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Control Formula for Nonlinear Systems Subject to Convex Input Constraints using Control Lyapunov Functions

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Abstract— **In this paper, we propose a two-step controller design method with control Lyapunov functions (CLFs) for nonlinear systems with convex input constraints. In the first step, we derive an input which minimizes the time derivative of a local CLF via nonlinear convex optimization. According to the Karush-Kuhn-Tucker condition (KKT-condition), we clarify the necessary and sufficient condition for the minimizing input. Then, we discuss the continuity of the minimizing input. We also consider the relation between the minimizing input and the asymptotically stabilizable domain. In the second step, we design a continuous asymptotically stabilizing controller based on the derived minimizing input for the system. Finally, we confirm the effectiveness of the proposed method through an example.**

I. INTRODUCTION

In recent years, control Lyapunov functions (CLFs) and CLF-based controller designs have attracted much attention in the nonlinear control theory [1], [2], [3], [4], [5], [6], [7], [8]. Particularly, CLF-based controller design for input constraints is considered to be an important problem and many stabilizing controllers have been proposed [3], [4], [5], [6], [7], [8]. First, Lin and Sontag proposed a control formula for nonlinear systems such that the 2-norm of the input was less than one [3]. Then, Malisoff and Sontag proposed a control formula for nonlinear systems such that k-norm ($k \leq 2$) of the input was less than one [4], and Kidane et al. proposed a continuous controller that stabilizes an origin in an asymptotically stabilizing domain, for nonlinear systems such that the k-norm ($k \geq 1$) of the input was less than one [5], [6]. The inverse optimal controller that guarantees the robustness and optimality is also provided by Nakamura et al. [7]. Moreover, Nakamura et al. considered a disturbance attenuation problem for nonlinear systems such that the norm of the input and the norm of the disturbance are less than one [8].

These CLF-based controller design methods for input constrained nonlinear systems consist of the following two steps:

(1) Derive an input that minimizes the time derivative of a local CLF under input constraints;

(2) Construct a continuous stabilizer based on the input obtained in step (1).

However, general input constraints are not considered in CLF-based controller designs in the previous works. In actual control problems, systems often violate norm input constraint [9]. Although applying a norm constraint as a sufficient condition is a practical method, it is too conservative. Then, it is important to consider a more general class of an input constraint and design a CLF based controller for nonlinear systems with such a general input constraint.

In this paper, we consider convex input constraints as a class of general input constraint, and propose a new controller design scheme for convex input constrained nonlinear systems based on the above two steps.

In step (1), we derive the minimizing input by nonlinear convex optimization. More precisely, we clarify the necessary and sufficient condition for the minimizing input under appropriate assumptions. Then, we discuss the continuity of the minimizing input. We also consider the relation between the minimizing input and the asymptotically stabilizable domain.

In step (2), we design a continuous asymptotically stabilizing controller for nonlinear systems with convex input constraints based on the minimizing input obtained in step (1). We design the continuous controller by using the result of [5].

Finally, we confirm the effectiveness of the proposed method through an example.

II. PRELIMINARIES

In this section, we introduce mathematical notations and some definitions. We consider the following input affine nonlinear system in this paper:

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where, $x \in \mathbb{R}^n$ is the state vector, $u \in U(x) \subseteq \mathbb{R}^m$ is an input vector, and $U(x)$ is a state-dependent input constraint. We assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous mappings, and $f(0) = 0$. In this paper, we consider a general convex input constraint $U(x)$ satisfying the following assumptions:

(A1) $U(x) = \{u \in \mathbb{R}^m | G_i(x, u) < 0 \ (i = 1, \ldots, l)\}$, where, each $G_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is differentiable on $\mathbb{R}^n \times \mathbb{R}^m$ and a convex function with respect to u for any $\bar{x} \in \mathbb{R}^n$;

(A2) $U(x)$ is uniformly bounded (see Definition 3 in Appendix);

(A3) $0 \in U(x), \ \forall x \in \mathbb{R}^n$.

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Note that (A1) and (A2) are slightly severe assumptions, but (A3) is satisfied under ordinary conditions.

We suppose that a local control Lyapunov function (CLF) defined as the following is given for system (1).

Definition 1 (Control Lyapunov Function (CLF))**.** *A smooth proper positive-definite function $V : X \to \mathbb{R}_{\geq 0}$, defined on a neighborhood of the origin $X \subset \mathbb{R}^n$ is said to be a local control Lyapunov function for system (1) if the condition*

$$\inf_{u \in U} \{L_f V + L_g V \cdot u\} < 0 \tag{2}$$

is satisfied for all $x \in X \backslash \{0\}$. Moreover, $V(x)$ is said to be a control Lyapunov function (CLF) for system (1) if $V(x)$ is a function defined on entire $\mathbb{R}^n$ and condition (2) is satisfied for all $x \in \mathbb{R}^n \backslash \{0\}$.

Note that $L_f V$ and $L_g V$ denote $(\partial V/\partial x) \cdot f(x)$ and $(\partial V/\partial x) \cdot g(x)$, respectively. If $V : X \to \mathbb{R}_{\geq 0}$ is a local CLF,

$$L_g V = 0 \implies L_f V < 0, \ ^\forall x \in X \backslash \{0\} \tag{3}$$

is satisfied.

The small control property defined as the following plays an important role in this paper.

Definition 2 (Small Control Property (SCP))**.** *A control Lyapunov function is said to satisfy the small control property if for any $\epsilon > 0$, there is $\delta > 0$ such that*

$$0 \neq \|x\| < \delta \Rightarrow \exists \|u\| < \epsilon \text{ s.t. } L_f V + L_g V \cdot u < 0. \tag{4}$$

III. INPUT THAT MINIMIZES THE TIME DERIVATIVE OF A LOCAL CLF

In this section, we consider the closure of input constraint $\bar{U}(x)$ instead of $U(x)$. We derive an input u that minimizes $\dot{V}(x, u)$ under the input constraint $u \in \bar{U}(x)$, where $V(x)$ is a local CLF for system (1). First, we clarify the necessary and sufficient condition of the minimizing input brought by nonlinear convex optimization. Then, we discuss the continuity of the minimizing input, and also consider the relation between the minimizing input and the domain in which the origin is asymptotically stabilizable.

