \mathcal{B}u_1(t-k) + \mathcal{D}\xi(t) \quad (19)$$

$$y(t) = \mathcal{C}x(t) + \mathcal{E}u_1(t-k) \quad (20)$$

$$y_p(t) = \mathcal{C}_p x(t) + (\mathcal{E}_\phi u_1)(t-k) \quad (21)$$

Clearly from equations: (12), (14) to (17) the combined state-dependent system models have the form:

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_0 & 0 \\ B_1\mathcal{C}_0 & A_1 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} \mathcal{B}_0 \\ B_1\mathcal{E}_0 \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} \mathcal{D}_0 & 0 \\ 0 & D_1 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} \mathcal{C}_0 & 0 \end{bmatrix}, \qquad \mathcal{E} = \mathcal{E}_0, \qquad \mathcal{C}_p = \begin{bmatrix} \mathcal{C}_{0\mathrm{p}} & \mathcal{C}_{1p} \end{bmatrix}$$

where $x(t) = \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix}$, $\xi(t) = \begin{bmatrix} \xi_0(t) \\ \xi_1(t) \end{bmatrix}$ (22)

For later use define the resolvent operator for the total state-dependent augmented system as:

$$\Phi = (zI - \mathcal{A}(t))^{-1} = (I - \mathcal{A}(t-1)z^{-1})^{-1} z^{-1} \quad (23)$$

E. Future Plant Outputs and States

Assume for the present that the future values of the control signal are known, so that the future values of the system matrices may be estimated. Then the future values of the states and outputs may be obtained as:

$$x(t+1) = \mathcal{A}(t)x(t) + \mathcal{B}(t)u_1(t-k) + \mathcal{D}(t)\xi(t) \quad (24)$$

The expression for the k_0 *steps-ahead* state-vector, where $k_0 \ge k$, may be obtained by generalizing the above result. These equations may be simplified letting:

$\coprod_{j=m}^{k} a(t+j) = a(t+k)a(t+k-1)\ldots a(t+m)$ and writing $\coprod_{j=0}^{k_0-1}\mathcal{A}(t+j) = \mathcal{A}(t+k_0-1)\mathcal{A}(t+k_0-2)\ldots\mathcal{A}(t)$ and so on. The k_0 *-steps* prediction of the state and output signals will therefore be defined from the relationships:

$$\begin{aligned} \hat{x}(t+k_0 \mid t) &= \coprod_{j=0}^{k_0-1}\mathcal{A}(t+j)\ \hat{x}(t \mid t) \\ &+ \coprod_{j=1}^{k_0-1}\mathcal{A}(t+j)\ \mathcal{B}(t)u_1(t-k) + \ldots \\ &\ldots + \coprod_{j=k_0-1}^{k_0-1}\mathcal{A}(t+j)\mathcal{B}(t+j-1)u_1(t+k_0-k-2) \\ &+ \mathcal{B}(t+k_0-1)u_1(t+k_0-k-1) \end{aligned} \quad (25)$$

$$\hat{y}(t+k_0 \mid t) = \mathcal{C}(t_0)\hat{x}(t_0 \mid t) + \mathcal{E}(t_0)u_1(t_0-k) \quad (26)$$

Simplifying using a *finite pulse response model*,

$$\begin{aligned} \mathcal{T}(k_0, z^{-1}) &= \coprod_{j=1}^{k_0-1}\mathcal{A}(t+j)\ \mathcal{B}(t)z^{-k} + \ldots \\ &+ \coprod_{j=k_0-1}^{k_0-1}\mathcal{A}(t+j)\ \mathcal{B}(t+j-1)z^{k_0-k-2} + \mathcal{B}(t+k_0-1)z^{k_0-k-1} \end{aligned} \quad (27)$$

and introducing the following notation:

$$\mathcal{A}^{k_0} = \coprod_{j=0}^{k_0-1}\mathcal{A}(t+j) = \mathcal{A}(t+k_0-1)\mathcal{A}(t+k_0-2)\ldots\mathcal{A}(t) \quad (28)$$

$$\hat{x}(t+k_0 \mid t) = \mathcal{A}^{k_0}\ \hat{x}(t \mid t) + \mathcal{T}(k_0, z^{-1})u_1(t) \quad (29)$$

$$\begin{aligned} \hat{y}(t+k_0 \mid t) &= \mathcal{C}(t+k_0)\mathcal{A}^{k_0}\ \hat{x}(t \mid t) \\ &+ \left(\mathcal{C}(t+k_0)\mathcal{T}(k_0, z^{-1}) + \mathcal{E}(t+k_0)z^{k_0-k}\right)u_1(t) \end{aligned} \quad (30)$$

where

$$\begin{aligned} \mathcal{T}(k_0, z^{-1}) &= \mathcal{A}(t+k_0-1)\mathcal{A}(t+k_0-2)\ldots.\mathcal{A}(t+1)\mathcal{B}(t)z^{-k} + \ldots \\ &\ldots + \mathcal{A}(t+k_0-1)\mathcal{B}(t+k_0-2)z^{k_0-k-2} + \mathcal{B}(t+k_0-1)z^{k_0-k-1} \end{aligned} \quad (31)$$

