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Robust fixed-order H_∞ Controller Design for Spectral Models by Convex Optimization

Alireza Karimi, Gorka Galdos and Roland Longchamp

***Abstract*— A new approach for robust fixed-order H_∞ controller design by convex optimization is proposed. Linear time-invariant single-input single-output systems represented by a finite set of complex values in the frequency domain are considered. It is shown that the H_∞ robust performance condition can be approximated by a set of linear or convex constraints with respect to the parameters of a linearly parameterized controller in the Nyquist diagram. Multimodel and frequency-domain uncertainty can be directly considered in the proposed approach by increasing the number of constraints. The proposed method is compared with the standard H_∞ control problem. It is shown by an example that for an unstable uncertain model, a PID controller can be designed with the proposed approach which gives better H_∞ performance than a 7th order unstable controller obtained by the standard H_∞ solution.**

I. Introduction

Spectral models (or frequency function models) can be easily identified from input/output data using Fourier or spectral analysis. These models are represented by a finite set of complex values and give some important information about the bandwidth and the static gain of the system. Although spectral models are largely used in practice, controller design methods based on this type of models are rather limited. The first systematic controller design methods were based on loop shaping with graphical tools in Bode diagrams or in Nichols chart and are discussed in classical textbooks for design and analysis of control systems. These approaches are very intuitive and work well for simple systems that can be approximated by a low-order model with relatively small delay. For unstable and non minimum phase systems and systems with parametric and frequency-domain uncertainty, more advanced methods should be used. A well-known method is the Quantitative Feedback Theory (QFT) [1] which is based on loop shaping in the Nichols chart. Frequency-domain approaches lead usually to low-order controllers and the design procedures need some expertise and are based on trial and error. Although recently optimization approaches are used to compute controllers in the QFT framework [2], [3], H_2 and H_∞ control criteria for spectral models have not been considered.

With new progress in numerical methods for solving convex optimization problems, new approaches for controller design with convex objectives and constraints have been developed. These techniques have been also applied to controller design for spectral models. In [4], [5] a convex optimization method for PID controller tuning by open-loop shaping in the frequency-domain is proposed. The infinity-norm of the difference between the desired open-loop transfer function and the achieved one weighted by a so-called target sensitivity function is minimized. For open-loop stable systems, it is shown through the small gain theorem that if the infinity norm is less than 1, then the nominal closed-loop system is stable. This is a sufficient condition which depends on the choice of the target sensitivity function. The condition for the stability of multiple models becomes more conservative as for each model a reasonable target sensitivity function should be available.

In [6] a robust fixed-order controller design using linear programming is proposed. The main feature of this method is that the stability and some robustness margins are guaranteed by linear constraints in the Nyquist diagram and the method is applicable to multiple models as well. However, the performance specifications are limited to the choice of a lower bound for crossover frequency and minimization of the integral of the tracking error. The results are improved by open-loop and closed-loop shaping using quadratic programming in [7].

In this paper, based on the idea proposed in [6], [7] a new approach for robust fixed-order controller design is developed. It is shown that robust fixed-order linearly parameterized controllers for Linear Time Invariant Single-Input Single Output (LTI-SISO) systems represented by nonparametric spectral models can be computed by convex optimization. The performance specification, like the standard H_∞ control problem, is a constraint on the infinity norm of the weighted sensitivity function. It should be mentioned that the set of all fixed-order stabilizing controllers is a nonconvex set. In this paper, a convex approximation of this set is given by a set of linear constraints in the Nyquist diagram. The proposed method can be used for PID controllers as well as for higher order linearly parametrized controllers in discrete or continuous time. The case of unstable open-loop systems can also be considered if a stabilizing controller is available. The main idea is to define new constraints such that the designed open-loop system has the winding number satisfying the Nyquist stability criterion. Another important feature is that, by contrast with the standard H_∞ problem, this approach can treat the case of multimodel uncertainty as well. The effectiveness of the proposed approach is illustrated by comparison with the standard H_∞ control design in a simulation example.