A. Existence of the minimizing input

In this subsection, we show that there exists an input that minimizes the derivative of the local CLF. The following lemma claims the existence of the minimizing input.

Lemma 1. *Consider system (1) with input constraint $u \in \bar{U}(x)$. Let $V(x)$ be a local CLF for system (1) and conditions (A1)-(A3) are assumed to be satisfied. Then, if there exists $\bar{u}$ the solution of the following problem (P1), $\bar{u}$ minimizes $\dot{V}(\bar{x}, u)$ for a fixed $\bar{x} \in \mathbb{R}^n$.*

(P1) Minimize $L_g V(\bar{x}) \cdot u$ subject to $u \in \bar{U}(\bar{x})$.

Proof: The time derivative of $V(x)$ can be represented by $\dot{V}(x, u) = L_f V(x) + L_g V(x) \cdot u$. For a fixed $\bar{x} \in \mathbb{R}^n$, $L_f V(\bar{x})$ and $L_g V(\bar{x})$ are considered to be a constant and a constant vector, respectively. Then, the input that minimizes $\dot{V}(\bar{x}, u)$ is identified on the input that minimizes $L_g V(\bar{x}) \cdot u$. ■

To solve (P1), we introduce the following Lagrangian $L_0(x, u, \lambda)$:

$$L_0(x, u, \lambda) = L_g V(x) \cdot u + \sum_{i=1}^{l} \lambda_i G_i(x, u), \tag{5}$$

where $\lambda = (\lambda_1, \ldots, \lambda_l)^{\mathrm{T}} \in \mathbb{R}^l$ is a vector of Lagrange multipliers.

By using $L_0(x, u, \lambda)$, we can clarify the existence of the solution of (P1) as the following.

Theorem 1. *Consider system (1) with input constraint $u \in \bar{U}(x)$. Let $V(x)$ be a local CLF for system (1) and conditions (A1)-(A3) are assumed to be satisfied. Then for each fixed $\bar{x} \in \mathbb{R}^n$, there exist $\bar{u} \in \bar{U}(x)$ and $\bar{\lambda} \in \mathbb{R}^l$ that satisfy the following conditions:*

$$\begin{aligned} &\frac{\partial L_0(\bar{x}, \bar{u}, \bar{\lambda})}{\partial u} = L_g V(x) + \sum_{i=1}^{l} \bar{\lambda}_i \frac{\partial G_i(\bar{x}, \bar{u})}{\partial u} = 0, \\ &\bar{\lambda}_i \geq 0, \ G_i(\bar{x}, \bar{u}) \leq 0, \ \bar{\lambda}_i G_i(\bar{x}, \bar{u}) = 0 \ (i = 1, \ldots, l). \end{aligned} \tag{6}$$

Moreover, $\bar{u}$ is a solution of (P1) for each $\bar{x} \in \mathbb{R}^n$.

Proof: By assumption (A1), each $G_i(\bar{x}, u)$ is a convex function of u. Additionally, $L_g V(\bar{x}) \cdot u$ is a continuous convex function with respect to u for a fixed $\bar{x}$. Then, the theorem is obtained straightforwardly by Lemmas 11 and 12 in the Appendix. ■

B. Continuity of the minimizing input $\bar{u}$

According to Theorem 1, there exists a minimizing input $\bar{u}$ for each $\bar{x}$. In this section, we discuss the continuity of $\bar{u}$ at $\bar{x}$.

To discuss the continuity of $\bar{u}$, we introduce an optimal value function $\phi : \mathbb{R}^n \to \mathbb{R}$ and anoptimal set mapping $\Phi : \mathbb{R}^n \to \bar{U}$ defined as follows:

$$\phi(x) = \min_{u \in \bar{U}(x)} \{L_g V(x) \cdot u\}, \tag{7}$$

$$\Phi(x) = \{u \in \bar{U}(x) | \phi(x) = L_g V(x) \cdot u\}. \tag{8}$$

Note that if $\Phi(x)$ is continuous with respect to x, $\bar{u}$ is also continuous with respect to x. To guarantee the continuity of $\Phi(x)$, we assume that the following condition (A4) is satisfied:

(A4) Solution $\bar{u}$ satisfying (6) is uniquely determined for any $\bar{x} \in \{x \in \mathbb{R}^n | L_g V(\bar{x}) \neq 0\}$.

In other words, (P1) has a unique solution $\bar{u}$ for each fixed $\bar{x}$. In this case, $\Phi(\bar{x}) = \{\bar{u}\}$. For simplicity, we express $\bar{u} = \Phi(\bar{x})$.

Remark 1. *There is no unique solution of (P1) for each $\bar{x} \in \{x \in \mathbb{R}^n | L_g V(\bar{x}) = 0\}$ because all $u \in \bar{U}(\bar{x})$ satisfy (6) by choosing $\bar{\lambda} = 0$.*

If conditions (A1)-(A4) are satisfied, the following lemma for the continuity of $\Phi(x)$ is obtained.

Lemma 2. *Consider system (1) with input constraint $u \in \bar{U}(x)$. Let $V(x)$ be a local CLF for system (1) and conditions (A1)-(A4) are assumed to be satisfied. Then, the optimal set mapping $\Phi(x)$ is continuous for all $x \in \{x \in \mathbb{R}^n | L_gV(\bar{x}) \neq 0\}$.*

Proof: The result follows from Lemma 13 in the Appendix straightforwardly. ∎

Summarizing Theorem 1 and Lemma 2, we can obtain the following theorem:

Theorem 2. *Consider system (1) with input constraint $u \in \bar{U}(x)$. Let $V(x)$ be a local CLF for system (1) and conditions (A1)-(A4) are assumed to be satisfied. Then, the following input $\bar{u} : \mathbb{R}^n \to \bar{U}(x)$ minimizes $\dot{V}(x, u)$ for all $x \in \mathbb{R}^n$ under input constraint $u \in \bar{U}(x)$.*

$$\bar{u}(x) = \begin{cases} \Phi(x) & (L_gV(x) \neq 0) \\ 0 & (L_gV(x) = 0) \end{cases}. \tag{9}$$

Moreover, $\bar{u}(x)$ is continuous for all $x \in \{x \in \mathbb{R}^n | L_gV(x) \neq 0\}$.