The total vector of state-estimates can be written as:

$$\hat{x}(t+j \mid t) = \begin{bmatrix} \hat{x}_0(t+j \mid t) \\ \hat{x}_1(t+j \mid t) \end{bmatrix} \quad \text{for} \quad j \ge 1. \quad (32)$$

III. KALMAN PREDICTOR

The *Kalman filter* equations introduced below are well known [12]. The result below is extended in an obvious way to accommodate the delays on input channels, through terms and bias signals. Recall the total system may be represented in the following state-dependent form that is similar to a known time-varying linear system if the past values of states are assumed known.

$$x(t+1) = \mathcal{A}x(t) + \mathcal{B}u_1(t-k) + \mathcal{D}\xi(t) \quad (33)$$

$$y(t) = \mathcal{C}x(t) + \mathcal{E}u_1(t-k) \quad (34)$$

$$z(t) = y(t) + v(t) = \mathcal{C}x(t) + \mathcal{E}u_1(t-k) + v(t) \quad (35)$$

$$e_0(t) = r(t) - z(t) = r(t) - \left(\mathcal{C}x(t) + \mathcal{E}u_1(t-k) + v(t)\right)$$
$$= \mathcal{C}_e x(t) - \mathcal{E}u_1(t-k) - v(t) \quad (36)$$

where $x(t) \in R^n$ and $\mathcal{C}_e = [-\mathcal{C}_0 \quad C_r \quad 0]$ denotes the output map taken from the total system states to the error channel and the *resolvent operator*: $\Phi = (zI - \mathcal{A}(t))^{-1}$.

A. Predictor Corrector Estimator Form

The standard discrete-time *Kalman filter* equations for a time-varying linear state-space model in predictor corrector form, assuming the exogenous signals have known means [12] are given as:

$$\hat{x}(t+1 \mid t) = A(t)\hat{x}(t \mid t) + B(t)u_1(t-k) + D(t)\overline{\xi}(t) \quad (37)$$

$$\hat{x}(t+1 \mid t+1) = \hat{x}(t+1 \mid t) + K_f(t+1)\left(e_0(t+1) - \hat{e}_0(t+1 \mid t)\right) \quad (38)$$

where $\hat{e}_0(t+1 \mid t) = C_e(t+1)\hat{x}(t+1 \mid t) - E(t+1)u_1(t+1-k) - \overline{v}(t)$ (39)

The *Kalman filter* gain and *Ricatti* equations for a system with process and noise covariance's Q and R:

$$K(t+1) = P(t+1 \mid t)C_e^T(t+1)[C_e(t+1)P(t+1 \mid t)C_e^T(t+1) + R(t+1)]^{-1} \quad (40)$$

A priori covariance:

$$P(t+1 \mid t) = A(t)P(t \mid t)A^T(t) + D(t)Q(t)D^T(t) \quad (41)$$

A posteriori covariance:

$$P(t+1 \mid t+1) = P(t+1 \mid t) - K(t+1 \mid t)C_e(t+1)P(t+1 \mid t) \quad (42)$$

Initial conditions: $\hat{x}(0 \mid 0) = m_0$ and $P(0 \mid 0) = P_0$

Bias terms: $\overline{\xi}(t) = E\{\xi(t)\}$ and $\overline{v}(t) = E\{v(t)\}$

It is possible to apply the above time-varying Kalman filter equations to the combined model for the linear and state-dependent sub-systems in equations (33) and (34). Recall that the plant includes a pure transport delay term and that the previous control actions are known. Assume that the states that affect the state-dependent matrices are not affected by the white noise disturbance input then the system matrices are computable for all times up to time *t*.

IV. NLGMV CONTROL PROBLEM

The cost-minimization problem may now be introduced for the system which is shown in Fig. 2. The optimal *NGMV* control problem involves the minimization of the variance of the signal $\{\phi_0(t)\}$ in Fig. 2. The signal to be minimized in a variance sense,

$$\phi_0(t) = P_c e(t) + (\mathcal{Z}_c x)(t) + (\mathcal{F}_c u)(t) \quad (43)$$

Thence, the cost-index to be minimized:

$$J = E\{\phi_0^T(t)\phi_0(t)\} = E\{trace\{\phi_0(t)\phi_0^T(t)\}\} \quad (44)$$

where $E\{\cdot\}$ denotes the unconditional expectation. The *fictitious signal* $\{\phi_0(t)\}$ that is minimized includes a *dynamic cost-function weighting*: $P_c(z^{-1})$, discussed in (9), that acts on the error signal. This weighting is represented by a linear state-space sub-system, as described in §2, with weighted output: $y_p(t) = P_c e(t)$. That is, if the linear states $x_1(t)$ are augmented with the weighting $P_c(z^{-1})$ dynamics, then the first component of $\phi_0(t)$ can be represented by (21) as: $y_p(t) = \mathcal{C}_p x(t) + (\mathcal{E}_\phi u_1)(t-k)$, through the definition of appropriate output maps in (17). The signal $\{\phi_0(t)\}$ also includes the state weighting term: $y_z(t) = (\mathcal{Z}_c x)(t)$ that can be nonlinear, and it enables a cost-weighting to be introduced on all the states.

