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Optimal Control of Linear Quantum Systems despite Feedback Delay

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Abstract— **In this paper, we investigate an optimal control problem of linear quantum systems despite feedback delays. The optimal controller, which is of the Smith predictor form, and an analytical expression of the best achievable performance are derived by applying existing results for control of classical (non-quantum) I/O delay systems. Then, we analyze the performance degrading effect caused by feedback delays in an illustrative example of a quantum free particle. In particular, we give a new insight for a typical experiment setup. This is accomplished by using the performance limit expression mentioned above.**

I. INTRODUCTION

Control of quantum systems is of significance for realizing quantum and nano technologies such as quantum computer. In particular, recent technological advances in quantum optics and atomic physics enable us to implement real-time feedback control of quantum systems. This has given birth to wide-ranging theoretical studies on quantum feedback control [2], [3], [5], [7]-[9], [11], [14] [16]-[18].

In this paper, we investigate feedback control of quantum systems whose dynamics are described by linear quantum stochastic differential equations. For this problem, Belavkin has derived the optimal filter and LQG controller; see e.g., [3]. Similarly to non-quantum cases, robust performance analysis/synthesis is necessary to implement control systems in a realistic environment. An important feature of control problems of linear quantum systems lies on the fact we can directly apply control theory for classical linear stochastic systems. In fact, H^∞ control theory [9], LEQG [8] and robust LQG control [16] have already been extended to quantum control problem settings.

Ideally we would design a controller based on these theoretical result. However, in the actual implementation, there are some drawbacks to be taken into account. For example, nano-mechanical dynamic systems have very fast dynamics, with time constants orders of magnitude less than the time necessary to compute the control input. From a practical viewpoint, this means that we need to formulate the control problem stated above taking feedback delays into consideration. The authors have analyzed the effect of delays in quantum spin control systems [11]. However, there exists no result which investigate the effect of delays in linear quantum feedback control systems.

In view of this, we solve an optimal control problem for linear quantum systems despite feedback delays. To be more precise, the optimal controller, which is of the Smith predictor form, is derived by using existing results for control of classical (non-quantum) I/O delay systems [12], [13].

It should be mentioned that an analytical expression of the performance degradation effect due to the delay can be achieved for some linear optimal control problems [12], [13]. In the classical case, this expression is often useful for characterizing *easily controllable plants* [6], [10]. By applying this way of thinking to quantum cases, we analyze the effect of the tuning of a parameter in a measurement apparatus. In [17], an optimal tuning policy was proposed for the delay free case. However the performace limit analysis enables us to conclude that the parameter tuning is not effective for some systems when the delay length is large.

This paper is organized as follows: linear quantum systems are introduced in the next section. In Section III, we state the control problem dealt with in this paper, and derive its optimal solution. In Section IV, we give numerical simulations and performance limitation analysis for the control of a quantum free particle.

NOTATION: Let $L^2_{n\times m}$ be the set of $\mathbb{R}^{n\times m}$-valued function f such that

$$\int_0^\infty \mathrm{trace}(f^\mathsf{T} f)(t)dt < \infty.$$

The subscript $n\times m$ is omitted as it is clear from the context. Function space L^2 is Hilbert space with the inner product defined by

$$(f,g)_{L^2} := \int_0^\infty \mathrm{trace}(g^\mathsf{T} f)(t)dt. \qquad (1)$$

II. LINEAR QUANTUM SYSTEMS

A. Quantum probability

To define the stochastic behavior of quantum systems, we need to introduce quantum probability space $(\mathscr{A},\mathbb{P})$, which is a noncommutative generalization of classical probability space $(\Omega,\mathcal{F},\mathbf{P})$. Here, $\mathscr{A}$ is a von Neumann algebra and $\mathbb{P}$ is a state, a linear functional on $\mathscr{A}$. In the following sections, we treat composite systems of a target quantum system and an environmental field. Physically interesting operators are defined as quantum random variables in $(\mathscr{A},\mathbb{P})$, and the statistical argument is taken by the state $\mathbb{P}$. For instance, we define the expectation by

$$\mathbb{E}[x] = \mathbb{P}[x]. \qquad (2)$$

For any self-adjoint operators, $\mathbb{E}[x]$ is a real number. Also, $\mathbb{E}[x]$ is positive for any positive operator.