C. Asymptotically stabilizable domain

By Theorem 2, we can design input $\bar{u}(x)$, which minimizes $\dot{V}(x, u)$. Here, we discuss the relation between $\bar{u}(x)$ and the asymptotically stabilizable domain. First, we guarantee the asymptotically stabilizable domain as the following.

Lemma 3. *Consider system (1) with input constraint $u \in U(x)$. We assume conditions (A1)-(A4) are satisfied. Let $V(x)$ be a local CLF for system (1) and $a_1 > 0$ be the maximum value satisfying the following condition:*

$$\begin{aligned} &\inf_{u \in U(x)} \{L_fV(x) + L_gV(x) \cdot u\} < 0, \\ &\forall x \in W_1 \backslash \{0\} := \{x | V(x) < a_1\} \backslash \{0\}. \end{aligned} \tag{10}$$

Then, the origin is asymptotically stabilizable in W_1. Moreover, if $a_1 = \infty$, the origin is globally asymptotically stabilizable.

We prove Lemma 3 in section IV by constructing a controller that stabilizes the origin in W_1 and it is continuous on $W_1 \backslash \{0\}$.

In Lemma 3, domain W_1 is defined under input constraint $u \in U(x)$. However, in the previous subsection, we considered input constraint $u \in \bar{U}(x)$ instead of $u \in U(x)$ and derived a minimizing input $\bar{u}(x)$. There exists a natural question of whether $\bar{u}(x)$ satisfies the actual input constraint $u \in U(x)$ in the case of $L_gV \neq 0$. Actually, we obtain $\bar{u}(x) \notin U(x)$ due to existence of $\lambda_i \neq 0$ (i.e. $G_i(x, u) = 0$) in (6). Then, we cannot use $\bar{u}(x)$ as a control input, but we can characterize W_1 as $\bar{u}(x)$ by using the following lemma.

Lemma 4. *Consider system (1) with input constraints $u \in U(x)$ and $u \in \bar{U}(x)$. We assume conditions (A1)-(A4) are satisfied. Let $V(x)$ be a local CLF for system (1), W_1 be a domain defined in Lemma 3, and $a_2 > 0$ be the maximum value satisfying the following condition:*

$$\begin{aligned} &\min_{u \in \bar{U}(x)} \{L_fV(x) + L_gV(x) \cdot u\} < 0, \\ &\forall x \in W_2 \backslash \{0\} := \{x | V(x) < a_2\} \backslash \{0\}. \end{aligned} \tag{11}$$

Then, $W_1 = W_2$.

To prove Lemma 4, we introduce the following Lemma 5:

Lemma 5. *Consider system (1) with input constraints $u \in U(x)$ and $u \in \bar{U}(x)$. We assume conditions (A1)-(A4) are satisfied. Let $V(x)$ be a local CLF for system (1), $\bar{u}(x)$ be an input defined in Theorem 2, and $\mu \in [0, 1)$ be a constant. Then,*

$$\mu\bar{u}(x) \in U(x). \tag{12}$$

Proof: According to conditions (A1) and (A3), $U(x)$ is an open convex set such that $0 \in U(x)$. In other words,

$$(1 - \mu) \cdot 0 + \mu \cdot \bar{u}(x) = \mu\bar{u}(x) \in \bar{U}(x) \tag{13}$$

is satisfied for any $\mu \in [0, 1)$. According to Lemma 14 in the Appendix by choosing $y = 0$, $z = \bar{u}$, we have

$$(1 - \mu) \cdot 0 + \mu \cdot \bar{u}(x) = \mu\bar{u}(x) \in U(x). \tag{14}$$

∎

Here, we prove Lemma 4.

Proof: It is sufficient to consider the case of $L_gV(x) \neq 0$. We can directly show $W_1 \subset W_2$ by using $U(x) \subset \bar{U}(x)$. Hence, we prove $W_2 \subset W_1$. Accroding to Theorems 1 and 2, the following condition is satisfied in W_2:

$$\begin{aligned} &\min_{u \in \bar{U}(x)} \{L_fV(x) + L_gV(x) \cdot u\} \\ &\quad = L_fV(x) + L_gV(x) \cdot \bar{u}(x) < 0, \ \forall x \in W_2. \end{aligned} \tag{15}$$

This implies that there exists $\alpha_1 > 0$ such that for each $x \in W_2$,

$$L_fV(x) + L_gV(x) \cdot \bar{u}(x) = -\alpha_1. \tag{16}$$

Then, there exist $\mu \in [0, 1)$ and $0 < \alpha_2 < \alpha_1$ such that

$$L_fV(x) + L_gV(x) \cdot \mu\bar{u}(x) = -\alpha_2 \tag{17}$$

for a fixed x. According to Lemma 5, $\mu\bar{u}(x) \in U(x)$ and $\inf_{u \in U(x)}\{L_fV + L_gV \cdot u\} < 0$ is satisfied. This implies $W_2 \subset W_1$. Therefore, we have $W_1 = W_2$. ∎

According to Lemma 4, we can use W_2 instead of W_1. In other words, we can directly relate $\bar{u}(x)$ to W_1 although $\bar{u}(x) \notin U(x)$. This idea is very important in the following discussion.

To clarify the relation between $\bar{u}(x)$ and W_1, we consider a function $P : \{x \in \mathbb{R}^n | L_gV \neq 0\} \to \mathbb{R}$ defined as the following:

$$P(x) = \frac{L_fV(x)}{-L_gV(x) \cdot \bar{u}(x)}. \tag{18}$$

Since the $\bar{u}(x)$ is continuous at $x \in \{x \in \mathbb{R}^n | L_gV \neq 0\}$, $P(x)$ is a continuous function.