If the state-dependent and linear state weightings are denoted $\mathcal{C}_{0z}$ and C_{1z}, then the state weighting:

$$y_z(t) = (\mathcal{Z}_c x)(t) = \mathcal{C}_{0z}x_0(t) + C_{1z}x_1(t) = \mathcal{C}_z x(t) \quad (45)$$

Combined *weighted error* and *state equation model*:

$$y_c(t) = y_p(t) + y_z(t) = P_c e(t) + (\mathcal{Z}_c x)(t) \quad (46)$$

In terms of the total state vector:

$$y_c(t) = P_c e(t) + (\mathcal{Z}_c x)(t) = \mathcal{C}_\phi x(t) + \mathcal{E}_\phi u_1(t-k) \quad (47)$$

where $\mathcal{C}_\phi = [\mathcal{C}_{\phi 0} \quad C_{\phi 1}] = [(\mathcal{C}_{0p} + \mathcal{C}_{0z}) \quad (C_{1p} + C_{1z})]$ (48)

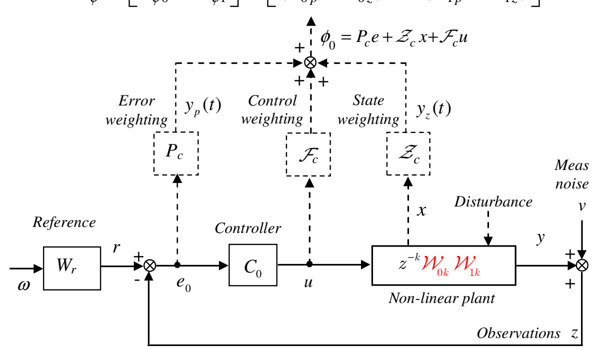

Fig. 2: Single Degree of Freedom Closed-Loop Control System

The final term in the criterion is the nonlinear dynamic control signal costing operator term: $(\mathcal{F}_c u)(t)$. If the smallest delay in each output channel of the plant is of magnitude k_0 steps this implies the control at time *t* affects the output at least k_0 steps later and the control signal costing is defined to have the form:

$$(\mathcal{F}_c u)(t) = z^{-k_0}(\mathcal{F}_{ck} u)(t) \quad (49)$$

where $k_0 = k$ if the state models include through terms, or $k_0 = k+1$ if the through terms are null and one additional explicit step delay is therefore present in the plant model. The importance of this distinction arises because the state models may often be found using NL system identification which often does not introduce a through term.

A. Solution of the NGMV Control Problem

The solution of the optimal control problem is straightforward and follows a *minimum variance* strategy working in the time-domain. It is obtained by introducing a prediction equation and by expanding the resulting expression for the signal that enters the cost-function. This signal may be referred to as a *minimized output* $\{\phi_0(t)\}$, since it is not a signal that exists physically. Recall signal,

$$\phi_0(t) = P_c e(t) + (\mathcal{Z}_c x)(t) + (\mathcal{F}_c u)(t) \tag{50}$$

From (47) the *minimized output*, may be written as:

$$\phi_0(t) = \mathcal{C}_\phi x(t) + \mathcal{E}_\phi u_1(t-k) + (\mathcal{F}_c u)(t) \tag{51}$$

The signal: $u_1(t)=\mathcal{W}_{1k}u(t)$ and the control weighting was defined: $(\mathcal{F}u)(t) = z^{-k_0}(\mathcal{F}_{ck}u)(t)$, so (51) becomes:

$$\phi_0(t) = \mathcal{C}_\phi x(t) + ((\mathcal{E}_\phi \mathcal{W}_{1k} + z^{-k_0+k}\mathcal{F}_{ck})u)(t-k) \tag{52}$$

B. The Prediction Equations

The prediction equation was obtained using equation (29) with the finite impulse response term, as:

$$\hat{x}(t+k_0 \mid t) = \mathcal{A}^{k_0}\hat{x}(t \mid t) + \mathcal{T}(k_0, z^{-1})u_1(t) \tag{53}$$

where from (22) and (28): $\mathcal{A} = \begin{bmatrix} \mathcal{A}_0 & 0 \\ \mathcal{A}_{21} & A_1 \end{bmatrix}$ and

$$\mathcal{A}^{k_0} = \coprod_{j=0}^{k_0-1}\mathcal{A}(t+j) = \mathcal{A}(t+k_0-1)\mathcal{A}(t+k_0-2)\ldots\mathcal{A}(t) \tag{54}$$