The main difference between quantum and classical probability spaces is that the quantum probability space can define

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random variables which do not commute. It is well-known that position and momentum operators in quantum mechanics satisfy the canonical commutation relation $[q,p]=qp-pq=i$. This equality corresponds to the fact we cannot determine both values simultaneously according to the Heisenberg uncertainty principle. This implies that position and momentum operators as random variables cannot be modeled in a common classical probability space. For details of quantum probability and stochastic calculus, see the review paper [1].

B. Linear quantum systems

The state variable

$$x_t := [x_{1,t}, \cdots, x_{n,t}]^\mathsf{T}$$

is a vector consisting of self-adjoint operators $x_{k,t}$ and the initial state $x_0 = x$ satisfies the commutation relation

$$[x, x^\mathsf{T}] := xx^\mathsf{T} - (xx^\mathsf{T})^\mathsf{T} = i\Theta, \tag{3}$$

where $i = \sqrt{-1}$ and Θ is a real antisymmetric matrix. Further, assume the state variables satisfy

$$[x_t, x_t^\mathsf{T}] = i\Theta \tag{4}$$

for any t. This always holds for real physical systems which do not interact with other environmental fields. Then, a large class of linear quantum systems is described by linear quantum stochastic differential equations

$$\begin{aligned} dx_t &= Ax_t dt + B_1 dw_t + B_2 u_t dt \\ dy_t &= Cx_t dt + D dw_t \end{aligned} \tag{5}$$

where A is a real $\mathbb{R}^{n\times n}$ matrix and B_1, B_2, C and D are all real matrices of proper dimension. Moreover, w_t is a quantum noise vector consisting of quantum Wiener processes and satisfies

$$dw_t dw_s^\mathsf{T} = \begin{cases} F_w dt, & \text{if } s = t \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

with a non-negative Hermitian matrix F_w such that

$$\frac{1}{2}(F_w + F_w^\mathsf{T}) = I \tag{7}$$

and $\mathbb{E}[w_{i,t}] = 0$ for each element $w_{i,t}$ of w_t. See [9] for an equivalent algebraic condition on system matrices for (4) to hold. In the following sections, we attempt to control this linear quantum system by using controllers implemented by classical devices[1].

C. Physical example

Let us give an illustrative example to provide a concrete image of our control systems. A system to be considered is a single one-dimensional free particle trapped in a harmonic potential. The quantum state of this particle is represented by the position operator q and momentum operator p. These noncommutative operators satisfy the canonical commutation relation

$$[q, p] = i. \tag{8}$$

[1]A mathematical representation for this assumption is $[y_s, y_t^\mathsf{T}] = 0$, $[y_s, x_s^\mathsf{T}] = 0$ for any $s \le t$; see e.g., [1], [9].

The particle interacts with a vacuum electromagnetic field to extract a position information by homodyne detection. When we input a linear potential to control, the system Hamiltonian is given by

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 - u_t q, \tag{9}$$

where m and ω are the mass of the particle and an angular frequency of the harmonic potential, respectively. When the interaction between the system and the probe field is described by the Hudson-Parthasarathy equation [1], the time evolution of the operators q and p is given by

$$\begin{aligned} d\begin{bmatrix} q_t \\ p_t \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{bmatrix} \begin{bmatrix} q_t \\ p_t \end{bmatrix} dt \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & -\sqrt{M} \end{bmatrix} dw_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t dt. \end{aligned} \tag{10}$$

The output equation with perfect detection efficiency is

$$dy_t = \begin{bmatrix} 2\sqrt{M} & 0 \end{bmatrix} \begin{bmatrix} q_t \\ p_t \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \end{bmatrix} dw_t. \tag{11}$$

Here, a real constant M is a measurement strength and the quantum noise vector satisfies

$$dw_t dw_s^\mathsf{T} = \begin{cases} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} dt, & \text{if } s = t \\ 0, & \text{otherwise.} \end{cases}$$

We can see equation (7) holds.

