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A Solution to the Tracking Control Problem for Switched Linear Systems with Time-Varying Delays

Qing-Kui Li, Georgi M. Dimirovski and Jun Zhao

***Abstract*— We investigate the tracking control problem for switched linear time-varying delays systems with stabilizable and unstabilizable subsystems. Sufficient conditions for the solvability of the tracking control problem are developed. The tracking control problem of a switched time-varying delays system with stabilizable and unstabilizable subsystems is solvable if the stabilizable and unstabilizable subsystems satisfy certain conditions and admissible switching law among them. Average dwell time approach and piecewise Lyapunov functional methods are utilized to the stability analysis and controller design. A simulation example shows the effectiveness of the proposed method.**

I. INTRODUCTION

Switched systems has attracted considerable attention due to the widespread application in control, communication network and biology engineering [3], [10]. These systems arise as models for phenomena which can be described by continuous or discrete time dynamics, and a rule specifying the switching among them. The motivation of such systems also arise from the better performance achieved via imposing a controller switching strategy [9], [10]. Stability analysis and control synthesis are two key problems in the study of switched systems. As useful tools, Lyapunov functions can deal with the stability problems for switched systems [1], [3], [9], [16], although certain switching laws incorporated with compatible information sometimes should be designed (see, e.g., [1], [5]).

On the other hand, time-delays, which are common phenomenon encountered in many engineering process, are known to be great sources of instability and poor performance. Therefore, how to deal with time delays has been a hot topic in the control area, see e.g., [4], [7] and [11]. For switched systems, because of the complicated behavior caused by the interaction between the continuous dynamics and discrete switching, the problem of time delays is more difficult to study. Only a few results have been reported in the literature such as the issues on stability analysis [13], [15], optimal control [14], and so on. The importance of the study of tracking control for switched systems with time-delays arises from the extensive applications in robot tracking control [17], guided missile tracking control, etc. However, to the authors' best knowledge, up to now, the issue of tracking control, which has been well addressed for non-switched systems without delay [12], has been rarely investigated for switched systems with time-delays. In paper [8], although the authors give a delay-dependent criteria, the results are based on the assumption that all the subsystems are stabilizable. Thus, the conditions given are restricted and the results are somewhat conservative.

We are interested in the tracking control problem for switched linear time-varying delays systems with stabilizable and unstabilizable subsystems. It is easy to find many applications involving such switched systems. For example, subsystems with actuators breakdown in the chemical process with multi-model cannot be discarded, one has to activate the healthy subsystems with dominant period.

In this paper, sufficient conditions for the solvability of the tracking control problem are developed. By introducing the integral controllers, some restricted assumptions imposing on the switched systems are avoided. The tracking control problem of a switched time-varying delays system with stabilizable and unstabilizable subsystems is solvable if the stabilizable and unstabilizable subsystems satisfy certain conditions and admissible switching law among them.

II. PROBLEM FORMULATION AND PRELIMINARIES

In this paper, we use $P > 0$ (< 0) to denote a positive (negative) definite matrix P, and $\lambda_{\max}(P)$ $(\lambda_{\min}(P))$ denote the maximum (minimum) eigenvalues of P. $\mathbb{R}^n$ denotes the n dimensional Euclidean space; $\mathscr{L}_2[0,\infty)$ is the space of square integrable functions on $[0,\infty)$. For given $\tau > 0$, let $\mathbb{R}_+ = [0,+\infty]$ and $C_n = C([-\tau,0],\mathbb{R}^n)$ be the Banach Space of continuous mapping from $([-\tau,0],\mathbb{R}^n)$ to $\mathbb{R}^n$ with topology of uniform convergence. Let $x_t \in C_n$ be defined by $x_t(\theta) = x(t+\theta), \theta \in [-\tau,0]$. $\|\cdot\|$ denotes the usual 2-norm and $\|x_t\|_{cl} = \sup_{-\tau\leq\theta\leq 0}\{\|x(t+\theta)\|, \|\dot{x}(t+\theta)\|\}$.

Consider the switched linear time-varying delays system

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + D_{\sigma(t)}x(t - d_{\sigma(t)}(t)) + B_{\sigma(t)}u(t), \\ x(t) = \phi(t),\ t \in [-\tau,0],\ x(0) = \phi(0) = 0, \\ y(t) = C_{\sigma(t)}x(t), \quad t \in [0,\infty), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$ and $y(t) \in \mathbb{R}^q$ are the state, the control input, and the output, respectively. $\phi(t)$ is the continuous vector valued function specifying the initial state of the system. The right continuous function $\sigma(t) : [0,\infty) \to \underline{N} \triangleq \{1,2,\cdots,N\}$ is the switching signal, corresponding to it, the switching sequence $\Sigma = \{x_0;(i_0,t_0),(i_1,t_1),\cdots,(i_j,t_j),\cdots|i_j \in \underline{N}\}$ means that

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Qing-Kui Li and Jun Zhao are with Key Lab of Process Industry Automation of Ministry of Education; School of Information Science and Engineering, Northeastern University, Shenyang, 110004, P. R. China zhaojun@ise.neu.edu.cn; sdlqk01@126.com

G. M. Dimirovski is with Dogus University, Faculty of Engineering, Istanbul, TR-34722, Turkey; and SS Cyril and Methodius University, Faculty of FEIT, Skopje, MK-1000, Macedonia gdimirovski@dogus.edu.tr

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the i_jth subsystem is active when $t \in [t_j, t_{j+1})$. For simplicity, we denote $\sigma := \sigma(t)$. A_i, D_i, B_i, and C_i, $(i \in \underline{N})$ are constant matrices of appropriate dimensions, $d_i(t)$ denote the time-varying delays satisfying the assumption below.