The predicted values of the state related terms in equation (52) may therefore be written as:

$$\mathcal{C}_\phi(t+k_0)\hat{x}(t+k_0 \mid t) = \mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t)$$
$$+\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0, z^{-1})u_1(t) \tag{55}$$

where $\mathcal{T}(k_0, z^{-1})$ was defined by (27). The k steps-ahead prediction of the signals: $y_c(t)$ and $\phi_0(t)$ follow from (46), (52) and (55), as follows:

$$\hat{y}_c(t+k_0 \mid t) = \mathcal{C}_\phi(t+k_0)\hat{x}(t+k_0 \mid t) + (\mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}u)(t+k_0-k)$$
$$= \mathcal{C}_\phi(t+k_0)\hat{x}(t+k_0 \mid t) + \mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}z^{k_0-k}u(t)$$

Note that when the through term in the plant model is null there is one more delay and $k_0 = k+1$, and from (47) in this case $\mathcal{E}_\phi(t+k_0)$ will also be null. Thus, the term $\mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}z^{k_0-k}$ can be replaced: $\mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}$ giving:

$$\hat{y}_c(t+k_0 \mid t) = \mathcal{C}_\phi(t+k_0)\hat{x}(t+k_0 \mid t) + \mathcal{E}_\phi(t+k_0)\mathcal{W}_{1k}u(t)$$
$$= \mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t)$$
$$+((\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0, z^{-1}) + \mathcal{E}_\phi(t+k_0))\mathcal{W}_{1k})u(t) \tag{56}$$

$$\hat{\phi}_0(t+k_0 \mid t) = \hat{y}_c(t+k_0 \mid t) + (\mathcal{F}_{ck}u)(t)$$
$$= \mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t)$$
$$+((\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0, z^{-1}) + \mathcal{E}_\phi(t+k_0))\mathcal{W}_{1k} + \mathcal{F}_{ck})u(t) \tag{57}$$

C. Solution of the NGMV Control Problem

The cost-function involves the minimization of the variance: $J = E\{\phi_0(t+k_0)^T\phi_0(t+k_0)\}$ that may be written in terms of the prediction $\hat{\phi}_0(t+k_0 \mid t)$ and the prediction error: $\tilde{\phi}_0(t+k_0 \mid t)$, noting these signals are orthogonal:

$$J = E\{\hat{\phi}_0(t+k_0 \mid t)^T\hat{\phi}_0(t+k_0 \mid t)\}$$
$$+E\{\tilde{\phi}_0(t+k_0 \mid t)^T\tilde{\phi}_0(t+k_0 \mid t)\} \tag{58}$$

The prediction error $\tilde{\phi}_0(t+k_0 \mid t)$ does not depend upon control action and hence the cost is clearly minimized by setting the predicted values of the signal: $\phi_0(t)$, for k_0 steps-ahead, to zero. Setting the predicted values of (57) to zero provides 2 possible expressions for the control.

Theorem 4.1: NGMV Control Law

Let the operator $\mathcal{N}_0$ represent the mapping from the signal $u_1(t)$ to the signal $\phi_0(t)$ to be minimized

$$(\mathcal{N}_0 u_1)(t) = ((P_c\mathcal{W}_{0k}(t+k_0) - \mathcal{Z}_c\tilde{\mathcal{W}}_{0k}(t+k_0))u_1)(t) \tag{59}$$

Assume that the weighting operators P_c, $\mathcal{Z}_c$ and $\mathcal{F}_c$ are chosen so that the *NL* operator: $(\mathcal{N}_0\mathcal{W}_{1k} - \mathcal{F}_{ck})$ has a finite-gain m_2 stable causal inverse, to ensure the system is closed-loop stable. The *NGMV* optimal controller to minimize the variance of the weighted error, states and control signals may then be computed as follows.

$$u(t) = \left(\mathcal{F}_{ck} + (\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0, z^{-1}) + \mathcal{E}_\phi(t+k_0))\mathcal{W}_{1k}\right)^{-1}$$
$$\times(-\mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t)) \tag{60}$$

or equivalently (as shown in Fig. 3):

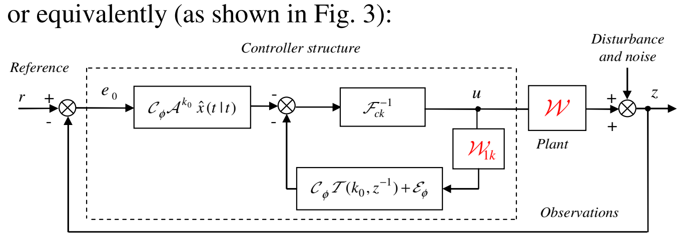

Fig. 3: Control Signal Generation and Control Modules

$$u(t) = -\mathcal{F}_{ck}^{-1}(\mathcal{C}_\phi(t+k_0)\mathcal{A}^{k_0}\hat{x}(t \mid t)$$
$$+(\mathcal{C}_\phi(t+k_0)\mathcal{T}(k_0, z^{-1}) + \mathcal{E}_\phi(t+k_0))(\mathcal{W}_{1k}u)(t)) \tag{61}$$

where $\mathcal{C}_{0p} = -E_p\mathcal{C}_0$, $C_{1p} = \begin{bmatrix} E_pC_r & C_p \end{bmatrix}$, $\mathcal{E}_\phi = -E_p\mathcal{E}_0$ and $\mathcal{C}_\phi = \begin{bmatrix} (\mathcal{C}_{0p} + \mathcal{C}_{0z}) & (C_{1p} + C_{1z}) \end{bmatrix}$. ●