III. Optimal controller and performance limit

A. Problem statement

As depicted in Figure 1, we investigate the feedback loop consisting of a plant, which is a quantum mechanical system, and a controller implemented by classical (non-quantum) devices. Due to the reason described in Section I, we consider the optimal control problem of linear quantum systems taking into account feedback delays. The real constants h_1 and h_2 are the length of the time delays in the input and output path of the physical controller.

To evaluate the system performance, we define an evaluation output z_t. Note that we can encompass the two delays into one delay with the length $h = h_1 + h_2$. Then, we give a solution for the following:

Problem 1: Consider the linear quantum system

$$\begin{aligned} dx_t &= Ax_t dt + B_1 dw_t + B_2 u_{t-h} dt \\ z_t &= C_1 x_t + D_{12} u_{t-h} \\ dy_t &= C_2 x_t dt + D_{21} dw_t \end{aligned} \tag{12}$$

where $h \ge 0$ is the delay length and quantum Wiener process w_t is independent of the initial condition. Then, find the causal, linear and time-invariant controller (from $\{y_s\}_{s\le t}$ to u_t) which makes $\lim_{t\to\infty} \mathbb{E}\left[\xi_t^\mathsf{T}\xi_t\right]$ exist for any internal variable ξ_t (hereafter we say internally stabilizing) and minimizes the cost functional

$$J := \lim_{t\to\infty} \mathbb{E}\left[z_t^\mathsf{T} z_t\right]. \tag{13}$$

We impose the following standard assumption:

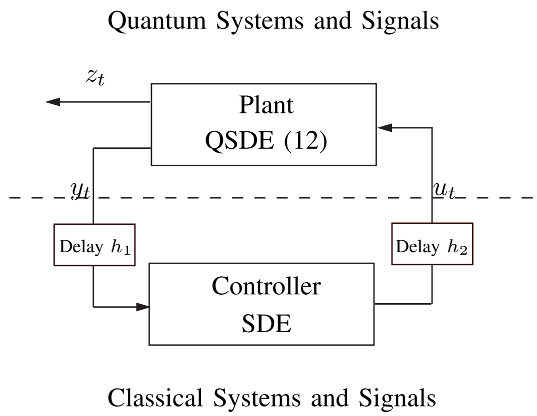

Fig. 1. Control of quantum systems by classical (non-quantum) controllers

Assumption 1:

1) (A, B_2) is stabilizable and (A, C_2) is detectable.
2) For any $\zeta \in \mathbb{R}$,
$$\begin{bmatrix} A - j\zeta I & B_2 \\ C_1 & D_{12} \end{bmatrix}, \begin{bmatrix} A - j\zeta I & B_1 \\ C_2 & D_{21} \end{bmatrix}$$
are row- and column-full rank, respectively.
3) $E_1 := D_{12}^\mathsf{T} D_{12}$ and $E_2 := D_{21} D_{21}^\mathsf{T}$ are nonsingular.

B. Optimal control

When the time delay can be neglected, the problem is called LQG optimal control problem and the *optimal* controller has been solved in both classical and quantum cases [3], [19]. To solve Problem 1, we first derive *suboptimal* controllers for the delay free case.

Theorem 1: Consider Problem 1 with $h = 0$. Let X, Y be the solutions of matrix Riccati equations

$$XA + A^\mathsf{T} X + C_1^\mathsf{T} C_1 - F^\mathsf{T} E_1 F = 0, \quad (14)$$
$$YA^\mathsf{T} + AY + B_1 B_1^\mathsf{T} - LE_2 L^\mathsf{T} = 0 \quad (15)$$

with

$$F := -E_1^{-1}(B_2^\mathsf{T} X + D_{12}^\mathsf{T} C_1) \quad (16)$$
$$L := -(YC_2^\mathsf{T} + B_1 D_{21}^\mathsf{T})E_2^{-1} \quad (17)$$

such that $A+B_2F$ and $A+LC_2$ are stable. Then, the optimal value of cost functional $E^* := \min J$ is given by

$$E^* = \mathrm{tr}(B_1 B_1^\mathsf{T} X) + \mathrm{tr}(F^\mathsf{T} D_{12}^\mathsf{T} D_{12} F Y). \quad (18)$$