The following lemma shows the relation between $\bar{u}(x)$ and W_1.

Lemma 6. *Consider system (1) with input constraint* $u \in U(x)$. *We assume conditions (A1)-(A4) are satisfied. Let* $V(x)$ *be a local CLF for system (1),* W_1 *be a domain defined in Lemma 3 and* $P(x)$ *be a function defined by (18). Then,*

$$P(x) < 1 \quad \forall x \in \{x \in W_1 | L_gV \neq 0\}. \tag{19}$$

Proof: We consider W_2 instead of W_1. First, we show that $L_gV(x) \cdot \bar{u}(x) < 0$ for any $x \in \{x \in \mathbb{R}^n | L_gV(x) \neq 0\}$. By condition (A3), there exists $\alpha_x > 0$ such that

$$\bar{B}(0, \alpha_x) = \{u | \, \|u\|_2 \leq \alpha_x\} \subset U(x) \tag{20}$$

for each fixed $x \in \{x \in \mathbb{R}^n | L_gV(x) \neq 0\}$. According to the extreme value theorem [13] and the result of [5],

$$\min_{u \in \bar{B}(0,\alpha_x)} L_gV(x) \cdot u = -\alpha_x \parallel L_gV(x) \parallel_2 < 0. \tag{21}$$

Moreover, we can obtain the following inequality according to $\bar{B}(0, \alpha_x) \subset U(x)$:

$$\min_{u \in \bar{U}(x)} L_gV(x) \cdot u \leq \min_{u \in \bar{B}(0,\alpha_x)} L_gV(x) \cdot u. \tag{22}$$

Substituting (21) into (22), we have

$$\min_{u \in \bar{U}(x)} L_gV(x) \cdot u = L_gV(x) \cdot \bar{u}(x) \leq -\alpha_x \| L_gV(x) \|_2 < 0. \tag{23}$$

It is clear that

$$\begin{aligned} & L_fV(x) + L_gV(x) \cdot \bar{u}(x) < 0, \\ & \Leftrightarrow L_fV(x) < -L_gV(x) \cdot \bar{u}(x), \\ & \Leftrightarrow \frac{L_fV(x)}{-L_gV(x) \cdot \bar{u}(x)} < 1. \end{aligned} \tag{24}$$

Condition (24) denotes condition (11) is equivalent to $P(x) < 1$. Thus, the domain in which $P(x) < 1$ contains domain W_2. According to Lemma 4, (19) is satisfied. ■

Remark 2. *If* $P(x) \geq 1$, *condition (11) is not satisfied because* $\dot{V}(x, u) \geq 0, \ \forall u \in U(x)$.

If $V(x)$ satisfies the small control property, the following lemma holds.

Lemma 7. *Consider system (1) with input constraint* $u \in U(x)$. *We assume conditions (A1)-(A4) are satisfied. Let* $V(x)$ *be a local CLF for system (1) that satisfies the small control property,* W_1 *be a domain defined in Lemma 3 and* $P(x)$ *be a function defined by (18). Then,*

$$\lim_{x \to 0} P(x) = 0. \tag{25}$$

Proof: We consider the compact subset Ξ defined on a small neighborhood of the origin $x = 0$. By condition (A3), there exists $\alpha > 0$ such that

$$\bar{B}(0, \alpha) \subset U(x), \ \forall x \in \Xi. \tag{26}$$

Note that α does not depend on x. We can obtain the following inequality by the same discussion of the proof of Lemma 6:

$$-L_gV(x) \cdot \bar{u}(x) \geq \alpha \parallel L_gV(x) \parallel > 0. \tag{27}$$

If $V(x)$ satisfies the small control property, there exists $\delta > 0$ such that $\|x\| < \delta$ and $L_fV < -L_gV \cdot u < \epsilon \|L_gV\|$ in Ξ. By the calculation of $|P(x)|$, we have

$$|P(x)| = \frac{|L_fV|}{|L_gV \cdot \bar{u}|} < \frac{\epsilon \|L_gV\|}{\alpha \|L_gV\|} = \frac{\epsilon}{\alpha}. \tag{28}$$

According to $\delta \to 0$ as $x \to 0$ and choosing $\epsilon \to 0$ as $\delta \to 0$, we achieve

$$\lim_{x \to 0} |P(x)| = \lim_{\epsilon \to 0} \frac{\epsilon}{\alpha} = 0. \tag{29}$$

Therefore, (25) is satisfied. ■

IV. CONTINUOUS CONTROLLER DESIGN

In the previous section, we derived the input $\bar{u}(x)$, which minimizes the time derivative of a local CLF $V(x)$. However, the use of $\bar{u}(x)$ as a control input for system (1) will be harmful for the following reasons:

1) $\bar{u}(x)$ is discontinuous at $x \in \{x \in \mathbb{R}^n | L_gV(x) = 0\}$;
2) $\bar{u}(x)$ becomes large even if x is in the neighborhood of the origin;
3) $\bar{u}(x)$ does not satisfy input constraint $u \in U(x)$.

To overcome these problems, in this section, we propose a new controller design scheme based on the result of [5]. The proposed controller stabilizes the origin in W_1, it satisfies the input constraint $u \in U(x)$, and it is continuous on $W_1 \backslash \{0\}$. Moreover, if $V(x)$ satisfies the small control property, the controller is continuous on the entire W_1.