Proof: The proof of optimality involves collecting the above results, subject to the assumptions. The proof that the controller obtained is stabilizing follows.

V. IMPLEMENTATION ISSUES AND STABILITY

The controller involves a time-varying Kalman filter with predictor stage and this may be illustrated in the more physically intuitive structure of the controller shown in Fig. 4. To justify this structure recall (61). For linear systems it is well known that stability is ensured when the combination of a control weighting and an error weighted plant model is strictly minimum-phase [7]. For *NL* systems a related operator equation must have a stable inverse. First write:

$$\hat{x}(t\,|\,t) = T_{f1}(z^{-1})e_0(t) + T_{f2}(z^{-1})u_1(t-k) \qquad (62)$$

To identify these operators and for the stability analysis neglect the input bias and all stochastic inputs other than the reference. Note from (38) and (39) for the system of interest:

$$\hat{x}(t\,|\,t) = (I - K_f(t)\mathcal{C}_e(t))\hat{x}(t\,|\,t-1) + K_f(t)\big(e_0(t) + \mathcal{E}(t)u_1(t-k)\big)$$

but from (37),

$$\hat{x}(t\,|\,t-1) = \mathcal{A}(t-1)\hat{x}(t-1\,|\,t-1) + \mathcal{B}(t-1)u_1(t-k-1)$$

Thence,

$$\hat{x}(t\,|\,t) = \big(I - (I - K_f(t)\mathcal{C}_e(t))z^{-1}\mathcal{A}(t)\big)^{-1}\big[K_f(t)e_0(t)$$
$$+\big((I - K_f(t)\mathcal{C}_e(t))\mathcal{B}(t-1)z^{-1} + K_f(t)\mathcal{E}(t)\big)u_1(t-k)\big] \qquad (63)$$

Follows the above linear operators in (62) have the form:

$$T_{f1}(z^{-1}) = \big(I - (I - K_f(t)\mathcal{C}_e(t))z^{-1}\mathcal{A}(t)\big)^{-1}K_f(t) \qquad (64)$$

$$T_{f2}(z^{-1}) = \big(I - (I - K_f(t)\mathcal{C}_e(t))z^{-1}\mathcal{A}(t)\big)^{-1}$$
$$\big((I - K_f(t)\mathcal{C}_e(t))\mathcal{B}(t-1)z^{-1} + K_f(t)\mathcal{E}(t)\big) \qquad (65)$$

Following expression is required based on these results,
$T_{f2}(z^{-1}) - T_{f1}(z^{-1})\mathcal{W}_{0k} = \Phi\mathcal{B}(t)$ (66)

A. Minimized Output

For the stability analysis an expression is required for the control action in terms of the closed-loop operators with input reference signal. Recall from equation (57):

$$\hat{\phi}_0(t+k_0\,|\,t) = \mathcal{C}_\phi\mathcal{A}^{k_0}\hat{x}(t\,|\,t)$$
$$+((\mathcal{C}_\phi\mathcal{T}(k_0,z^{-1}) + \mathcal{E}_\phi)\mathcal{W}_{1k}' + \mathcal{F}_{ck})u(t)$$

Substituting from (62) and letting:

$$T_{f2}(z^{-1}) = \Phi\mathcal{B}(t) + T_{f1}(z^{-1})\mathcal{W}_{0k},$$

$$\hat{\phi}_0(t+k_0\,|\,t) = \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}e_0(t) + ((\mathcal{C}_\phi\mathcal{T}(k_0,z^{-1})$$
$$+\mathcal{E}_\phi + \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f2}z^{-k})\mathcal{W}_{1k}' + \mathcal{F}_{ck})u(t) \qquad (67)$$
$$= \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}r(t) + \big(\mathcal{C}_\phi(\mathcal{T}(k_0,z^{-1})$$
$$+\mathcal{A}^{k_0}\Phi\mathcal{B}(t)z^{-k})\mathcal{W}_{1k}' + \mathcal{E}_\phi\mathcal{W}_{1k}' + \mathcal{F}_{ck})u(t) \qquad (68)$$

A second operator relationship required obtained similarly:

$$\Phi(t+k_0)\mathcal{B}(t+k_0)z^{k_0-k} = \mathcal{T}(k_0,z^{-1}) + \mathcal{A}^{k_0}\Phi(t)\mathcal{B}(t)z^{-k} \qquad (69)$$