Let $\gamma > E^*$ be any prespecified performance level. Then, all controllers satisfying $J < \gamma$ are given by

$$d\hat{x}_t = (A + LC_2 + B_2 F)\hat{x}_t dt - L dy_t + B_2 \xi_t dt \quad (19)$$
$$u_t = F\hat{x}_t + u_{K,t} \quad (20)$$
$$d\eta_t = -C_2 \hat{x}_t dt + dy_t \quad (21)$$
$$\xi_t = \int_0^t \psi(t-\tau) d\eta_\tau \quad (22)$$

where ψ is any function in L^2 such that

$$\|\psi\|_{L^2}^2 < \gamma - E^*. \quad (23)$$

Proof: In the non-quantum case, the *standard H^2 control problem* is usually proven via frequency domain approach. In order to avoid naive discussion on Lapalace transformability of quantum variables, we prove this theorem within the time-domain framework. For simplicity, we consider finite-dimensional controllers only.

Since the linearity of the plant and controller dynamics, z_t can be represented by

$$dx_{cl,t} = A_{cl} x_{cl,t} dt + B_{cl} dw_t \quad (24)$$
$$z_t = C_{cl} x_{cl,t} \quad (25)$$

where $x_{cl,t} := \begin{bmatrix} x_t & x_{K,t} \end{bmatrix}^\mathsf{T}$ with the internal state $x_{K,t}$ of the controller and A_{cl}, B_{cl}, C_{cl} are real matrices of appropriate dimensions. Therefore,

$$x_{cl,t} = e^{A_{cl}t} x_{cl,0} + \int_0^t \bar{K}(t-s) dw_s \quad (26)$$

with

$$\bar{K}(t) := e^{A_{cl}t} B_{cl}.$$

It should be emphasized that z_t is a quantum (noncommutative operator valued) stochastic process.

By using quantum Ito's rule (6) and (7), we obtain

$$\begin{aligned}
\mathbb{E}[x_{cl,t}^T x_{cl,t}] &= \mathbb{E}\left[x_{cl,0}^\mathsf{T} e^{A_{cl}^\mathsf{T} t} e^{A_{cl}t} x_{cl,0}\right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_0^t dw_s^\mathsf{T} \bar{K}^\mathsf{T}(t-s)\bar{K}(t-\tau) dw_\tau\right] \\
&= \mathrm{tr}\left(e^{A_{cl}t}\mathbb{E}[x_{cl,0}x_{cl,0}^\mathsf{T}]e^{A_{cl}^\mathsf{T} t}\right) \\
&\quad + \int_0^t \mathrm{tr}\left((\bar{K}^\mathsf{T}\bar{K})(s)F_w\right) ds \\
&= \mathrm{tr}\left(e^{A_{cl}t}\mathbb{E}[x_{cl,0}x_{cl,0}^\mathsf{T}]e^{A_{cl}^\mathsf{T} t}\right) \\
&\quad + \frac{1}{2}\int_0^t \mathrm{trace}((\bar{K}^\mathsf{T}\bar{K})F_w + (\bar{K}^\mathsf{T}\bar{K})F_w^\mathsf{T})(s) ds \\
&= \mathrm{tr}\left(e^{A_{cl}t}\mathbb{E}[x_{cl,0}x_{cl,0}^\mathsf{T}]e^{A_{cl}^\mathsf{T} t}\right) \\
&\quad + \int_0^t \mathrm{tr}\left(\bar{K}^\mathsf{T}\bar{K}\right)(s) ds.
\end{aligned}$$

Clearly, the system is internally stable if and only if $e^{A_{cl}t} \to 0$ and $\bar{K} \in L^2$. Thus, since the cost functional is

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}[z_t^\mathsf{T} z_t] &= \lim_{t\to\infty} \mathrm{tr}\left(C_{cl}e^{A_{cl}t}\mathbb{E}[x_{cl,0}x_{cl,0}^\mathsf{T}]e^{A_{cl}^\mathsf{T} t}C_{cl}^\mathsf{T}\right) \\
&\quad + \lim_{t\to\infty}\int_0^t \mathrm{tr}\left(K^\mathsf{T} K\right)(s) ds
\end{aligned}$$

with

$$K(t) = C_{cl}\bar{K}(t).$$

Problem 1 is equivalent to that of finding the controller which minimizes $\|K\|_{L^2}$ subject to the constrain $e^{A_{cl}t} \to 0$. The latter problem is free from any noncommutative variables, and exactly the same as the standard control problem for the classical case. Hence, the remaining of the proof is the same as that in [19]. ■