First, we show the basic idea of the proposed controller. We consider a $\bar{U}^*(x) = \{b_1(x)u | u \in U(x)\} \subset U(x)$ where $b_1 : \{x \in W_1 | L_gV(x) \neq 0\} \to (0, 1)$ is a continuous function. Note that $\bar{U}^*(x)$ is homothetic to $\bar{U}(x)$. This implies $\bar{U}^*(x)$ is considered to be a virtual input constraint. According to Theorem 2, input $u^*(x)$ that minimizes $\dot{V}(x, u)$ under the virtual input constraint $u \in \bar{U}^*(x)$ can be obtained as the following:

$$u^*(x) = \begin{cases} b_1(x)\bar{u}(x) & (L_gV(x) \neq 0) \\ 0 & (L_gV(x) = 0). \end{cases} \tag{30}$$

We consider this $u^*(x)$ as a control input. Then, the input constraint $u^*(x) \in U(x)$ is satisfied because Lemma 5 holds. The time derivative of the local CLF with input $u^*(x)$ is

$$\dot{V}(x, u^*) = L_fV(x) + b_1(x)L_gV(x) \cdot \bar{u}(x). \tag{31}$$

To stabilize the origin in W_1 with $u^*(x)$, we have to choose $\bar{U}^*(x)$ such that: $\dot{V}(x, u^*) < 0 \ \forall x \in \{x \in W_1 | L_gV(x) \neq 0\}$.

Here, we consider construction of $b_1(x)$ by using $P(x)$ defined by (18). More precisely, the problem to be solved is to design $b_1(x)$ such that

1) If $P(x) \to 1$, $\bar{U}^*(x) \to \bar{U}(x)$ ($b_1(x) \to 1$) such that $\dot{V}(x, u^*) < 0$;
2) If $P(x)$ is small, $\bar{U}^*(x)$ becomes small ($b_1(x) \to 0$).

For instance, we can construct $b_1(x)$ as the following:

Lemma 8. *Consider system (1) with input constraint* $u \in U(x)$. *We assume conditions (A1)-(A4) are satisfied. Let*

$V(x)$ be a local CLF for system (1), $\bar{u}(x)$ be an input defined by Theorem 2, W_1 be a domain defined by Lemma 3 and $P(x)$ be a function defined by (18). Additionally, we choose $b_1(x)$ as

$$b_1(x) = \frac{P(x) + |P(x)|}{2} + b_2(x), \tag{32}$$

where $b_2(x) : \{x \in W_1 | L_gV \neq 0\} \to (0, 1 - \frac{1}{2}(P + |P|))$ is a continuous function.

Then, input (30) satisfies the $\dot{V}(x,u) < 0,\ \forall x \in W_1$.

Proof: According to Lemma 6, we have $P(x) < 1,\ \forall x \in \{x \in W_1 | L_gV(x) \neq 0\}$. The time derivative of a local CLF $\dot{V}(x,u^*)$ is obtained as the following:

$$\begin{aligned}\dot{V}(x,u^*) &= L_fV + \left(\frac{1}{2}(P(x) + |P(x)|) + b_2(x)\right) L_gV \cdot \bar{u}(x) \\ &= \begin{cases} b_2(x)L_gV \cdot \bar{u}(x) & (0 < P < 1) \\ L_fV + b_2(x)L_gV \cdot \bar{u}(x) & (P \leq 0). \end{cases}\end{aligned} \tag{33}$$

Note that $L_gV \cdot \bar{u}(x) < 0$ and $P(x) \leq 0 \Rightarrow L_fV < 0$. Therefore, $\dot{V}(x,u^*) < 0\ \forall x \in W_1$ is satisfied. ■

According to Lemma 8, we can obtain the following theorem.

Theorem 3. *Consider system (1) with input constraint $u \in U(x)$. We assume conditions (A1)-(A4) are satisfied. Let $V(x)$ be a local CLF for system (1), $\bar{u}(x)$ be an input defined by Theorem 2, W_1 be a domain defined by Lemma 3, $P(x)$ be a function defined by (18), $c > 0$ and $q \geq 1$ be constants. Then, the input*

$$u(x) = \begin{cases} \dfrac{P+|P|+c\|L_gV\|_q}{2+c\|L_gV\|_q}\bar{u}(x) & (L_gV \neq 0) \\ 0 & (L_gV = 0) \end{cases} \tag{34}$$

stabilizes the origin in domain W_1. Moreover, input (34) is continuous on $W_1 \backslash \{0\}$, and it is also continuous at the origin if $V(x)$ satisfies the small control property.

To prove Theorem 3, we introduce the following lemma.

Lemma 9. *Consider system (1) with input constraint $u \in U(x)$. We assume conditions (A1)-(A4) are satisfied. Let $V(x)$ be a local CLF for system (1) that satisfies the small control property, W_1 be a domain defined by Lemma 3, and $P(x)$ be a function defined by (18). Then,*

$$\lim_{x \to 0} \frac{P+|P|+c\|L_gV\|_q}{2+c\|L_gV\|_q}\bar{u}(x) = 0. \tag{35}$$

Proof: When $\|x\| \to 0$, $\|L_gV\|_q \to 0$. Additionally, when $x \to 0$, $|P(x)| \to 0$ by Lemma 7. Then, we achieve (35). ■

Lemma 9 is important for the continuity of input (34) at the origin.

Now, we can prove Theorem 3.

Proof: First, we show the continuity of input (34). By conditions (A1)-(A4) and Theorem 2, input (34) is continuous on $\{x \in W_1 | L_gV(x) \neq 0\}$. Moreover, due to (3), $P(x) + |P(x)| = 0$ is satisfied in the neighborhood of $L_gV(x) = 0$ except at the origin. Then,

$$\lim_{L_gV \to 0} \frac{P + |P| + c\|L_gV\|_q}{2 + c\|L_gV\|_q} = 0 \tag{36}$$

is satisfied except at the origin. Hence, input (34) is continuous on $W_1 \backslash \{0\}$.

If $V(x)$ satisfies the small control property, it is obvious that input (34) is continuous at the origin by Lemmas 7 and 9.

Next, we show that input (34) satisfies the input constraint $u \in U(x)$. The input constraint is satisfied when $L_gV(x) = 0$ because $u(x) = 0 \quad \forall x \in \{x \in W_1 | L_gV = 0\}$. When $L_gV \neq 0$, we can derive

$$0 < \frac{P+|P|+c\|L_gV(x)\|_q}{2+c\|L_gV\|_q} < 1 \tag{37}$$

by using Lemma 6. According to Lemma 5, we can obtain $u(x) \in U(x)$.