Assumption 1. $0 < d_i(t) \leq \tau$ for a constant τ, $i \in \underline{N}$.

Our purpose is to design a control law $u = u(t)$ and a class of switching signal $\sigma = \sigma(t)$, such that the output of system (1) tracks the reference input $y_r = r(t)$.

Consider the following integral controller

$$\begin{aligned} \dot{z}(t) &= C_\sigma x(t) - r(t), \\ u(t) &= K_\sigma x(t) + L_\sigma z(t), \end{aligned} \tag{2}$$

where $z(t)$, $r(t) \in \mathbb{R}^q$. Let $e(t) \triangleq y(t) - y_r(t)$, it is obvious that the controller contains the tracking error integral.

Augmenting system (1) with (2), we have

$$\begin{cases} \dot{x}(t) = (A_\sigma + B_\sigma K_\sigma)x(t) + D_{\sigma(t)}x(t - d_\sigma(t)) \\ \qquad\quad +B_\sigma L_\sigma z(t), \\ \dot{z}(t) = C_\sigma x(t) - r(t). \end{cases} \tag{3}$$

Let

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \ \bar{A}_\sigma = \begin{bmatrix} A_\sigma + B_\sigma K_\sigma & B_\sigma L_\sigma \\ C_\sigma & 0 \end{bmatrix},$$

$$\bar{D}_\sigma = \begin{bmatrix} D_\sigma & 0 \\ 0 & 0 \end{bmatrix}, \ \varpi(t) = \begin{bmatrix} 0 \\ -r(t) \end{bmatrix}.$$

Augmented system (3) can be rewritten as

$$\dot{\bar{x}}(t) = \bar{A}_\sigma \bar{x}(t) + \bar{D}_\sigma \bar{x}(t - d_\sigma(t)) + \varpi(t). \tag{4}$$

Definition 1 (cf. [2]). For system (1), assume the feedback controller (3) has been applied, then the closed-loop system is said to be stable if the resulting closed-loop system obtained with $\omega = 0$, $y_r = 0$ is stable. For the convenience of our talking, in this case, we say that system (1) is stabilizable.

In the development to follow, we take the following standard assumptions.

Assumption 2. Rank $\begin{bmatrix} A_i & B_i \\ C_i & 0 \end{bmatrix} = n + q$ for all $i \in \underline{N}$.

Assumption 3. For tracking control problem, suppose that not all the subsystems of system (1) are stabilizable.

Definition 2. The system (1) is said to be exponentially stabilizable under control law $u = u(t)$ and switching signal $\sigma = \sigma(t)$, if the solution $x(t)$ of switched system (1) through $(t_0, \phi) \in \mathbb{R}_+ \times C_n$ satisfies

$$\|x(t)\| \leq \kappa \|x_{t_0}\|_{cl} e^{-\lambda(t-t_0)}, \ \forall t \geq t_0$$

for some constants $\kappa \geq 0$ and $\lambda > 0$.

Definition 3. For augmented system (3), $\bar{K}_\sigma \triangleq [K_\sigma \ \ L_\sigma]$ is said to define an asymptotic (or exponential) switching tracking control for system (1), if the following conditions are satisfied:

(i) Internal stability. The system

$$\dot{\bar{x}}(t) = \bar{A}_\sigma \bar{x}(t) + \bar{D}_\sigma \bar{x}(t - d_\sigma(t))$$

is asymptotically (or exponentially) stable.

(ii) Asymptotic (or exponential) tracking. Given any initial state $x_0 \in R^n$ and $z_0 \in R^q$, and any $y_r \in \mathscr{Y}$, where $\mathscr{Y}$ is a set of $\mathbb{R}^q$-valued functions on $[0, \infty]$, then $y(t) \to y_r(t)$ as $t \to \infty$, where $y(t) = C_\sigma x(t)$ and $x(\cdot)$ is the solution of (3) with $x(0) = x_0$.

Definition 4. System (1) is said to satisfy weighted H_∞ tracking performance, if the following conditions are satisfied:

(i) Internal stability. Stated in definition 2.

(ii) Optimal tracking performance. Given the performance index as

$$J_L = \int_0^\infty e^{-\sigma t} \left[x^T(t) Q_1 x(t) + \left(\int_0^t e(t)dt \right)^T Q_2 \left(\int_0^t e(t)dt \right) + u^T(t) R u(t) \right] dt, \tag{5}$$

the tracking performance index V_L can be minimized and meet certain upper bound, where $Q_1 \in R^{n \times n}$ and $Q_2 \in R^{q \times q}$ are positive semidefinite matrices and $R \in R^{p \times p}$ is positive definite matrix.