The obtained suboptimal controller parameterization is exactly the same as that for classical linear systems. By

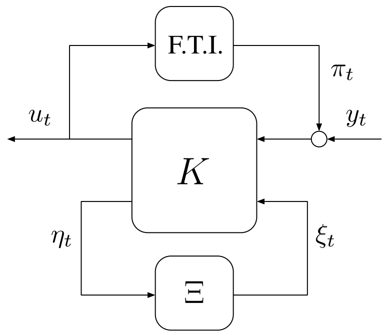

Fig. 2. Parametrization of all suboptimal controllers

choosing free L^2 function ψ appropriately, we can obtain controllers which are implementable inspite of the I/O delay.

Theorem 2: Consider Problem 1. With the same notation as that in Theorem 1, the optimal value of the cost functional J is given by

$$E_h := E^* + \int_0^h \mathrm{tr}(Fe^{A\tau}LL^\mathsf{T}e^{A^\mathsf{T}\tau}F^\mathsf{T})d\tau$$

Let $\gamma > E_h$ be any prespecified performance level. Then, all controllers satisfying $J < \gamma$ are given by

$$\begin{aligned} d\hat{x}_t &= (A + B_2F + e^{Ah}LC_2e^{-Ah})\hat{x}_t dt \\ &\quad - e^{Ah}L(dy_t + \pi_t dt) + B_2\xi_t dt && (27) \\ u_t &= F\hat{x}_t + \xi_t && (28) \\ d\eta_t &= -C_2e^{-Ah}\hat{x}_t dt + (dy_t + \pi_t dt) && (29) \\ \xi_t &= \int_0^t \Xi(t-\tau)d\eta_\tau && (30) \end{aligned}$$

and the finite-time integration system

$$\pi_t = C_2 \int_{t-h}^t e^{A(t-h-\tau)}B_2u_\tau d\tau \tag{31}$$

where $\Xi \in L^2$ is any function satisfying

$$\|\Xi\|_{L^2}^2 < \gamma - E_h.$$

The optimal controller which satisfies $J = E_h$ is given by $\Xi = 0$.

Proof: It is known ([12]) that the suboptimal controllers are causal despite feedback delay if and only if ψ satisfies

$$\psi(t) = Fe^{At}L, \quad 0 \le t < h. \tag{32}$$

When $\psi \in L^2$ satisfies (32), the suboptimal controllers given in Theorem 1 can be rewritten in the desired form. ■

IV. Control of one-dimensional free particle

A. Simulation

In this section, we consider feedback control of a single one-dimensional free particle given in Section II-C, the system with the matrices

$$A = \begin{bmatrix} 0 & \frac{1}{m} \\ -m\omega^2 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\sqrt{M} \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 2\sqrt{M} & 0 \end{bmatrix}, D_{21} = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{33}$$

The control objective is to stabilize the particle position and momentum at the origin with small error variance. We apply the optimal control law derived in Section III-B with

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; D_{12} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{34}$$

and with the time delay of the length h in the output path of the physical controller.

Since it is impossible to perform numerical simulation of noncommutative operator-valued dynamics, we use the so-called *stochastic master equation*. According to [1], the equation is given by

$$\begin{aligned} d\rho_t &= -i\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2q^2, \rho_t\right]dt + iu_{t-h}[q,\rho_t]dt \\ &\quad + M\left(q\rho_t q - \frac{1}{2}q^2\rho_t - \frac{1}{2}\rho_t q^2\right)dt \\ &\quad + \sqrt{M}\{q\rho_t + \rho_t q - 2\mathrm{tr}(q\rho_t)\rho_t\}dW_t, \end{aligned} \tag{35}$$

where ρ_t is a conditional density operator which has the best statistical information available of the quantum systems and W_t is a one-dimensional classical Wiener process. The output equation of the system is given by