Finally, we show that input (34) asymptotically stabilizes the origin in W_1. If $L_gV(x) = 0$, it is obvious that $L_fV < 0$ by (2). Then, $\dot{V}(x,u) < 0$ is satisfied. If $L_gV \neq 0$ and $P(x) \leq 0$, we can obtain $L_fV \leq 0$. Note that $L_gV \cdot u(x) < 0$, and we have $\dot{V}(x,u) < 0$. If $L_gV \neq 0$ and $0 < P(x) < 1$, the input (34) is denoted by

$$u = \left\{P + \frac{c(1-P) \parallel L_gV \parallel_q}{2 + c \parallel L_gV \parallel_q}\right\}\bar{u}(x). \tag{38}$$

Then, the derivative $\dot{V}(x,u)$ is

$$\dot{V}(x) = \frac{c(1-P) \parallel L_gV \parallel_q}{2 + c \parallel L_gV \parallel_q} L_gV \cdot \bar{u}(x) < 0. \tag{39}$$

Note that $P(x) < 1$, and $\dot{V}(x,u) < 0,\ \forall x \in W_1 \backslash \{0\}$ are confirmed. Therefore, input (34) stabilizes the origin in W_1. ■

By Theorem 3, we can prove Lemma 3 in the previous section.

Proof: The controller (34) stabilizes the origin in W_1 under the input constraint $u \in U(x)$. Then, the closed-loop system becomes a continuous ordinary differential equation with continuous right-hand side. Therefore, W_1 is an asymptotically stabilizable domain. ■

V. NUMERICAL EXAMPLE

We proposed stabilizing controller (34) in section IV. In this section, we confirm the effectiveness of the proposed controller (34) through an example. We consider the following nonlinear control system:

$$\Sigma \begin{cases} \dot{x}_1 = (\sin x_1)^2 + u_1, \\ \dot{x}_2 = u_2 \end{cases} \tag{40}$$

with input constraint $(u_1 + 0.5)^2 + u_2^2 - 1 < 0$. The constraint denotes the center shifted circle (Fig. 1.) We consider the following function as a CLF for system Σ:

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2. \tag{41}$$

We can calculate $L_fV(x)$ and $L_gV(x)$ as

$$\begin{aligned} L_fV(x) &= x_1(\sin x_1)^2, \\ L_gV(x) &= [L_{g1}V, L_{g2}V] = [x_1, x_2]. \end{aligned} \tag{42}$$

Note that $V(x)$ satisfies the small control property. By the direct calculation of (6), we obtain the minimizing input $\bar{u}(x)$ as follows:

$$\bar{u}(x) = [\bar{u}_1(x), \bar{u}_2(x)]^{\mathrm{T}}, \tag{43}$$

$$\bar{u}_1(x) = \begin{cases} -\left(0.5 + \dfrac{x_1}{\sqrt{x_1^2+x_2^2}}\right) & (L_gV(x) \neq 0) \\ 0 & (L_gV(x) = 0) \end{cases}, \tag{44}$$

$$\bar{u}_2(x) = \begin{cases} -\dfrac{x_2}{\sqrt{x_1^2+x_2^2}} & (L_gV(x) \neq 0) \\ 0 & (L_gV(x) = 0) \end{cases}. \tag{45}$$

Note that $P(x)$ is calculated as

$$P(x) = \frac{x_1(\sin x_1)^2}{\frac{1}{2}x_1 + \sqrt{x_1^2+x_2^2}}. \tag{46}$$

Then, we can design the following controller by using Theorem 3:

$$u(x) = \begin{cases} \dfrac{P(x) + |P(x)| + c\|L_gV\|_q}{2 + c\|L_gV\|_q} \cdot \bar{u}(x) & (L_gV \neq 0) \\ 0 & (L_gV = 0) \end{cases}. \tag{47}$$

Let $c = 1$, $q = 2$, and $x(0) = [3, 1]^{\mathrm{T}}$. We show time responses of Σ with controller (47) in Fig. 3. In Fig. 3 (a) and (b), we can observe that the state and the input successfully converge to 0. Figure 3 (c) illustrates a trajectory of inputs, and the input constraint $(u_1 + 0.5)^2 + u_2^2 - 1 < 0$ is in a dashed circle. We can confirm controller (47) satisfies the input constraint by the figure.

On the other hand, we can replace input constraint $(u_1 + 0.5)^2 + u_2^2 - 1 < 0$ with a sufficient condition $u_1^2 + u_2^2 - 0.5^2 < 0$ (Fig. 2.) It is clear that $u_1^2 + u_2^2 - 0.5^2 < 0 \Rightarrow (u_1 + 0.5)^2 + u_2^2 - 1 < 0$. Note that the sufficient condition is a 2-norm constraint. Thus, by the result of [6], we can construct a continuous stabilizing controller as the following:

$$u(x) = \begin{cases} \dfrac{P'(x) + |P'(x)| + c\|L_gV\|_q}{2 + c\|L_gV\|_q} \cdot \bar{u}'(x) & (L_gV \neq 0) \\ 0 & (L_gV = 0) \end{cases}, \tag{48}$$

$$\bar{u}'(x) = [\bar{u}'_1(x), \bar{u}'_2(x)]^{\mathrm{T}}, \tag{49}$$

$$\bar{u}'_i(x) = \begin{cases} -\dfrac{x_i}{2\sqrt{x_1^2+x_2^2}} & (L_gV(x) \neq 0) \\ 0 & (L_gV(x) = 0) \end{cases} \quad (i = 1, 2), \tag{50}$$

$$P'(x) = \frac{2x_1(\sin x_1)}{\sqrt{x_1^2+x_2^2}}. \tag{51}$$

We compare controller (47) with controller (48). Figure 4 shows time responses of Σ with controller (48). Parameters c, q and the initial state $x(0)$ are the same as the controller (47) case. In Fig 4, we can permit that x_1 and u_1 do not converge to 0, and a trajectory of the controller (48) converges to point $(-0.5, 0)$ by Fig. 4 (c). This implies that the sufficient condition $u_1^2 + u_2^2 < 0.5^2$ is too conservative to asymptotically stabilize system Σ. Therefore, we can confirm the effectiveness of proposed controller (47).