Remark 1. For the switching tracking problem, asymptotic (or exponential) tracking is an ideal case for system (1), usually this ideal case cannot be achieved and thus one can consider the tracking performance.

Remark 2. As part of our construction of a switching tracking control, we shall specify the matrices K_i and L_i $(i \in \underline{N})$, note that the switching tracking controller involves dynamic compensation through the introduction of the vector $z(t)$. This is indispensable because it is not possible to use linear feedback control to achieve switching tracking without the introduction of such a compensator.

Remark 3. When a system satisfies Definition 3 or Definition 4, we say that the tracking control problem is solvable.

Definition 5[5]. For any $T_2 > T_1 \geq 0$, let $N_\sigma(T_1, T_2)$ denote the number of switching of $\sigma(t)$ over (T_1, T_2). If $N_\sigma(T_1, T_2) \leq N_0 + \frac{T_2 - T_1}{T_\alpha}$ holds for $T_\alpha > 0, N_0 \geq 0$, then T_α is called average dwell time.

III. Controller design and performance analysis

In this section, we will show how to design feedback gain K_i, L_i and switching law $\sigma(t) = i \, (i \in \underline{N})$, for switched time-varying delays system (1). We first consider the non-switched system,

$$\begin{cases} \dot{x}(t) = Ax(t) + Dx(t - d(t)) + \omega(t), \\ x(t) = \phi(t), \ t \in [-\tau, 0], \ x(0) = \phi(0) = 0, \\ y(t) = Cx(t), \quad t \in [0, \infty), \end{cases} \tag{6}$$

where A, D, C are constant matrices with appropriate dimensions. We have the following lemmas.

Lemma 1. Suppose that system (6) satisfies Assumption 1. For given positive constants α and γ, if there exist positive definite matrices P, S, matrix K, and any matrices Y, T, M with appropriate dimensions, such that

$$\begin{bmatrix} \varphi_{11} + Q & \varphi_{12} & \varphi_{13} & A^T S & -Y & K^T \\ * & \varphi_{22} & \varphi_{23} & D^T S & -T & 0 \\ * & * & \varphi_{33} & S & -M & 0 \\ * & * & * & -\tau^{-1} S & 0 & 0 \\ * & * & * & * & -\tau^{-1} e^{-\alpha\tau} S & 0 \\ * & * & * & * & * & -R^{-1} \end{bmatrix} < 0 \tag{7}$$

holds, for the Lyapunov functional candidate

$$V(t)=x^T(t)Px(t)+\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)e^{-\alpha(t-s)}S\dot{x}(s)dsd\theta \tag{8}$$

along the trajectory of the system (6), there hold the following inequalities

$$V(x_t)\le e^{-\alpha(t-t_0)}V(x_{t_0})-\int_{t_0}^{t}e^{-\alpha(t-s)}\Gamma(s)ds, \tag{9}$$

where Q is a given positive semidefinite matrix, and

$$\begin{aligned}\Gamma(s) &= x^T(s)Qx(s)+u^T(s)Ru(s)-\gamma^2\omega^T(s)\omega(s)\\ &\quad (u(s)=Kx(s)),\end{aligned} \tag{10}$$

$$\begin{aligned}&\varphi_{11}=A^TP+PA+\alpha P+Y^T+Y, && \varphi_{13}=M^T+P,\\ &\varphi_{12}=PD+T^T-Y, && \varphi_{23}=-M^T,\\ &\varphi_{22}=-T^T-T, && \varphi_{33}=-\gamma^2 I.\end{aligned}$$

Proof. By Schur complement lemma, the condition (7) is equivalent to the following inequality

$$\begin{bmatrix}\Omega_{11}+K^TRK+Q & \Omega_{12} & \Omega_{13} & -Y\\ * & \Omega_{22} & \Omega_{23} & -T\\ * & * & \Omega_{33} & -M\\ * & * & * & -\tau^{-1}e^{-\alpha\tau}S\end{bmatrix}<0. \tag{11}$$

where

$$\begin{aligned}&\Omega_{11}=\varphi_{11}+\tau A^TSA, && \Omega_{13}=\varphi_{13}+\tau A^TS,\\ &\Omega_{12}=\varphi_{12}+\tau A^TSD, && \Omega_{23}=\varphi_{23}+\tau D^TS,\\ &\Omega_{22}=\varphi_{22}+\tau D^TSD, && \Omega_{33}=\varphi_{33}+\tau S.\end{aligned}$$

Multiplying both sides of (11) by symmetric matrix $diag(I,I,I,d(t)I)$, and noticing $0<d(t)\le\tau$, we have

$$\Theta\triangleq\begin{bmatrix}\Omega_{11}+K^TRK+Q & \Omega_{12} & \Omega_{13} & -d(t)Y\\ * & \Omega_{22} & \Omega_{23} & -d(t)T\\ * & * & \Omega_{33} & -d(t)M\\ * & * & * & -d(t)e^{-\alpha\tau}S\end{bmatrix}<0. \tag{12}$$