The *minimized output* (68) may therefore be written, writing $t_0 = t + k_0$ as:

$$\hat{\phi}_0(t+k_0\,|\,t) = \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}r(t) + \big[(\mathcal{C}_\phi(t_0)\Phi(t_0)\mathcal{B}(t_0)z^{k_0-k}$$
$$+\mathcal{E}_\phi(t_0))\mathcal{W}_{1k}' + \mathcal{F}_{ck}\big]u(t) \qquad (70)$$

and

$$\big(\mathcal{Z}_c x\big)(t+k_0) - P_c\,y(t+k_0)$$
$$= \big(\mathcal{C}_\phi(t+k_0)\Phi(t+k_0)\mathcal{B}(t+k_0)z^{k_0-k} + \mathcal{E}_\phi(t+k_0)\big)\mathcal{W}_{1k}'u(t)$$
$$= \big(\mathcal{Z}_c\tilde{\mathcal{W}}_{0k}(t+k_0) - P_c\mathcal{W}_{0k}(t+k_0)\big)\mathcal{W}_{1k}'u(t)$$

where $\tilde{\mathcal{W}}_{0k}$ is operator between $u_1(t-k)$ and states and

$$\mathcal{C}_\phi(t_0)\Phi(t_0)\mathcal{B}(t_0)z^{k_0-k} + \mathcal{E}_\phi(t_0) = \mathcal{Z}_c\tilde{\mathcal{W}}_{0k}(t_0) - P_c\mathcal{W}_{0k}(t_0) \qquad (71)$$

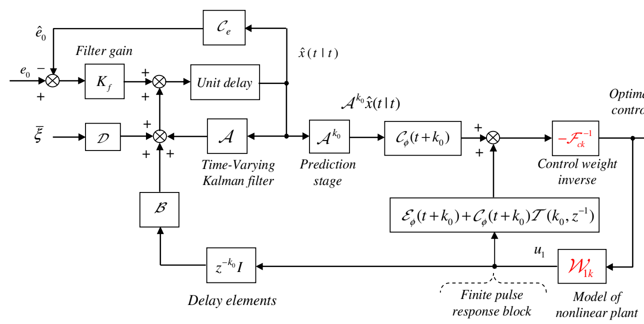

Fig. 4: NGMV Optimal Controller in Kalman Form

Define: $\mathcal{N}_0 = (P_c\mathcal{W}_{0k}(t+k_0) - \mathcal{Z}_c\tilde{\mathcal{W}}_{0k}(t+k_0))$ (72)

which represents the transfer between the signal u_1 and the output to be minimized: $\{\phi_0(t)\}$. Then, (70) becomes:

$$\hat{\phi}_0(t+k_0\,|\,t) = \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}(z^{-1})r(t) + \big((\mathcal{Z}_c\tilde{\mathcal{W}}_{0k}(t_0) - P_c\mathcal{W}_{0k}(t_0))\mathcal{W}_{1k}' + \mathcal{F}_{ck}\big)u(t)$$
$$= \mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}(z^{-1})r(t) + (-\mathcal{N}_2\mathcal{W}_{1k}' + \mathcal{F}_{ck})u(t) \qquad (73)$$

B. Stability Analysis

To simplify the stability analysis recall that the external inputs, except the reference signal *r*(t), were assumed null in the previous section. Using (73) the condition for optimality $\hat{\phi}_0(t+k\,|\,t) = 0$ leads to the optimal control:

$$u(t) = \mathcal{F}_{ck}^{-1}\big(-\mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}(z^{-1})r(t) + (\mathcal{N}_2\mathcal{W}_{1k}'u)(t)\big) \qquad (74)$$

Rearranging, the desired expressions become:

$$u(t) = \big(\mathcal{N}_2\mathcal{W}_{1k}' - \mathcal{F}_{ck}\big)^{-1}\mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}(z^{-1})r(t) \qquad (75)$$

$$(\mathcal{W}u)(t) = \mathcal{W}\big(\mathcal{N}_2\mathcal{W}_{1k}' - \mathcal{F}_{ck}\big)^{-1}\mathcal{C}_\phi\mathcal{A}^{k_0}T_{f1}(z^{-1})r(t) \qquad (76)$$

where the delay-free plant model: $\mathcal{W}_k' = \mathcal{W}_{0k}\mathcal{W}_{1k}$ and $\tilde{\mathcal{W}}_{0k}$ denotes the transfer-operator between $u_1(t-k)$ and the states of the *state-dependent* sub-system. To show that the system is stable, recall that the series connection of two finite gain m_2 stable systems is m_2 stable. The assumption stated in the theorem was that the cost-weightings are chosen, so that the operator: $(\mathcal{N}_2\mathcal{W}_{1k}' - \mathcal{F}_{ck})^{-1}$ is *finite gain stable*. The state weighting term will often be omitted from the cost. In this case the requirement is for the operator: $\big((P_c\mathcal{W}_k(t+k_0) - \mathcal{F}_{ck}\big)^{-1}$ to be *finite gain stable*.