This paper is organized as follows: In Section II the class of models, controllers and the control objectives are defined.

The authors are with the Laboratoire d'Automatique of Ecole Polytechnique Fédérale de Lausanne (EPFL), 1015 Lausanne, Switzerland.

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Corresponding author: alireza.karimi@epfl.ch

Section III introduces the control design methodology based on the linear and convex constraints in the Nyquist diagram. Simulation results and comparison with the standard H_∞ design are given in Section IV. Advantages and disadvantages of the proposed method are discussed in Section V. Finally, Section VI gives some concluding remarks.

II. Problem Formulation

A. Class of models

The class of continuous-time LTI-SISO systems with bounded infinity norm is considered. However, the results can be applied directly to the discrete-time systems. It is assumed that the plant model belongs to a set $\mathcal{G}$ that is the convex combination of m spectral models with a sufficiently large number of frequency points N:

$$\mathcal{G} = \left\{ \sum_{i=1}^{m} \lambda_i G_i(j\omega_k) : \sum_{i=1}^{m} \lambda_i = 1; k = 1, N \right\} \tag{1}$$

where λ_i are real positive numbers. By sufficiently large number of frequency points we mean that the open-loop frequency response of the system in the Nyquist diagram between two adjacent frequency points can be well approximated by linear interpolation. The set $\mathcal{G}$ represents multimodel and unstructured frequency-domain uncertainty. In the sequel, for the sake of simplicity, we consider a nominal model $G \in \mathcal{G}$ and represent the robust performance conditions for a single nominal model with frequency-domain uncertainty. It will be shown that the multimodel uncertainty can be considered by repeating the constraints for each model.

B. Class of controllers

Linearly parameterized controllers are given by :

$$K(s) = \rho^T \phi(s) \tag{2}$$

where $\rho^T = [\rho_1, \rho_2, \ldots, \rho_n]$, $\phi^T(s) = [\phi_1(s), \phi_2(s), \ldots, \phi_n(s)]$, n is the number of controller parameters and $\phi_i(s)$ are stable transfer functions with possible poles on the imaginary axis chosen from a set of orthogonal basis functions. It is clear that PID controllers belong to this set. The main property of this parameterization is that every point on the Nyquist diagram of $L(j\omega) = K(j\omega)G(j\omega)$ can be written as a linear function of the controller parameters ρ:

$$\begin{aligned} K(j\omega_k)G(j\omega_k) &= \rho^T \phi(j\omega_k)G(j\omega_k) \\ &= \rho^T \mathcal{R}(\omega_k) + j\rho^T \mathcal{I}(\omega_k) \end{aligned} \tag{3}$$

where $\mathcal{R}(\omega_k)$ and $\mathcal{I}(\omega_k)$ are respectively the real and imaginary parts of $\phi(j\omega_k)G(j\omega_k)$.

C. Design Specifications

Let the sensitivity function $\mathcal{S}(s) = [1 + L(s)]^{-1}$, the complementary sensitivity function $\mathcal{T}(s) = L(s)[1+L(s)]^{-1}$ and the crossover frequency ω_c be defined. The proposed approach can consider very simple specifications for the design of simple PID controllers as well as standard performance specifications for H_∞ control problems. For simple controller design, a lower bound on the modulus margin (the inverse of the infinity norm of the sensitivity function that ensures a lower bound on the gain and the phase margin) and a desired value for the crossover frequency can be considered. While more advanced control problems in which the performance and robust stability are defined by constraints on the infinity norm of the weighted sensitivity functions can also be treated for fixed-order controller design.