Differentiating the Lyapunov functional candidate (8) along the trajectory of (6) and noticing $d(t)\le\tau$, we have

$$\begin{aligned}\dot{V}(x_t)\le& 2x^T(t)P(Ax(t)+Dx(t-d(t))+\omega(t))\\ &+\tau\dot{x}^T(t)S\dot{x}(t)-\int_{t-d(t)}^{t}\dot{x}^T(s)e^{-\alpha\tau}S\dot{x}(s)ds\\ &-\alpha\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)e^{-\alpha(t-s)}S\dot{x}(s)dsd\theta.\end{aligned} \tag{13}$$

Note that

$$\begin{aligned}\tau\dot{x}^T(t)S\dot{x}(t)=&\,x^T(t)\tau A^TSAx(t)+2x^T(t)\tau A^TS\omega(t)\\ &+2x^T(t)\tau A^TSDx(t-d(t))\\ &+x^T(t-d(t))\tau D^TSDx(t-d(t))\\ &+2x^T(t-d(t))\tau D^TS\omega(t)+\omega^T(t)\tau S\omega(t).\end{aligned} \tag{14}$$

From the Leibniz-Newton formula, for any matrices Y,T,M with appropriate dimensions, we have

$$\begin{aligned}&2[x^T(t),x^T(t-d(t)),\omega(t)]\begin{bmatrix}Y\\ T\\ M\end{bmatrix}\\ &\times[x(t)-x(t-d(t))-\int_{t-d(t)}^{t}\dot{x}(s)ds]=0.\end{aligned} \tag{15}$$

From (10), we have

$$\Gamma(t)=\begin{bmatrix}x(t)\\ \omega(t)\end{bmatrix}^T\begin{bmatrix}Q+K^TRK & 0\\ 0 & -\gamma^2 I\end{bmatrix}\begin{bmatrix}x(t)\\ \omega(t)\end{bmatrix}. \tag{16}$$

Substituting (14) into (13) and taking (15), (16) into account, it holds that

$$\begin{aligned}&\dot{V}(x_t)+\alpha V(x_t)+\Gamma(t)\\ &\le\begin{bmatrix}x(t)\\ x(t-d(t))\\ \omega(t)\end{bmatrix}^T\begin{bmatrix}\Omega_{11}+Q+K^TRK & \Omega_{12} & \Omega_{13}\\ * & \Omega_{22} & \Omega_{23}\\ * & * & \Omega_{33}\end{bmatrix}\begin{bmatrix}x(t)\\ x(t-d(t))\\ \omega(t)\end{bmatrix}\\ &-2\left[x^T(t)Y+x^T(t-d(t))T+\omega^T(t)M\right]\int_{t-d(t)}^{t}\dot{x}(s)ds\\ &-\int_{t-d(t)}^{t}\dot{x}^T(s)e^{-\alpha\tau}S\dot{x}(s)ds.\end{aligned}$$

Let $\xi(t,s)=\left[x^T(t)\ x^T(t-d(t))\ \omega^T(t)\ \dot{x}^T(s)\right]^T$. Taking (12) into account, we have

$$\begin{aligned}&\dot{V}(x_t)+\alpha V(x_t)+\Gamma(t)\\ &\le\frac{1}{d(t)}\times\int_{t-d(t)}^{t}\xi^T(t,s)\Theta\xi(t,s)ds\le 0.\end{aligned}$$

According to the theory of the first order linear nonhomogeneous differential inequality, (9) holds obviously. The proof of Lemma 1 is complete. □

Lemma 2. Suppose that system (6) satisfies Assumption 1. For given positive constants β and γ, if there exist positive definite matrices P,S, matrix K, and any matrices Y,T,M with appropriate dimensions, such that

$$\begin{bmatrix}\varphi_{11}+Q & \varphi_{12} & \varphi_{13} & A^TS & -Y & K^T\\ * & \varphi_{22} & \varphi_{23} & D^TS & -T & 0\\ * & * & \varphi_{33} & S & -M & 0\\ * & * & * & -\tau^{-1}S & 0 & 0\\ * & * & * & * & -\tau^{-1}S & 0\\ * & * & * & * & * & -R^{-1}\end{bmatrix}<0 \tag{17}$$

holds, for the Lyapunov functional candidate

$$V(t)=x^T(t)Px(t)+\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)e^{\beta(t-s)}S\dot{x}(s)dsd\theta \tag{18}$$

along the trajectory of the system (6), there holds the following inequality

$$V(x_t)\le e^{\beta(t-t_0)}V(x_{t_0})-\int_{t_0}^{t}e^{\beta(t-s)}\Gamma(s)ds, \tag{19}$$

where Q is a given positive semidefinite matrix, and

$$\begin{aligned}\Gamma(s)=&\,x^T(s)Qx(s)+u^T(s)Ru(s)-\gamma^2\omega^T(s)\omega(s)\\ &(u(s)=Kx(s)),\end{aligned} \tag{20}$$

$$\varphi_{11} = A^T P + PA - \beta P + Y^T + Y, \quad \varphi_{13} = M^T + P,$$
$$\varphi_{12} = PD + T^T - Y, \quad \varphi_{23} = -M^T,$$
$$\varphi_{22} = -T^T - T, \quad \varphi_{33} = -\gamma^2 I.$$

Proof. The proof is same as in Lemma 1 and omitted. □

Lemma 1 and Lemma 2 provide the methods for the estimation of Lyapunov functional candidate. For tracking control problem, a switched time-varying delays system with stabilizable and unstabilizable subsystems is solvable if the stabilizable and unstabilizable subsystems satisfy certain conditions and admissible switching law among them, respectively. Now we give the method of switching tracking controller design for switched time-varying delays system.