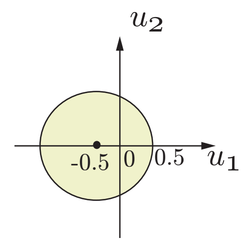

Fig. 1. $(u_1+0.5)^2+u_2^2-1<0$.

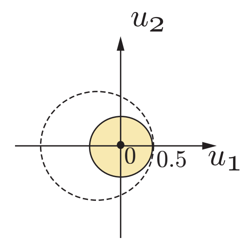

Fig. 2. $u_1^2+u_2^2-0.5^2<0$

VI. CONCLUSION

In this paper, we have proposed a controller design method consisting of two steps for nonlinear systems with convex input constraints.

In the first step, we derived an input which minimizes the time derivative of a local CLF by using nonlinear convex optimization. Then, we discussed the continuity of the minimizing input. We also considered the relation between the minimizing input and asymptotically stabilizable domain.

In the second step, we designed a continuous asymptotically stabilizing controller for nonlinear systems with convex input constraints based on a derived minimizing input.

We confirmed the effectiveness of the proposed method through an example.

However, the proposed controller (34) did not guarantee the robustness in the sense of sector margins. For nonlinear systems with norm input constraints, an inverse optimal controller that guarantees a sector margin has been proposed [7]. Therefore, we would like to design a controller that guarantees a sector margin for nonlinear systems with convex input constraints in our future work.

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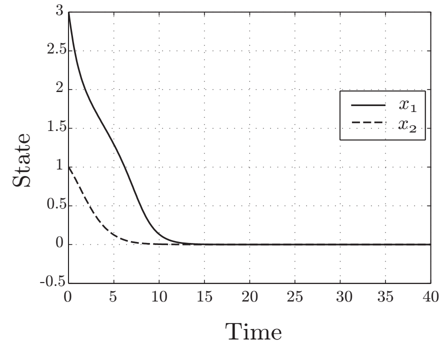
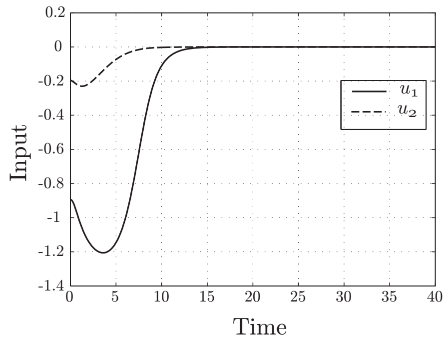
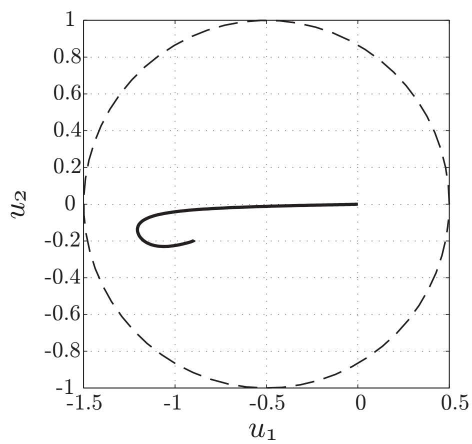


(a) Responses of state.

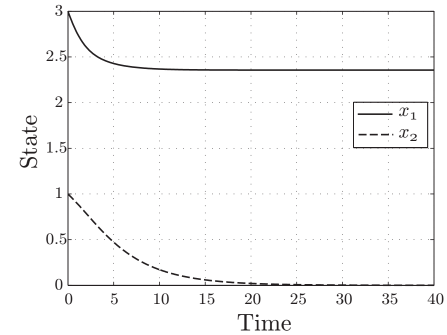


(b) Responses of input.

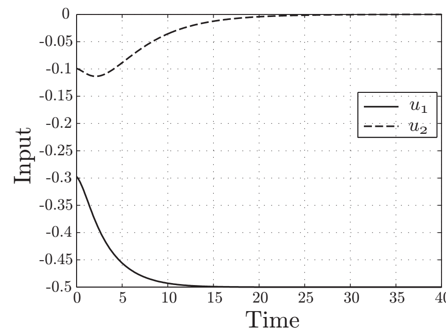


(c) Trajectory of controller (47).

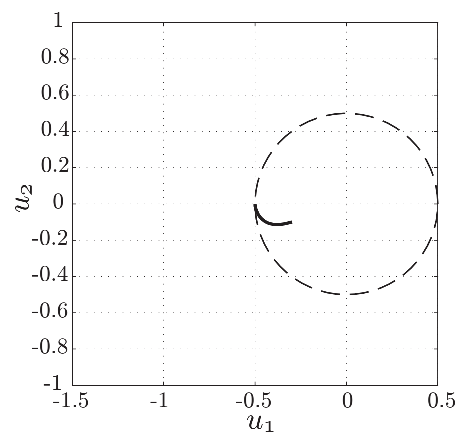
Fig. 3. Responses of state and input with controller (47)



(a) Responses of state.



(b) Responses of input.



(c) Trajectory of controller (48).

Fig. 4. Responses of state and input with controller (48)

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APPENDIX

A. Nonlinear Optimization and Convex Analysis

In this subsection, we introduce some definitions and lemmas of nonlinear convex optimization [10], [11], [12].

We consider the following optimization problem:

Problem A.

$$\text{Minimize } F(y,z) \ \text{ subject to } y \in S(z),$$
$$where,\ S(z) = \{y \in \mathbb{R}^n | G_i(y,z) \leq 0\ (i = 1, \ldots, l)\}. \tag{52}$$

where $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is an objective function, $G_i : \mathbb{R}^n \times$

$\mathbb{R}^m \to \mathbb{R}$ $(i = 1, \ldots, l)$ are constraint functions. $z \in \mathbb{R}^m$ is a parameter variable. $S : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n)$ denotes a constraint mapping, where $\mathcal{P}(\mathbb{R}^n)$ is the power set of $\mathbb{R}^n$.