A very standard robust control problem is to design a controller that satisfies $\|W_1\mathcal{S}\|_\infty < 1$ for a set of models, where $W_1(s)$ is the performance weighting filter. If the set of models is represented by multiplicative uncertainty, i.e. $G_m(s) = G(s)[1 + W_2(s)\Delta(s)]$ with $\|\Delta\|_\infty < 1$, the necessary and sufficient condition for robust performance is given by [8]:

$$\| |W_1\mathcal{S}| + |W_2\mathcal{T}| \|_\infty < 1 \tag{4}$$

There is no analytical solution to this problem, however, in the standard H_∞ framework a solution to the following approximate problem can be found:

$$\left\| \begin{matrix} W_1\mathcal{S} \\ W_2\mathcal{T} \end{matrix} \right\|_\infty < \frac{1}{\sqrt{2}} \tag{5}$$

This solution is conservative and leads to high order controllers. The proposed approach in this paper is based on some linear or convex constraints on the Nyquist diagram such that the following constraints are satisfied :

$$|W_1(j\omega_k)\mathcal{S}(j\omega_k)| + |W_2(j\omega_k)\mathcal{T}(j\omega_k)| < 1 \tag{6}$$

for $k = 1, \ldots, N$. For models with additive uncertainty, i.e. $G_a(s) = G(s) + W_3(s)\Delta(s)$ with $\|\Delta\|_\infty < 1$, the robust performance condition is given by:

$$|W_1(j\omega_k)\mathcal{S}(j\omega_k)| + |W_3(j\omega_k)K(j\omega_k)\mathcal{S}(j\omega_k)| < 1 \tag{7}$$

for $k = 1, \ldots, N$. The use of linear or convex constraints instead of the above non-convex constraints leads also to a conservative solution. It will be shown that this conservatism can be significantly reduced if a desired open-loop transfer function $L_d(s)$ is available and a norm of $L(s) - L_d(s)$ is minimized under the robust performance constraints.

The choice of $L_d(s)$ has already been discussed in open-loop shaping design methods and we do not intend to investigate this choice in this contribution. However, some simple choices are recalled that usually lead to good results for simple models. For example $L_d(s) = \omega_c/s$ is an appropriate choice for low-order stable systems. If a desired reference model $M(s)$ for the closed-loop system is available, $L_d(s)$ can be chosen equal to $M(s)[1 - M(s)]^{-1}$. The choice of $L_d(s)$ is more important for unstable systems. In this case the winding number of $L_d(s)$ around the critical point in the Nyquist diagram should satisfy the Nyquist stability criterion. For this purpose, the number of unstable poles of the plant model should be known or a stabilizing controller $K_0(s)$ should be available. It should be mentioned that a nonrealistic choice of $L_d(s)$ (with respect to plant model and controller structure) will only increase the conservatism of the approach and never leads to a destabilizing controller. A reasonable approach, known as windsurfing [9], is to start with a modest choice of $L_d(s)$ (with a small bandwidth) and increase iteratively the closed-loop bandwidth.

III. Robust Controller Design in Nyquist Diagram

A. Robust performance constraints

The basic idea is to approximate the nonconvex robust performance constraints in (6) and (7) by linear constraints. This way, the controller design is represented by a convex feasibility problem. We start by multiplying the robust performance condition in (6) by $|1+L(j\omega_k)|$ to obtain:

$$|W_1(j\omega_k)| + |W_2(j\omega_k)L(j\omega_k)| < |1+L(j\omega_k)| \quad \text{for} \quad k=1,N \tag{8}$$

Note that $|1+L(j\omega_k)|$ is the distance between the critical point and $L(j\omega_k)$. Hence, this constraint is satisfied if and only if there is no intersection in the Nyquist diagram between a circle centered at the critical point with a radius of $|W_1(j\omega_k)|$ and a circle centered at $L(j\omega_k)$ with a radius of $|W_2(j\omega_k)L(j\omega_k)|$ at each frequency ω_k [8]. Now, consider a straight line d_k^* which is tangent to the circle with radius $|W_1(j\omega_k)|$ and orthogonal to the line between the critical point and $L(j\omega_k)$. Therefore, the robust performance condition in (6) is satisfied if and only if the circle centered at $L(j\omega_k)$ does not intersect d_k^* and is completely in the side that excludes the critical point (at the right hand side in Fig. 1). This condition cannot be represented as a convex constraint because d_k^* is a function of the controller parameters. However, d_k^* can be approximated by d_k which is tangent to the circle with radius $|W_1(j\omega_k)|$ but orthogonal to the line connecting the critical point to $L_d(j\omega)$ (see Fig. 1). It should be noted that the equation of d_k at each frequency ω_k depends only on $W_1(j\omega_k)$ and $L_d(j\omega_k)$. If we name x and y, respectively, the real and imaginary parts of a point on the complex plane, the equation of d_k at each frequency ω_k becomes :