Consider the switched time-varying delays system (1). Under the Assumption 3, for tracking control problem, not all the subsystems are stabilizable, without loss of generality, we suppose that the i-th subsystem ($1 \le i \le r$) is stabilizable (where the positive integer r satisfies $1 \le r < N$), accordingly, we suppose that the corresponding augmented subsystems of (4) satisfy Lemma 1, whereas the other subsystems of (1) are unstabilizable and the corresponding augmented subsystems are satisfy Lemma 2.

For each subsystem of the augmented system (4), which is satisfies Lemma 1, the Lyapunov functional candidate can be chosen as

$$V_i(\bar{x}_t) = \bar{x}^T(t) P_i \bar{x}(t) + \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{\bar{x}}^T(s) e^{-\alpha(t-s)} S_i \dot{\bar{x}}(s) ds d\theta, \tag{21}$$

where $i \in \{1, 2, \cdots, r\}$, and P_i, S_i are positive definite matrices.

Also, for each subsystem of the augmented system (4), which is satisfies Lemma 2, the Lyapunov functional candidate can be chosen as

$$V_j(\bar{x}_t) = \bar{x}^T(t) P_j \bar{x}(t) + \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{\bar{x}}^T(s) e^{\beta(t-s)} S_j \dot{\bar{x}}(s) ds d\theta, \tag{22}$$

where $j \in \{r+1, \cdots, N\}$, and P_j, S_j are positive definite matrices.

Consider the following piecewise Lyapunov functional candidate

$$V(\bar{x}_t) = V_{\sigma(t)}(t), \quad \sigma(t) \in \underline{N}. \tag{23}$$

From Lemma 1-2, it is easy to show the properties of the Lyapunov functional candidate (23) as

(i) There exist scalars $\alpha_1 > 0$, $\alpha_2 > 0$, such that

$$\alpha_1 \|\bar{x}\|^2 \le V_{i,j \in \underline{N}}(\bar{x}_t) \le \alpha_2 \|\bar{x}\|_{cl}^2, \ \forall \bar{x} \in R^{n+q}.$$

(ii) There exists a constant scalar $\mu \ge 1$ such that

$$V_k(\bar{x}_t) \le \mu V_l(\bar{x}_t) \quad \forall \bar{x} \in R^{n+q}, \ k, l \in \underline{N}. \tag{24}$$

(iii) The Lyapunov functional candidate (23) whose derivative along the trajectory of the corresponding subsystem satisfies

$$V(\bar{x}_t) \le \begin{cases} e^{-\alpha(t-t_0)} V_i(\bar{x}_{t_0}) & \texttt{if } i \le r, \\ e^{\beta(t-t_0)} V_j(\bar{x}_{t_0}) & \texttt{if } j > r, \end{cases} \tag{25}$$

while $\omega(t) \equiv 0$.

Now, for any piecewise constant switching signal $\sigma(t)$ and any $0 \le t_0 < t$, we let $T^-(t_0, t)$ (resp., $T^+(t_0, t)$) denote the total activation time of stabilizable (resp., unstabilizable) subsystems during (t_0, t). Then, we choose a scalar $\lambda^* \in (0, \alpha)$ arbitrarily to propose the following switching law:

(**S1**): Let $t_0 < t_1 < t_2 < \cdots < t_i$ be a specified sequence of time instants. Determined the switching signal $\sigma(t)$ so that

$$\inf_{t \ge t_0} \frac{T^-(t_0, t)}{T^+(t_0, t)} \ge \frac{\beta + \lambda^*}{\alpha - \lambda^*} \tag{26}$$

holds on time interval (t_0, t). Meanwhile, we choose $\lambda^* \le \alpha$ as the average dwell time scheme: for any $t > t_0$,

$$N_\sigma(t_0, t) \le N_0 + \frac{t - t_0}{\tau}, \quad \tau > \tau^* = \frac{\ln \mu}{\lambda^*}. \tag{27}$$

Under the switching law (S1) for any t_0, t satisfying $t_{i-1} < t_0 \le t_i < t_{i+1} < \cdots < t_k \le t$, we have

$$\beta T^+(t_0, t) - \alpha T^-(t_0, t) \le \beta(t - t_k) - \lambda^*(t - t_0) + \beta(t_i - t_0). \tag{28}$$

Therefore,

$$e^{\beta T^+(t_0,t) - \alpha T^-(t_0,t)} \le e^{c - \lambda^*(t - t_0)}, \tag{29}$$

where $c = 2\beta T_0$, $T_0 = \max\{t - t_k, t_i - t_0\}$.

We are now in a position to present the procedure of construction of tracking control for switched time-varying delays system.

Step 1. Augmenting System (1) by the integral controller (2); by the LMIs ToolBox calculating the matrices inequalities (7) for the stabilizable subsystems of (1), and (17) for the unstabilizable subsystems of (1). The controller gains K_i, L_i ($i \in \underline{N}$) are determined, meanwhile, the Lyapunov functional candidates (21) and (22) can be obtained.