$$dy_t = 2\sqrt{M}\mathrm{tr}(q\rho_t)dt + dW_t. \tag{36}$$

Note that ρ_t is an operator, and that the simulation of (35) is still complex. However, since the system Hamiltonian is quadratic in terms of position and momentum operators and (35) preserves the Gaussian nature of the state, the conditional expectations $\hat{q}_t = \mathrm{tr}(q\rho_t)$, $\hat{p}_t = \mathrm{tr}(p\rho_t)$ and error covariances $\Sigma_t = (\Sigma_t^{ij})$:

$$\Sigma_t^{ij} := \frac{1}{2}\mathrm{tr}\left(\rho_t(x^ix^j + x^jx^i)\right) - \bar{x}_t^i\bar{x}_t^j \tag{37}$$

suffice to describe the system state with $x^1 := q$ and $x^2 := p$. Using (35), we obtain the following linear equation describing the time evolution of $\hat{q}_t$ and $\hat{p}_t$.

$$d\begin{bmatrix} \hat{q}_t \\ \hat{p}_t \end{bmatrix} = A\begin{bmatrix} \hat{q}_t \\ \hat{p}_t \end{bmatrix}dt + B_2u_{t-h}dt + \Sigma C_2^\mathsf{T}dW_t \tag{38}$$

Here Σ is the best achievable error covariance and is the stabilizing solution of the matrix Riccati equation

$$A\Sigma + \Sigma A^\mathsf{T} + B_1B_1^\mathsf{T} - \Sigma C_2^\mathsf{T}C_2\Sigma^\mathsf{T} = 0. \tag{39}$$

In the simulation, $\hat{q}_t$, $\hat{p}_t$ are considered as the plant variables.

Under the above setting, the simulation results are shown in Fig. 3. We chose the system parameters as $m = \omega = \alpha = 1, h = 0.8$. In Fig. 3, the solid and dot line represent the state trajectories with respect to the optimal controllers in Theorem 2 and 1, respectively. When the controller (19)-(20) is used, while it is optimal one for delay free case, the stabilization fails. On the other hand, the controller (27)-(28) guarantees the stability to the origin. Thus we can see the effectiveness of the delay compensating controller.

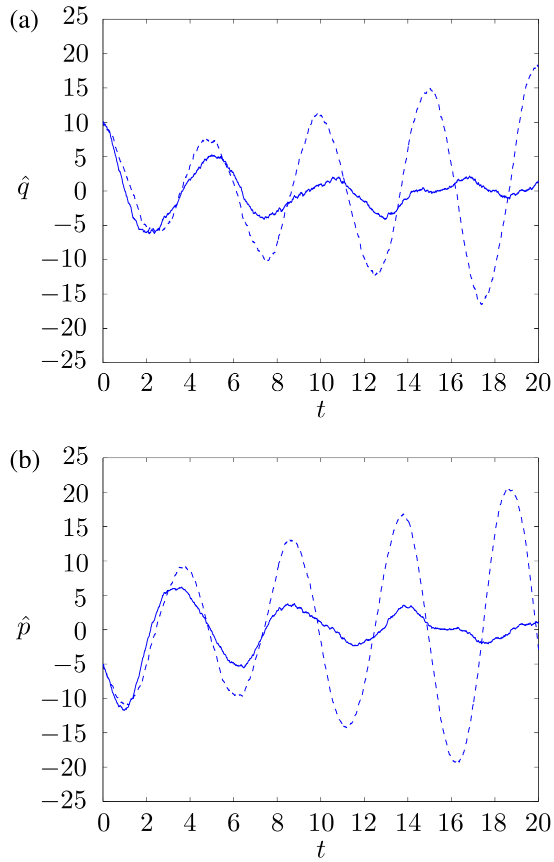

Fig. 3. Time evolution of the (a) expected position and (b) expected momentum of the free particle subject to feedback delay. Solid line and dashed line represent the state trajectories when using the optimal controller (27)-(28) and (19)-(20), respectively.