$$|W_1(j\omega_k)[1+L_d(j\omega_k)]| - I_m\{L_d(j\omega_k)\}y - [1+R_e\{L_d(j\omega_k)\}][1+x] = 0 \tag{9}$$

where $R_e\{\cdot\}$ and $I_m\{\cdot\}$ represent real and imaginary parts of a complex value, respectively. Therefore, the condition that $L(j\omega_k)$ for all ω_k is located in the side of d_k that excludes the critical point can be given by the following linear constraints :

$$|W_1(j\omega_k)[1+L_d(j\omega_k)]| - I_m\{L_d(j\omega_k)\}\rho^T\mathcal{I}(j\omega_k) - [1+R_e\{L_d(j\omega_k)\}][1+\rho^T\mathcal{R}(j\omega_k)] < 0 \quad \text{for} \quad k=1,\ldots,N$$

There exists two alternatives in order that this condition to be satisfied for all models in the uncertainty set represented by a circle centered at $L(j\omega_k)$. The first alternative is to approximate the uncertainty circle by a polygon of $m>2$ vertices. Then, the robust performance condition in (6) is

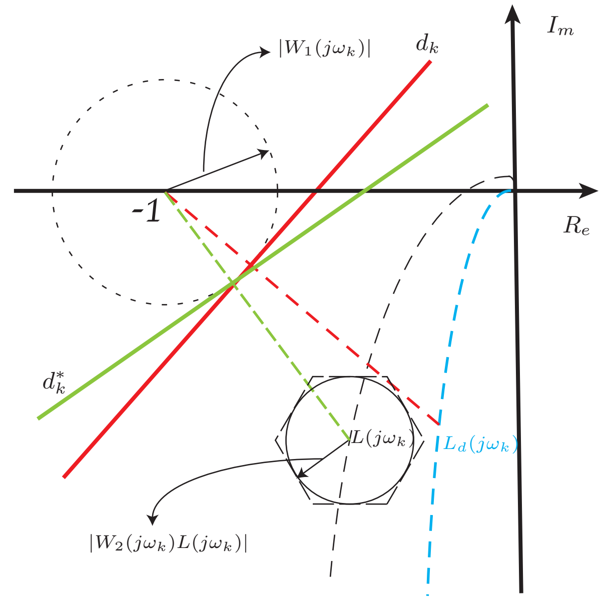

Fig. 1. Linear constraints for robust performance in Nyquist diagram

satisfied if all vertices are located in the right side of d_k. This can be represented by the following constraints :

$$|W_1(j\omega_k)[1+L_d(j\omega_k)]| - I_m\{L_d(j\omega_k)\}\rho^T\mathcal{I}_i(j\omega_k) - [1+R_e\{L_d(j\omega_k)\}][1+\rho^T\mathcal{R}_i(j\omega_k)] < 0 \quad \text{for} \quad k=1,\ldots,N \quad \text{and} \quad i=1,\ldots,m \tag{10}$$

where $\mathcal{R}_i(j\omega_k)$ and $\mathcal{I}_i(j\omega_k)$ are the real and the imaginary parts of $\phi(j\omega_k)G_i(j\omega_k)$ with

$$G_i(j\omega_k) = G(j\omega_k)[1+\frac{|W_2(j\omega_k)|}{\cos\pi/m}e^{j2\pi i/m}] \tag{11}$$

It can be observed that the number of linear constraints are multiplied by m when the uncertainty circle is approximated by a polygon of m vertices.