Step 2. For given α, β which satisfy (7) and (17), specify the activation time period ratio of stabilizable subsystems to unstabilizable ones by (26).

Step 3. From Lyapunov functional candidates (21) and (22), determine the parameter μ which satisfies (24), from this calculate the average dwell time by (27). Therefore, switching tracking controller can be constructed.

The following theorems provide theoretical basis for the switching tracking controller design above.

Theorem 1. Consider the switched time-varying delays system (1) satisfying Assumption 1-3. Suppose that the subsystems ($1 \le i \le r$) of system (4) satisfy the conditions of Lemma 1, and the others satisfy the conditions of Lemma 2, $y_r = r(t) \in \mathscr{L}_2[0, \infty]$. Then, under the switching law (S1) and the average dwell time scheme (27), the switching tracking control problem is solvable for switched time-varying delays system (1). The closed-loop system (3) or (4) achieves the upper bound of performance index

$$J_L \le \frac{\alpha e^c}{\lambda^*} \left[V(\bar{x}_0) + \gamma^2 \int_0^\infty \varpi^T(s) \varpi(s) ds \right].$$

Proof. *Internal stability.* Consider the augmented system (4). Note that with $\varpi(t) \equiv 0$, for any $\sigma(t)$ under the switching law (S1), the piecewise Lyapunov functional candidate (23) on every switching point t_k satisfies (24). We have,

$$\begin{aligned}V(\bar{x}_t) \le& e^{\beta T^+(t_k,t)-\alpha T^-(t_k,t)}V_{\sigma(t_k)}(\bar{x}_{t_k})\\ &\le \cdots \le e^{\beta T^+(t_0,t)-\alpha T^-(t_0,t)}\mu^k V_{\sigma(t_0)}(\bar{x}_{t_0}).\end{aligned}$$

Thus,

$$V(\bar{x}_t) \le \mu^{N_\sigma(t_0,t)} \cdot e^{\beta T^+(t_0,t)-\alpha T^-(t_0,t)}V_{\sigma(t_0)}(\bar{x}_{t_0}), \quad (30)$$

where $N_\sigma(t_0,t)$ is the switching numbers in (t_0,t). Taking (27) and (29) into account, we get

$$\begin{aligned}V(\bar{x}_t)\le&\mu^{N_\sigma(t_0,t)} \cdot e^{\beta T^+(t_0,t)-\alpha T^-(t_0,t)}V_{\sigma(t_0)}(\bar{x}_{t_0})\\ \le& e^{N_0 \ln\mu - c} \cdot e^{-(\lambda^* - \frac{\ln\mu}{\tau})(t-t_0)}V_{\sigma(t_0)}(\bar{x}_{t_0})\\ \le& c_0 \cdot e^{-2\lambda(t-t_0)}V_{\sigma(t_0)}(\bar{x}_{t_0}), \quad (31)\end{aligned}$$

where $c_0 = e^{N_0 \ln\mu - c}$, $\lambda = \frac{1}{2}(\lambda^* - \frac{\ln\mu}{\tau})$.
According to (21), (22) and (31), we have

$$a\|\bar{x}(t)\|^2 \le V(\bar{x}_t),\ V_{\sigma(t_0)}(\bar{x}_{t_0}) \le b\|\bar{x}(t_0)\|_{cl}^2, \quad (32)$$

where

$$a = \min_{i\in\underline{N}} \lambda_{\min}(P_i),\ b = \max_{i\in\underline{N}} \lambda_{\max}(P_i) + \tau^2 \max_{i\in\underline{N}} \lambda_{\max}(S).$$

Combining (31) and (32) gives rise to

$$\|\bar{x}(t)\|^2 \le \frac{1}{a}V(\bar{x}_t) \le \frac{bc_0}{a} \cdot e^{-2\lambda(t-t_0)}\|\bar{x}_{t_0}\|_{cl}^2.$$

Therefore, $\|\bar{x}(t)\| \le \sqrt{\frac{bc_0}{a}} \cdot e^{-\lambda(t-t_0)}\|\bar{x}_{t_0}\|_{cl}$, which means that system (3) or (4) is exponentially stable with $\varpi \equiv 0$. The proof of internal stability is complete.

Optimal tracking performance. When $\varpi \neq 0$, considering the Lyapunov functional candidates from Lemma 1-2, for any $t \in [t_k, t_{k+1})$, we have

$$V(\bar{x}_t) \le \begin{cases} e^{-\alpha(t-t_k)}V_{\sigma(t_k)}(\bar{x}_{t_k}) - \int_{t_k}^t e^{-\alpha(t-s)}\Gamma(s)ds, \\ \qquad \texttt{if } \sigma(t_k) = i \le r, \\ e^{\beta(t-t_k)}V_{\sigma(t_k)}(\bar{x}_{t_k}) - \int_{t_k}^t e^{\beta(t-s)}\Gamma(s)ds, \\ \qquad \texttt{if } \sigma(t_k) = j > r. \end{cases} \quad (33)$$