B. Optimization of a parameter in detector device

First, we provide a parameter adjustable measurement scheme. Homodyne detector is a measurement apparatus which enables to measure the field quadratures, in our case, of a probe lazer. In Section II-C, the measurement process was implicitly assumed to be the homodyne detection with a special parameter ($\phi = 0$ defined below). In more general settings, the output equation of (11) is given by ([1], [18])

$$dy_t = \begin{bmatrix} 2\sqrt{M} & 0 \end{bmatrix} \begin{bmatrix} q_t \\ p_t \end{bmatrix} dt + \begin{bmatrix} 1 & 0 \end{bmatrix} d\tilde{w}_t \tag{40}$$

where

$$d\tilde{w}_t d\tilde{w}_s^{\mathsf{T}} = \begin{cases} \begin{bmatrix} 1 & ie^{-i\phi} \\ -ie^{i\phi} & 1 \end{bmatrix} dt, & \text{if } s = t \\ 0, & \text{otherwise.} \end{cases} \tag{41}$$

Here $\phi \in [0, 2\pi)$ is a detector parameter which designers can change. Since the quantum noise matrix satisfies

$$\begin{aligned} S_\phi &:= \frac{1}{2}\left(\begin{bmatrix} 1 & ie^{-i\phi} \\ -ie^{i\phi} & 1 \end{bmatrix} + \begin{bmatrix} 1 & ie^{-i\phi} \\ -ie^{i\phi} & 1 \end{bmatrix}^{\mathsf{T}} \right) \\ &= \begin{bmatrix} 1 & \sin\phi \\ \sin\phi & 1 \end{bmatrix}, \end{aligned} \tag{42}$$

the quantum noise $d\tilde{w}_t$ introduce here does not satisfy (7) when $\phi \neq 0$ because $S_\phi \neq I$. However, all of the results in Section III are applicable to the general homodyne detection scheme by modifying the system matrices as follows [16]:

$$B_{1,\phi} = B_1 S_\phi^{1/2}, \; D_{21,\phi} = D_{21} S_\phi^{1/2}. \tag{43}$$

where $S_\phi^{1/2}$ is the square matrix of the positive semi-definite matrix S_ϕ.

In the paper [17], it was shown that the achievable performance in the quantum LQG control depends on the measurement, and that the optimal measurement can be identified by solving a semidefinite program. This fact is distinctive in feedback control of quantum systems and is a consequence of the property of the quantum noise. The following theorem clarifies the effect of the feedback delays on the optimal homodyne detection.

Theorem 3: Consider Problem 1 with the system matrices defined by (33), (43). Then there exist constants $\mathsf{A}, \mathsf{B}, \mathsf{E}$ and ϑ such that the best achievable performance $E_{h,\phi}$ is given by

$$E_{h,\phi} = \mathsf{E} + \mathsf{B}h + \mathsf{A}\sin(\omega h + \vartheta). \tag{44}$$

Moreover, A and B are independent of the choice of ϕ.

Proof: Notice that we have

$$\begin{aligned} B_{1,\phi} D_{21,\phi}^{\mathsf{T}} &= \begin{bmatrix} 0 \\ -\sqrt{M}\sin\phi \end{bmatrix}, \\ B_{1,\phi} B_{1,\phi}^{\mathsf{T}} &= B_1 B_1^{\mathsf{T}}, \\ D_{21,\phi} D_{21,\phi}^{\mathsf{T}} &= 1. \end{aligned}$$

Let Y_ϕ be the stabilizing solution to

$$Y_\phi^{\mathsf{T}} + AY_\phi + B_1 B_1^{\mathsf{T}} - L_\phi L_\phi^{\mathsf{T}} = 0 \tag{45}$$

with

$$L_\phi := -(Y_\phi C_2 + B_{1,\phi} D_{21,\phi}^{\mathsf{T}}).$$

Then, by Theorem 2, the best achievable performance is given by

$$E_{h,\phi} := E_\phi^* + \int_0^h (Fe^{A\tau} L_\phi)^2 d\tau \tag{46}$$

where E_ϕ^* is the positive constant defined by (18) with $Y = Y_\phi$. On the other hand, direct computation yields

$$\begin{aligned} &Fe^{A\tau} L_\phi \\ &= \sqrt{\left\{ l_1^2 + \left(\frac{l_2}{m\omega} \right)^2 \right\} \{ f_1^2 + (m\omega f_2)^2 \}} \sin(\omega\tau + \theta), \end{aligned} \tag{47}$$

where

$$F = \begin{bmatrix} f_1 & f_2 \end{bmatrix}, \; L_\phi = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

and θ satisfies

$$\tan\theta = \frac{m\omega(f_1 l_1 + f_2 l_2)}{f_1 l_2 - (m\omega)^2 f_2 l_1}.$$

By combining this and (46), we obtain the first claim.