The second alternative is to consider for each frequency only one constraint for the closest point of the circle to d_k. It is clear that if this point is at the side of d_k that excludes the critical point, then the whole uncertainty circle is in the correct side. The coordinates of the closest point of the uncertainty circle from d_k can be computed as :

$$x = \rho^T\mathcal{R}(j\omega_k) - |W_2(j\omega_k)\rho^T\phi(j\omega_k)G(j\omega_k)| \times \frac{1+R_e\{L_d(j\omega_k)\}}{|1+L_d(j\omega_k)|} \tag{12}$$

$$y = \rho^T\mathcal{I}(j\omega_k) - |W_2(j\omega_k)\rho^T\phi(j\omega_k)G(j\omega_k)| \times \frac{I_m\{L_d(j\omega_k)\}}{|1+L_d(j\omega_k)|} \tag{13}$$

Using these coordinates and the equation of d_k in (9) the robust performance constraints obtained are no longer linear but convex with respect to the controller parameter vector

ρ :

$$
\begin{aligned}
&|W_1(j\omega_k)[1+L_d(j\omega_k)]| - I_m\{L_d(j\omega_k)\}\rho^T\mathcal{I}(j\omega_k) + \\
&\quad |W_2(j\omega_k)\rho^T\phi(j\omega_k)G(j\omega_k)[1+L_d(j\omega_k)]| - \\
&\quad [1+R_e\{L_d(j\omega_k)\}][1+\rho^T\mathcal{R}(j\omega_k)] < 0 \\
&\qquad \text{for} \quad k=1,\ldots,N
\end{aligned}
\tag{14}
$$

This alternative has less constraints and no conservatism but leads to a bit more complex convex optimization problem (convex constraints instead of linear constraints).

Remarks:

1) The same approach can be applied while an additive uncertainty model is available. In this case the robust performance condition in (7) can be represented by linear constraints in (10) or by convex constraints in (14) with the difference that $|W_2(j\omega_k)| = |W_3(j\omega_k)|/|G(j\omega_k)|$.
2) Individual shaping of the sensitivity functions is also possible using the constraints in (6) and (7) and putting one of the filters equal to zero. For example, in many applications we need to put some constraints on the magnitude of the input sensitivity function $\mathcal{U}(s) = K(s)[1+L(s)]^{-1}$ in order to reduce the control effort. This can be done by defining a weighting frequency function $W_3(j\omega_k)$, usually a high pass filter, and using the constraints in (7) with $W_1(j\omega) \equiv 0$.
3) Multimodel uncertainty can be directly taken into account in the proposed approach. We only should repeat the constraints for each model in the model set.
4) The robust performance can be improved by defining the following constraint :
$$\||W_1\mathcal{S}| + |W_2\mathcal{T}|\|_\infty < \gamma \tag{15}$$
and minimizing γ. In the proposed approach an upper bound for γ can be computed by an iterative bisection algorithm. At each iteration for a fixed γ_i, we replace W_1 and W_2 with W_1/γ_i and W_2/γ_i and we solve the feasibility problem represented by the linear constraints in (10) or convex constraints in (14). If the problem is feasible γ_{i+1} will be chosen smaller than γ_i and if the problem is infeasible γ_{i+1} will be increased.

B. Optimization criterion

Up to now, it has been shown that the robust performance condition can be represented in the Nyquist diagram by a set of linear or convex constraints. Hence, fixed-order robust performance control problem becomes a feasibility problem with linear or convex constraints. The major drawback of the proposed approach with respect to the standard H_∞ control problem is the need for a desired open-loop frequency function $L_d(j\omega)$. However, it should be noted that the performance specification is defined by the weighting frequency function $W_1(j\omega)$ and $L_d(j\omega)$ plays only an intermediate role to reduce the conservatism of the solution and not the solution itself. This means that even without knowing $L_d(j\omega)$, a straight line with a fixed slope for all frequencies can divide the Nyquist plane into two half planes and leads to a set of linear constraints for robust performance [6]. Therefore, $L_d(j\omega)$ just adjusts the slope of d_k to enlarge the set of admissible controllers defined by the constraints. As a result, a non properly chosen $L_d(j\omega)$ may lead to a infeasible solution. By a non properly chosen $L_d(j\omega)$ we mean a frequency function which is not coherent with the performance specification (with a bandwidth much larger than that specified by W_1) and is far from achievable for the plant model with given uncertainty set and restricted order and structure of the controller. For example, if we consider an integrator in the controller but we do not put it in $L_d(j\omega)$ we will have evidently a non properly chosen $L_d(j\omega)$. A suitable choice of $L_d(j\omega)$ is a simple choice that satisfies the Nyquist stability criterion and has essentially the poles on the imaginary axis of the controller and the plant model.