Also observe that the internal feedback loop in the controller in Fig. 3 does not contain any subsystem that is unstable. It only contains $\mathcal{W}'_{1k}$ which is assumed finite gain stable and the consequence is that the inverse dynamics will not attempt to cancel any unstable parts of the plant model.

VI. EXAMPLE

In this section, we illustrate the *NGMV* control theory presented in this paper with simulations of a simple chemical process. The process is an irreversible exothermic first order reaction, which takes place in a continuous stirred tank reactor (*CSTR*), as shown in Fig. 5.

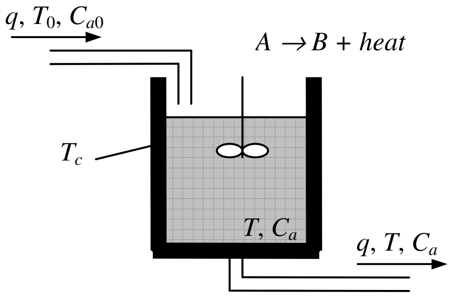

Fig. 5: CSTR Process (input variable is cooling jacket temperature T_c and output is the concentration C_a)

It is assumed that the liquid in the reactor is perfectly mixed and the feed flow is equal to the product outflow. The cooling jacket temperature T_c is regarded as an input to the process and the product concentration C_a is regarded as the output. Similar types of processes have been extensively studied in the literature as they present very interesting and challenging control problems. The *CSTR* processes exhibit some rich non-linear behavior, involving multiple steady-state solutions and both stable and unstable equilibrium points, and show clearly non-linear dynamic responses. Consider a normalized dimensionless model given in [13], where the values of the parameters are listed in Table 1:

Table 1: CSTR model parameters

Parameter	Meaning	Value
D_a	Damköhler number	0.072
φ	dimensionless activation energy	20.0
B	heat of reaction coefficient	8.0
β	heat transfer coefficient	0.3

$$\dot{x}_1 = -x_1 + D_a(1-x_1)\exp\left(x_2/(1+x_2/\varphi)\right)$$
$$\dot{x}_2 = -x_2 + BD_a(1-x_1)\exp\left(x_2/(1+x_2/\varphi)\right) + \beta(u-x_2)$$
$$y = x_1.$$

By defining a new scaled input: $u_s = (u+4)/8$, both the input and output signals are contained in the range (0,1). A polynomial *ARMA* model was identified from simulation data in [13], followed by a sigmoid function. Such a structure with output nonlinearity is not suitable for the state-dependent *NGMV* theory, and thus we identified a new nonlinear *ARMAX* model, using the tools provided with the *System Identification Toolbox* for *Matlab*. The dimensionless sample time of 0.5 units was used to generate the data for identification. A particular feature of the *CSTR* model is the presence of two stable regions at the two ends of the output range, separated by an unstable region, from which the system is repelled. The effort was made to include the whole operating range in the estimation data. The following model was found to provide a good balance between accuracy and complexity:

$$\begin{aligned} y(t) &= \theta_0 + \theta_1 y(t-1) + \theta_2 y(t-2) + \theta_3 u_s(t-1) + \theta_4 u_s(t-2) \\ &+ \theta_5 y^2(t-1) + \theta_6 y(t-1)y(t-2) + \theta_7 y(t-1)u_s(t-1) \\ &+ \theta_8 y(t-1)u_s(t-2) + \theta_9 y^3(t-1) + \theta_{10} y^2(t-1)y(t-2) \end{aligned}$$

with the estimated parameter values:

$\theta_0 = 0.0129$, $\theta_1 = -0.0390$, $\theta_2 = 0.5112$, $\theta_3 = 0.0219$, $\theta_4 = 0.0290$, $\theta_5 = 7.8027$, $\theta_6 = -6.5853$, $\theta_7 = 0.1232$, $\theta_8 = 0.0877$, $\theta_9 = -7.9623$, $\theta_{10} = 7.0073$

The model verification is shown in Fig. 6 and a good match is confirmed between the plant and model responses for both the estimation and validation data. For the purpose of the *NGMV* control, the above polynomial *NARMAX* model can be converted to a state-dependent representation by defining the following states: $x_1(t) = y(t)$, $x_2(t) = y(t-1)$, $x_3(t) = u_s(t-1)$.