Under the switching law (S1), for the switching signal $\sigma(t)$ the piecewise Lyapunov functional candidate (23) satisfies

$$\begin{aligned}V(\bar{x}_t)\le& e^{\beta T^+(t_k,t)-\alpha T^-(t_k,t)}V_{\sigma(t_k)}(\bar{x}_{t_k})\\ &-\int_{t_k}^t e^{\beta T^+(s,t)-\alpha T^-(s,t)}\Gamma(s)ds\\ \le& \mu e^{\beta T^+(t_k,t)-\alpha T^-(t_k,t)}V_{\sigma(t_k^-)}(x_{t_k^-})\\ &-\int_{t_k}^t e^{\beta T^+(s,t)-\alpha T^-(s,t)}\Gamma(s)ds \le \cdots\\ \le& \mu^k e^{\beta T^+(t_0,t)-\alpha T^-(t_0,t)}V_{\sigma(t_0)}(\bar{x}_0)\\ &-\mu^k \int_{t_0}^{t_1} e^{\beta T^+(s,t)-\alpha T^-(s,t)}\Gamma(s)ds - \cdots\\ &-\mu^0 \int_{t_k}^t e^{\beta T^+(s,t)-\alpha T^-(s,t)}\Gamma(s)ds\\ =& e^{\beta T^+(t_0,t)-\alpha T^-(t_0,t)+N_\sigma(t_0,t)\ln\mu}V_{\sigma(t_0)}(\bar{x}_{t_0})\\ &-\int_{t_0}^t e^{\beta T^+(s,t)-\alpha T^-(s,t)+N_\sigma(s,t)\ln\mu}\Gamma(s)ds.\end{aligned}$$

For the augmented system (3) or (4), there has $\Gamma(t) = \bar{x}^T(t)Q\bar{x}(t) + u^T(t)Ru(t) - \gamma^2\varpi^T(t)\varpi(t)$, in which $Q = diag\{Q_1, Q_2\}$, $u(t) = K\bar{x}(t)$, $K = [k_\sigma, L_\sigma]$. Multiplying both sides of (34) by $e^{-N_\sigma(t_0,t)\ln\mu}$ gives rise to

$$\begin{aligned}&e^{-N_\sigma(t_0,t)\ln\mu}V(\bar{x}_t) + \int_{t_0}^t e^{\beta T^+(s,t)-\alpha T^-(s,t)-N_\sigma(t_0,s)\ln\mu}\\ &\times\left[\bar{x}^T(s)Q\bar{x}(s) + u^T(s)Ru(s)\right]ds\\ \le& e^{\beta T^+(t_0,t)-\alpha T^-(t_0,t)}V(\bar{x}_0) \quad (34)\\ &+\gamma^2 \int_{t_0}^t e^{\beta T^+(s,t)-\alpha T^-(s,t)-N_\sigma(t_0,s)\ln\mu}\varpi^T(s)\varpi(s)ds.\end{aligned}$$

For the convenience of discussion, we let $t_0 = 0$. Under the switching law (S1) and the average dwell time scheme (27) with $\sigma < \lambda^*$, we can obtain

$$\begin{aligned}&\int_0^t e^{-\alpha(t-s)-\sigma s}\left[\bar{x}^T(s)Q\bar{x}(s) + u^T(s)Ru(s)\right]ds\\ \le& e^{c-\lambda^* t}V(\bar{x}_0) + \gamma^2 \int_0^t e^{c-\lambda^*(t-s)}\varpi^T(s)\varpi(s)ds. \quad (35)\end{aligned}$$

Integrating the above inequality from $t = 0$ to ∞ leads to

$$\begin{aligned}&\int_0^\infty e^{-\sigma s}\left[\bar{x}^T(s)Q\bar{x}(s) + u^T(s)Ru(s)\right]ds\\ &\le \frac{\alpha e^c}{\lambda^*}\left[V(\bar{x}_0) + \gamma^2 \int_0^\infty \varpi^T(s)\varpi(s)ds\right]. \quad (36)\end{aligned}$$

Note that
$\bar{x}^T(t)Q\bar{x}(t) = x^T(t)Q_1x(t) + \left(\int_0^t e(t)dt\right)^T Q_2 \left(\int_0^t e(t)dt\right)$, so we have

$$J_L \le \frac{\alpha e^c}{\lambda^*}\left[V(\bar{x}_0) + \gamma^2 \int_0^\infty \varpi^T(s)\varpi(s)ds\right].$$

The proof is complete. □

Theorem 2. Consider the switched time-varying delays system (1) satisfying Assumption 1-3, and the time-varying delays also satisfy $0 < \dot{d}_i(t) \le d < 1$. Suppose that the subsystems $(1 \le i \le r)$ of system (4) satisfy the conditions of Lemma 1, and the others satisfy the conditions of Lemma 2, $y_r = r(t) \in \mathscr{D}$, where $\mathscr{D}$ is a set of constants or step inputs on $[0,\infty]$. Then, under the switching law (S1) and the average dwell time scheme (27), there holds $y(t) \to y_r(t)$ as $t \to \infty$, that is, the switching tracking control problem is solvable for system (1).

Proof. Adopting the method in [12] and repeating the procedures in the proof of Lemma 1, 2 and Theorem 1 with the condition $0 < \dot{d}_i(t) \le d < 1$ gives directly the result. □

Remark 4. In Theorem 1 and Theorem 2, if the subsystems which satisfy the conditions of Lemma 1 are $1 \le i \le N$, i.e., $r = N$, this case degenerates into the case of paper [8].