If $L_d(j\omega)$ is chosen such that it represents some desired control specifications, then it is judicious to minimize a norm of $L - L_d$ under the robust performance constraints. We propose either a quadratic programming approach in which an approximation of the two norm of $L - L_d$ is minimized under some linear constraints :

$$
\begin{aligned}
&\min_{\rho} \sum_{k=1}^{N} |\rho^T\phi(j\omega_k)G(j\omega_k) - L_d(j\omega_k)|^2 \\
&\text{Subject to:} \\
&|W_1(j\omega_k)[1+L_d(j\omega_k)]| - I_m\{L_d(j\omega_k)\}\rho^T\mathcal{I}_i(j\omega_k) - \\
&[1+R_e\{L_d(j\omega_k)\}][1+\rho^T\mathcal{R}_i(j\omega_k)] < 0 \\
&\text{for} \quad k=1,\ldots,N \quad \text{and} \quad i=1,\ldots,m
\end{aligned}
\tag{16}
$$

or a convex optimization approach in which an approximation of the infinity norm of $L - L_d$ is minimized under some convex constraints :

$$
\begin{aligned}
&\min_{\rho} \max_{k} |\rho^T\phi(j\omega_k)G(j\omega_k) - L_d(j\omega_k)| \\
&\text{Subject to:} \\
&|W_1(j\omega_k)[1+L_d(j\omega_k)]| - I_m\{L_d(j\omega_k)\}\rho^T\mathcal{I}(j\omega_k) + \\
&|W_2(j\omega_k)\rho^T\phi(j\omega_k)G(j\omega_k)[1+L_d(j\omega_k)]| - \\
&[1+R_e\{L_d(j\omega_k)\}][1+\rho^T\mathcal{R}(j\omega_k)] < 0 \\
&\text{for} \ \ k=1,\ldots,N
\end{aligned}
\tag{17}
$$

It is interesting to notice that a large value of the criterion for the optimal solution shows that the choice of $L_d(j\omega)$ has not been appropriate and with a better choice better performance may be achieved. Based on this observation a practical algorithm for improving the control performance can be suggested. We can start with a simple $L_d(j\omega)$ and compute a first controller, say $K_0(s)$, then we can compute a new $L_d(j\omega)$ equal to $K_0(j\omega)G(j\omega)$ and run the optimization problem with tighter specifications (e.g. larger $|W_1|$). In this new optimization the conservatism is significantly reduced because L and L_d and consequently d_k^* and d_k are close to each other at all frequencies.

C. Unstable systems

One of the main interest of the proposed approach with respect to other frequency-domain methods is that it can be applied to the unstable systems. The essential condition is that the desired open-loop frequency function $L_d(j\omega)$ should satisfy the Nyquist stability criterion. It means that $L_d(j\omega)$ has to encircle the critical point n_p times, where n_p is the number of unstable poles of $G(s)$ (knowing that the controller $K(s)$ has no poles in the right half plane). Under this condition, $L(j\omega)$ will encircle the critical point n_p times too if the constraints in (10) or (14) are satisfied. The reason is as follows : if $L_d(j\omega)$ encircles n_p times the critical point, then the vector $1 + L_d(j\omega)$ and d_k which is orthogonal to this vector will turn n_p times around the critical point. Hence, since $L(j\omega)$ and all models in the uncertainty circle are always in the side of d_k that excludes the critical point, they will also encircle the critical point n_p times.