It should be emphasized that ϕ contribute to A and B only through

$$l_1^2 + \left(\frac{l_2}{m\omega} \right)^2. \tag{48}$$

Hence, it is sufficient to show the second claim that (48) does not depend on ϕ. From the definition of L_ϕ,

$$l_1 = -2\sqrt{M}y_{11},\ l_2 = -\sqrt{M}(2y_{12} - \sin\phi)$$

where

$$Y_\phi = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix}.$$

Simple calculation yields

$$y_{11}^2 = \bar{y}_{11}^2 - \frac{1}{2mM}\left\{\left(\bar{y}_{12} + \frac{m\omega^2}{4M}\right) - \frac{1}{2}\sin\phi\right\} \pm \frac{1}{2mM}\Delta$$
$$y_{12} - \frac{1}{2}\sin\phi = -\frac{m\omega^2}{4M} \pm \Delta$$
$$\Delta = \sqrt{\left(\bar{y}_{12} + \frac{m\omega^2}{4M}\right)^2 - \frac{m\omega^2}{4M}\sin\phi}$$

where

$$Y_0 = \begin{bmatrix} \bar{y}_{11} & \bar{y}_{12} \\ \bar{y}_{12} & \bar{y}_{22} \end{bmatrix}.$$

Here the symbol $\pm$ represents that these equalities hold for $+$ or $-$. Then, we obtain the following:

$$\begin{aligned} l_1^2 + \left(\frac{l_2}{m\omega}\right)^2 &= 4My_{11}^2 + \left(\frac{2\sqrt{M}}{m\omega}\right)^2\left(y_{12} - \frac{1}{2}\sin\phi\right)^2 \\ &= 4M\left\{\bar{y}_{11}^2 + \left(\frac{\bar{y}_{12}}{m\omega}\right)^2\right\} \end{aligned}$$

This completes the proof. ■

To illustrate this theorem, performance limit $E_{h,\phi}$ is illustrated in Fig. 4 for $m = \omega = \alpha = 1$ and $\phi = 0,\ \pi/9,\ \pi/6$. Roughly speaking, the first statement says that $E_{h,\phi}$ increases *linearly* with respect to the delay length h. This is a natural result of the fact that A has only pure imaginary eigenvalues. Since E and θ depend on ϕ, we can improve the performance limit for any fixed delay (including delay free case). For example, $\phi = \pi/6$ achieves better performance than $\phi = 0$ for any h.

However, the second statement give us a new and nontrivial insight: *the growth rate* B *is independent of a homodyne detector parameter* ϕ. This implies that the benefits of the optimal measurement is lost when the delay length h is much larger than $1/\omega$.

V. Conclusion

In this paper, we investigated optimal control problem of linear quantum systems despite feedback delays. The optimal controller and an analytical expression of the best achievable performance are derived by applying existing results for control of classical (non-quantum) I/O delay systems. Based on this result, we gave a new insight for a typical experimental setup of controlling quantum free particles.

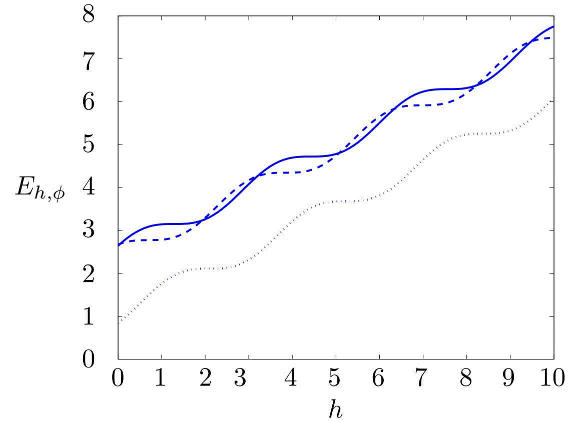

Fig. 4. Optimal performance deterioration with different detector parameters. The solid line, dashed and dotted line represent the optimal control performance when $\phi = 0$, $\phi = \pi/9$ and $\phi = \pi/6$, respectively.

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