IV. Numerical Example

We illustrate the main results by a numerical example.

Consider the switched linear time varying delays system (1) with a stabilizable and an unstabilizable subsystem with

$$A_1 = \begin{bmatrix} -4 & -2.5 \\ 1.2 & -1.5 \end{bmatrix},\ D_1 = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ -0.3 \end{bmatrix};$$

$$A_2 = \begin{bmatrix} -2 & 0.5 \\ 3.2 & 3.5 \end{bmatrix}, D_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} -0.1 \\ -0.2 \end{bmatrix};$$

$$C_1 = \begin{bmatrix} -5 & -0.5 \\ 0.1 & -1 \end{bmatrix}, C_2 = \begin{bmatrix} -0.1 & 0.7 \\ 0.3 & 1.2 \end{bmatrix}, r(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix};$$

and $d_\sigma(t) = 2.4 - 0.1e^{-t}$, $\alpha = 0.6$, $\beta = 0.4, \tau = 2.4$, $Q = I_{4\times 4}$, $R = 0.1$, solving (7) and (17) gives piecewise Lyapunov functional (21) and (22) respectively with

$$P_1 = \begin{bmatrix} \hat{P}_1^{-1} & 0 \\ 0 & \hat{P}_1^{-1} \end{bmatrix}, P_2 = \begin{bmatrix} \hat{P}_2^{-1} & 0 \\ 0 & \hat{P}_2^{-1} \end{bmatrix},$$

$$S_1 = \begin{bmatrix} \hat{S}_1 & 0 \\ 0 & \hat{S}_1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} \hat{S}_2 & 0 \\ 0 & \hat{S}_2 \end{bmatrix},$$

where

$$\hat{P}_1 = \begin{bmatrix} 0.2405 & -0.0857 \\ -0.0857 & 0.1672 \end{bmatrix}, \hat{P}_2 = \begin{bmatrix} 0.2907 & -0.0865 \\ -0.0865 & 0.1877 \end{bmatrix},$$

$$\hat{S}_1 = \begin{bmatrix} 3.7770 & 2.6622 \\ 2.6622 & 6.1308 \end{bmatrix}, \hat{S}_2 = \begin{bmatrix} 2.7482 & -0.0531 \\ -0.0531 & 2.7368 \end{bmatrix}.$$

Consequently, the controller gains are given as

$$K_1 = \begin{bmatrix} 0.52951.0927 \end{bmatrix}, L_1 = \begin{bmatrix} 0.52951.0927 \end{bmatrix};$$

$$K_2 = \begin{bmatrix} 4.02921.8681 \end{bmatrix}, L_2 = \begin{bmatrix} 4.02921.8681 \end{bmatrix}.$$

Solving (24) gives $\mu = 1.2951$, and according (27), we have $\tau_\alpha^* = \frac{\ln \mu}{\alpha} = 0.4310$. Take $\lambda^* = 0.5 < \alpha$, the activation ratio of stabilizable subsystems to unstabilizable subsystems is $\frac{T^-(t_0,t)}{T^T(t_0,t)} = 9$, by using average dwell time method provided by Theorem 2, we obtained that system (1) is solvable, the simulation results are depicted in Fig.1-Fig.2.

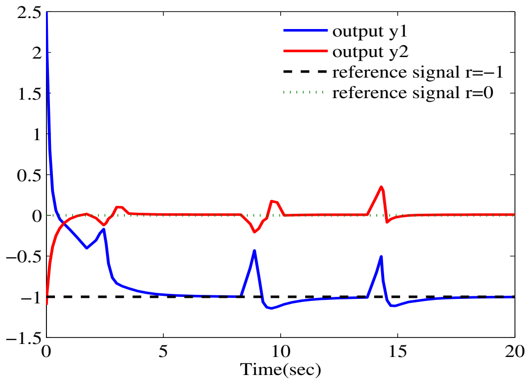

Fig. 1. Output y tracking the reference input $r(t)$.

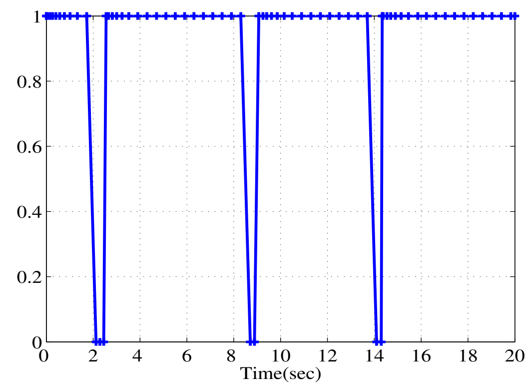

Fig. 2. Switching signal.

V. CONCLUSIONS

In this paper, we have investigated tracking control problem for switched linear time-varying delays systems with stabilizable and unstabilizable subsystems. Average dwell time approach and piecewise Lyapunov functional methods are utilized to the stability analysis and controller design, and with free weighting matrix scheme, switching control laws are obtained. A simulation example shows the effectiveness of the proposed switching control laws.

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