If the unstable poles of the plant model are known, a good choice of $L_d(j\omega)$ includes these poles. If these poles are unknown, $L_d(j\omega)$ should contain the same number of unstable poles as the plant model. Finally, if a stabilizing controller $K_0(s)$ is known, an appropriate choice is $L_d(j\omega) = K_0(j\omega)G(j\omega)$. In this case, L_d does not represent a *desired* open-loop transfer function so it is not necessary to minimize a norm of $L - L_d$ in the optimization problem and only a feasibility problem can be solved instead.

IV. Simulation results

This example is taken from [10] where a robust performance problem is defined for an unstable plant. Consider the family of plants described by the following multiplicative uncertainty model:

$$P(s) = \frac{(s+1)(s+10)}{(s+2)(s+4)(s-1)}[1 + W_2(s)\Delta(s)] \quad (18)$$

where

$$W_2(s) = 0.8\frac{1.1337s^2 + 6.8857s + 9}{(s+1)(s+10)} \quad (19)$$

The nominal performance is defined by $\|W_1\mathcal{S}\|_\infty < 1$ with :

$$W_1(s) = \frac{2}{(20s+1)^2} \quad (20)$$

The objective is to compute a controller $K(s)$ that optimizes the robust performance by minimizing γ in (15).

The standard H_∞ solution that solves an approximate problem leads to $\gamma_{\text{opt}} = 0.844$ for this problem with the controller $K(s) = N_\infty/D_\infty$ where

$$\begin{aligned} N_\infty = 7.409e6s^6 + 1.266e8s^5 + 6.335e8s^4 + 1.152e9s^3 \\ + 6.911e8s^2 + 5.442e7s + 9.37e5 \end{aligned} \quad (21)$$

and

$$\begin{aligned} D_\infty = s^7 + 9.07e5s^6 + 1.901e7s^5 + 1.043e8s^4 + 4.416e7s^3 \\ - 4.682e7s^2 - 4.962e6s - 1.262e5 \end{aligned} \quad (22)$$

This 7th-order controller is unstable and has a pair of complex conjugate poles very close to the imaginary axis.

Now, the proposed method is applied to design a PID controller represented by :

$$K(s) = [K_p, K_i, K_d][1, \frac{1}{s}, \frac{s}{1+T_f s}]^T$$

where the time constant of the derivative part of the PID controller T_f is set to 0.01 s. The frequency response of the model is computed at $N = 50$ logarithmically spaced frequency points between 10^{-3} and 10^3 rad/s. The uncertainty circle at each frequency is approximated by an outbounding polygon with $m = 8$ vertices. The plant model contains one unstable pole and the controller an integrator, so the desired open-loop transfer function is chosen as

$$L_d(s) = \beta\frac{s+1}{s(s-1)} \quad (23)$$

where $\beta > 1$ satisfies the Nyquist stability condition for $L_d(s)$. In this example, we choose $\beta = 2$. It should be noted that this choice of $L_d(s)$ is not compatible with desired performances so the difference between $L(s)$ and $L_d(s)$ will not be minimized. In order to obtain the controller giving the minimal value for γ, the bisection algorithm explained in Remark 4 is used with the linear constraints in (10) that leads to

$$\||W_1\mathcal{S}| + |W_2\mathcal{T}|\|_\infty = 0.7233$$

The resulting PID controller is :

$$K_0(s) = \frac{2.426s^2 + 6.675s + 11.11}{0.01s^2 + s} \quad (24)$$

It is interesting to observe that this PID controller gives better performance than the H_∞ controller. Moreover, it is stable and easily implementable on a real system. The performance can be further improved using a new $L_d(s)$ based on $K_0(s)$. With this new $L_d(s)$ the optimal controller is given by :

$$K(s) = \frac{3.416s^2 + 26.28s + 25.08}{0.01s^2 + s} \quad (25)$$

which leads to $\gamma_{\text{opt}} = 0.7213$.