We define a uniform boundedness for $S(z)$ as follows.

Definition 3. *Consider constraint mapping $S(z)$. $S(z)$ is said to be uniformly bounded on a neighborhood of $\bar{z} \in \mathbb{R}^m$, if there exists $\Omega \subseteq \mathbb{R}^m$ such that $\cup_{z\in\Omega} S(z) \subseteq \mathbb{R}^n$ is bounded.*

Problem A is an optimization problem of minimizing $F(y, z)$ with respect to y for each z. We discuss the optimal solution of problem A. We assume that the following conditions are satisfied.

- (S1) $F(y, z)$ and each $G_i(y, z)$ $(i = 1, \ldots, l)$ are differentiable on $\mathbb{R}^n \times \mathbb{R}^m$ and a convex function with respect to y for any $z \in \mathbb{R}^m$.
- (S2) $S(z)$ is uniformly bounded for each fixed $z \in \mathbb{R}^m$.
- (S3) $0 \in \mathrm{int}S(z),\ \forall z \in \mathbb{R}^m$.

Then, the following lemma for the continuity of $S(z)$ is obtained.

Lemma 10. *Consider problem A. We assume that conditions (S1)-(S3) are satisfied.*

Then, $S(z)$ is continuous on $\forall z \in \mathbb{R}^m$.

For the existence of the optimal solution of problem A and the boundedness of the solution, we introduce the following lemma.

Lemma 11. *Consider problem A. We assume that conditions (S1)-(S3) are satisfied.*

Then, $F(y, \bar{z})$ is bounded at each fixed $\bar{z} \in \mathbb{R}^m$ and there exists an optimal solution $\bar{y}$ for problem A.

Proof: We fix $\bar{z} \in \mathbb{R}^m$. According to condition (S2), $S(\bar{z})$ is a compact set. On the other hand, $F(y, \bar{z})$ is continuous on $S(\bar{z}) \times \{\bar{z}\}$ by condition (S1). By using the extreme value theorem, we can prove the existence of the optimal solution and the boundedness of $F(y, \bar{z})$. ∎

To clarify the condition of the optimal solution of problem A, we introduce Lagrangian defined as the following.

Definition 4 (Lagrangian)**.** *Consider problem A. A function $L_0 : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ defined as the following is called Lagrangian for problem A:*

$$L_0(y, z, \lambda) = F(y, z) + \sum_{i=1}^{l} \lambda_i G_i(y, z), \tag{53}$$

where $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{R}^l$ is a vector Lagrange multipliers.

We can obtain the following Lemma for the necessary and sufficient condition for problem A.

Lemma 12 (Karush-Kuhn-Tucker condition)**.** *Consider problem A. We assume conditions (S1)∼(S3) are satisfied.*

Then, for each fixed $\bar{z} \in \mathbb{R}^m$, if there exists $(\bar{y}, \bar{\lambda})$ that satisfy the following conditions, $\bar{y}$ is the optimal solution of problem A.

$$\frac{\partial L_0(\bar{y}, \bar{z}, \bar{\lambda})}{\partial y} = \frac{\partial F(\bar{y}, \bar{z})}{\partial y} + \sum_{i=1}^{l} \bar{\lambda}_i \frac{\partial G_i(\bar{y}, \bar{z})}{\partial y} = 0,$$
$$\bar{\lambda}_i \geq 0,\ G_i(\bar{y}, \bar{z}) \leq 0,\ \bar{\lambda}_i G_i(\bar{y}, \bar{z}) = 0\ (i = 1, \ldots, l). \tag{54}$$

Remark 3. *In general, it is required that problem A satisfies some sort of constraint qualification for (54) to be a necessary and sufficient condition for problem A. In problem A, the following Slater constraint qualification is satisfied for all $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$.*

Definition 5 (Slater constraint qualification)**.** *In problem A, we say that Slater constraint qualification is satisfied for $\bar{z} \in \mathbb{R}^m$ and $\bar{y} \in S(\bar{z})$, if the following conditions are satisfied:*

- (1) *Each $G_i(y, z)$ $(i \in \{i | G_i(\bar{y}, \bar{z}) = 0\})$ is a convex function with respect to y;*
- (2) *There exists y^0 such that $G_i(y^0, \bar{z}) < 0$ $(i = 1, \ldots, l)$.*

Then, we discuss the continuity of the optimal solution of problem A with respect to z. We introduce the following optimal value function $\phi(z)$ and optimal set mapping $\Phi(z)$.

Definition 6. *A function $\phi : \mathbb{R}^m \to \mathbb{R}$ and a mapping $\Phi : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^n)$ defined as the following are called an optimal value function and an optimal set mapping, respectively:*

$$\phi(z) = \min_{y \in S(z)} \{F(y, z)\}, \tag{55}$$

$$\Phi(z) = \{y \in S(z) | \phi(z) = F(y, z)\}. \tag{56}$$

Here, we are interested in the continuity of $\Phi(z)$. To discuss the continuity of $\Phi(z)$, we employ the following condition:

- (S4) Solution $\bar{y}$ satisfying (54) is uniquely determined for $\bar{z} \in \mathbb{R}^m$.

Then, we can obtain the following lemma for the continuity of $\Phi(z)$.

Lemma 13. *Consider problem A. We assume conditions (S1)∼(S3) are satisfied. Then, if condition (S4) is satisfied for $z = \bar{z}$, $\Phi(z)$ is continuous at $\bar{z}$.*

We introduce the following important lemma for convex sets.

Lemma 14. *Let $\mu \in [0, 1)$, $T \subseteq \mathbb{R}^n$ be a convex set such that $0 \in \mathrm{int}T$, and $\bar{T}$ denotes a closure of T.*

Then, for any $y \in \mathrm{int}T$ and $z \in \bar{T}$,

$$(1 - \mu)y + \mu z \in \mathrm{int}T. \tag{57}$$