The state-dependent model follows then as:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} x_1^2+x_1+1 & x_1^2+x_1+1 & x_1+1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} x_1+1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_s(t)$$

$$y(t) = x_1(t)$$

A. Control design and simulation results

The major challenge is to control the system around an unstable equilibrium point, i.e. for middle concentration values. As the nominal unstable operating point we choose the point corresponding to the control input of $u_s(t) = 0.5$. Based on the model equation, the corresponding output steady-state values: $y_{01} = 0.1437$, $y_{02} = 0.3652$ and $y_{03} = 0.7658$. Out of these 3 solutions, y_{02} is the unstable equilibrium and the control objective will consist of regulating the system around this value. The *NGMV* controller was designed for this problem using the following design parameters:

Reference: $W_r(z^{-1}) = 0.05/(1-0.95z^{-1})$

Error: $P_c(z^{-1}) = (1-0.1z^{-1})^2/(1-0.97z^{-1})^2$

Control: $F_{ck}(z^{-1}) = 0.55\cdot(1-0.3z^{-1})^2/(1-0.9z^{-1})^2$

The goal of the simulation was to move the process from the stable equilibrium $y_{01} = 0.1437$ to the unstable one: $y_{02} = 0.3652$, and keep it there under a step output disturbance. The results of the simulation are shown in Fig. 7 and despite a rather oscillatory controller response to the disturbance these objectives are in fact achieved.

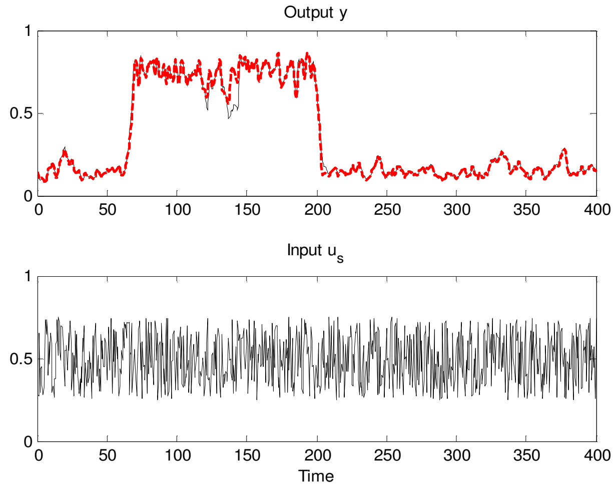

Fig. 6: NARX model validation: plant (solid), model (dashed)
(The estimation data used for identification: up to time 200)

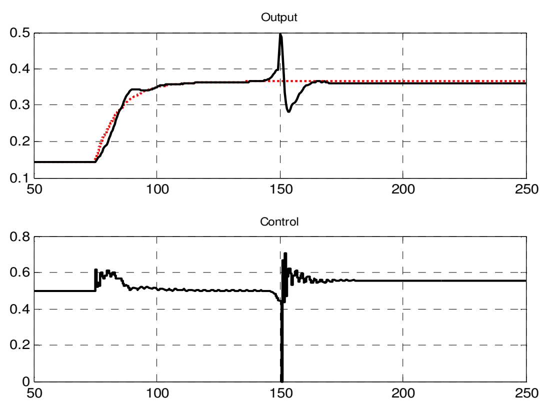

Fig. 7: NGMV output tracking and regulation around an unstable operating point. Upper plot: output (solid), set-point (dotted)

The final test consisted of tracking the concentration set-point across the operating range. Since the dynamics and nonlinearity of the system vary wildly for different operating conditions, the approach used was to define a separate set of the control weighting parameters for each operating regime (four of them were specified) and use a simple switching scheme when moving from one to another.

In particular, the control weighting was parameterized as: $F_{ck}(z^{-1}) = \rho \cdot \left((1-\beta z^{-1})^2 / (1-\alpha z^{-1})^2\right)$ and the parameters ρ, β and α were tuned separately for each region corresponding to a single step change (the switching variable was the output concentration). The values were then collected in the look-up tables and switched by what can be considered as a gain-scheduled type of control law. The result of one such simulation is shown in Fig. 8 and again good tracking is achieved across the operating range.

VII CONCLUDING REMARKS

A relatively simple controller for nonlinear multivariable systems was introduced that extends the family of *NGMV* controllers to much more general systems. The inclusion of the state-dependent sub-system model provides the main innovation and this has the advantage that it may be used to represent open-loop unstable plants with input or output nonlinearities.

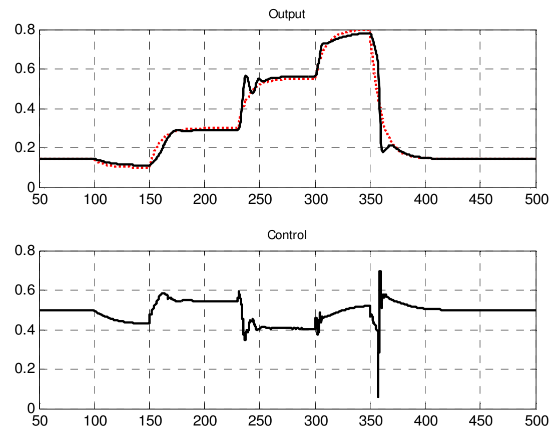

Fig. 8: *NGMV* control: output tracking across the operating range with a gain-scheduled control weighting

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