In order to study the sensitivity of the solutions to the choice of $L_d(s)$, the value of β in (23) is changed from 2 to 97 with a step size of 5. For each value of β the minimum of γ is computed. The mean value of optimal γ's is 0.7549 and its standard deviation 0.0228. This shows that although the optimal solution depends on the choice of $L_d(s)$, it is not very sensitive to this choice. Moreover, the results obtained by this approach, whatever the choice of β between 2 and 97, are better than the standard H_∞ optimal solution.

V. Discussion

It should be mentioned that the problem of robust fixed-order controller design is a non-convex NP-hard problem and all solutions to this problem, including ours, are based on some approximations. For example, if we consider the standard H_∞ control problem for design of a fixed-order controller for a system with multiple models and frequency-domain uncertainty, we have the following approximations :

1) Approximation of the structured multimodel uncertainty with unstructured frequency-domain uncertainty.

2) Approximation of the frequency-domain uncertainty with a reduced-order weighting filter.
3) Approximation of the real robust performance condition in (4) with the condition given in (5).
4) Approximation of the resulting high-order controller with a fixed-order controller. In this operation, it is difficult to even guarantee the stability and performance for the reduced-order controller.

The proposed method considers directly the multimodel and frequency-domain uncertainty and designs directly a fixed-order controller. However, it seems that this method has some drawbacks which are discussed below :

1) The plant and uncertainties are defined only in N frequency points, so the performance and stability conditions are satisfied only in N points. It is clear that N should be sufficiently large such that the Nyquist diagram of $L(j\omega_k)$ is a good approximation of $L(j\omega)$. For discrete-time controller design, since the frequency domain is limited to the half of sampling frequency, by increasing N the quality of approximation can be improved. This will increase the number of constraints but will not make a serious problem for linear and quadratic programming methods which are able to deal with more than hundred thousand of linear constraints. For continuous-time controller design, the choice of N and the sampling frequency should be done cautiously. This will need some information about the plant and the desired closed-loop specifications.
2) The controller is linearly parameterized so the denominator of the controller is fixed and it should be chosen prior to design. In practice, some of the poles of the controller are usually fixed to achieve certain closed-loop performances. For example a pole at origin, an integrator, or a pair of complex poles in a certain frequency are fixed in order to reject the disturbances (internal model principle). Therefore, this condition is not restrictive for low-order controller design. For higher order controller design the use of a set of orthogonal basis function is proposed. It is known that by increasing the controller order any stable transfer function can be approximated with such a set. On the other hand, this restriction ensures the stability of the controller which is required in many applications and cannot be guaranteed by a full controller parameterization.
3) The robust performance condition in (4) is approximated by a set of linear constraints in (10) or convex constraints in (14). It is discussed in the paper that the quality of this approximation depends on the choice of a desired open-loop transfer function.

It is too difficult (if not impossible) to compare, by a theoretical analysis, the overall approximation or conservatism of different approaches to fixed-order controller design. In this paper we tried to show the effectiveness of the proposed approach by means of a simulation example and compare it with the standard H_∞ method.

VI. Conclusions

A new fixed-order robust controller design method in the Nyquist diagram for spectral models has been developed. The method is based on an approximation of the robust performance condition in the H_∞ framework that leads to linear or convex constraints with respect to linearly parameterized controllers. The advantages of this approach are summarized below:

1) The method uses only the frequency response of the system and no parametric model is required. The frequency response of the model and the uncertainty at each frequency can be obtained directly by discrete Fourier transform from a set of periodic data, so the method can be considered as completely "data-driven". Of course, the method can be applied as well if a parametric model with an uncertainty set is available.
2) The method is very simple, at least as simple as open-loop shaping methods in Bode diagram or in Nichols chart currently used in textbooks for undergraduate courses in control systems. For instance, it can be used to design of PID controllers ensuring a given modulus margin and optimizing for a desired crossover frequency by a quadratic programming optimization approach. Moreover, the case of multimodel uncertainty can be handled easily just by increasing the number of linear constraints while the mentioned classical frequency-domain approaches cannot deal with this type of uncertainty.
3) Higher order controllers for unstable systems with H_∞ type specifications can also be designed within the same framework.

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