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The relation between redundancy and convergence rate in distributed multi-agent formation control

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Abstract— **In an earlier paper we had proposed a one step ahead optimization based distributed control law for autonomous agents, modeled as double integrators, that achieves a formation specified by relative position between agents. The law requires minimal information exchange between the agents and minimal knowledge on the part of each agent of the overall formation objective, and is fault tolerant and scalable, being easily reconfigurable in the face of the loss or arrival of an agent, and the loss of a communication link. In this earlier paper we had provided a framework to incorporate redundancy that allows a network to survive faults caused by the loss of agents and communications links. In this paper we consider a different aspect of redundancy: Specifically the impact it has on control performance of the law as quantified by the speed with which a desired formation is achieved.**

Index Terms— **Co-operative Control, Stability, Fault Tolerance, Decentralized Control, Autonomus Agents, Rate of Convergence**

I. INTRODUCTION

The cooperative control of mobile agents has become an important area of research over the last decade, [1]-[27]. Aspects of the problem include: control with little or no centralized intervention, poor information quality, and performance of cooperative tasks.

In this paper, we revisit the work of [2] involving agents modeled as double integrators in each cartesian dimension. The goal in [2] is to induce the agents to organize themselves into formations prescribed by the relative positions between them. To this end, [2] devises a control law that requires minimal information exchange between the agents and minimal knowledge on the part of each agent of the overall formation objective. The law itself is fault tolerant and scalable, being easily reconfigurable in the face of the loss or arrival of an agent, and the loss of a communication link. A key contribution of [2] is to provide a framework to incorporate redundancy that allows a network to survive faults caused by the loss of agents and communications links. In this paper we consider a different aspect of redundancy: Specifically the impact it has on control performance of the law in [2] as quantified by the speed with which a desired formation is achieved.

The area of multiagent control encapsulates a wide expanse of subjects. This includes string stability, [3], [5]; flocking motivated by biology, [6]-[10]; behavior of self propelled particles, [11]; the rendezvous problem, [12]-[15]; robotics, [16]- [19]; and formation control the subject of this paper.

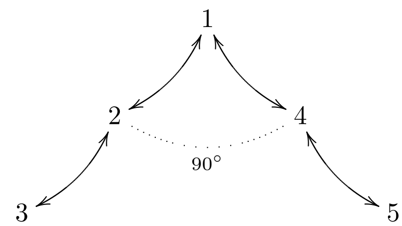

Fig. 1. Formation Topology with no Redundancy

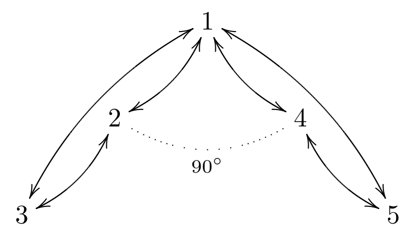

Fig. 2. Formation Topology with Redundancy

Two notable trends in the formation control literature involve formations defined by inter-agent distances, and formations defined by relative positions. Papers like [21], [22] and [28]-[29] serve as examples of the first class.

In the second class are papers like [24], and [27] that separately propose a desired formation and a state exchange architecture and ask whether the latter suffices to achieve a formation. On the other hand, [2] reverses the question and asks: given a desired formation specified by *relative positions rather than interagent distances*, what state exchange architecture suffices to achieve ? It also focuses on control laws that incorporate redundancies that permit the formation to survive the loss of agents and/or communication links.

Since the take off point of this paper is [2], we briefly reprise its salient points, which involve a *Formation Topology* that is used to provide a potentially redundant geometric description of a formation. The formation topology comprises in part of an undirected graph, with agents as nodes. An arc exists between two agents if the relative positions between them is explicitly provided in the formation description. Figures 1 and 2 are examples of two formation topologies describing the same formation, the latter providing a redundant description. In the redundant description 5 survives the loss of 4 whereas in fig. 1 loss of 4, isolates 5.

This contrasts with the *Communication Topology* which defines the state information flow required to implement a cooperative control law. The relationship between the

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two topologies is explored in [2], which proposes a cost function that incorporates the formation topology. A one step ahead optimal control law obtained on its basis has many features. Foremost among them is the fact that the communication topology required to implement it is *identical* to the underlying formation topology. A redundant and fault tolerant formation topology has a denser communication topology.

The key attractive properties of the approach of [2] is that to execute its control law, an agent needs to know only the constraints it is explicitly involved in, and the states of only those agents it shares arcs with. Consequently, the loss of an agent or a communication link only requires reconfiguration of the control law of the agents the lost agent shared an arc with, or on which the lost link impinged. Scalability is similarly accommodated with minimal reconfiguration.

While redundancy in the formation topology was studied in [2] purely from the perspective of fault tolerance, as can be imagined redundancy also play a role in how fast convergence is achieved. Accordingly in this paper we quantify the role of redundancy in formation convergence.

II. Problem Description

Consider the problem of a two dimensional N-agent formation topology, we will partition the global, $4N$ state vector x of the formation as

$$x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T \tag{1}$$

where x_1 and x_2 contain the positions and velocities respectively. In particular, denoting $x_{l,j}$ as the j-th element of x_l, we have

$x_{1,i}$ is the x position of agent i,
$x_{2,i}$ is the x velocity of agent i,
$x_{1,i+N}$ is the y position of agent i, and
$x_{2,i+N}$ is the y velocity of agent i

Each vehicle will be internally modeled as a double integrator with a sampling interval of 1-second. The system of agents can be represented as:

$$x(k+1) = \Phi x(k) + \Gamma u(k) \tag{2}$$

where

$$\Phi = \begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix}, \tag{3}$$

and,

$$\Gamma = \begin{bmatrix} I_n \\ 2I_n \end{bmatrix}. \tag{4}$$

Where $n = 2N$ for simplicity of notation.

Observe the following fact that follows directly from (2-4).

Fact 2.1: Consider agent i. The corresponding states associated with agent i are $j \in \{i, i+N, i+2N, i+3N\}$. The computation of the j-th element of $\Phi x(k)$ requires only the states associated with agent i.

A. The Formation Topology

There are two views of the formation topology. In graph theory terms, each agent is modeled as a node. An undirected edge exists between agents i and j if relative position constraints are specified between them. If an x-position constraint is specified between a pair of agents, then we assume that a y-position constraint has also been specified.

Since each such constraints takes the form of

$$x_i - x_j = c_{xij}$$

and

$$x_{i+N} - x_{j+N} = c_{yij},$$

an algebraic description of the formation topology is provided by a matrix vector pair $[A, b]$. Specifically, the formation topology can be represented by the following equation:

$$Ax = b \tag{5}$$

A can be further partitioned as:

$$A = \begin{bmatrix} A_p & 0 \end{bmatrix} \tag{6}$$

and

$$A_p = A_{ps} \bigoplus A_{ps} \tag{7}$$

where $A_{ps} \in R^{L \times N}$, L being the number of arcs in the formation topology. There are as many rows in A_{ps} as there are arcs in the formation topology, one row for each arc. If an arc exists between agents i and j, then the corresponding row of A_{ps} is a vector all but the i and j-th element of which is zero, the i-th elements is 1 and j-th element is -1.

We make the following assumptions on the pair $[A, b]$.

Assumption 2.1: (i) The matrix A_{ps} has rank $N-1$. (ii) Further b is in the range space of A.

It is well known that (i) ensures that the formation topology viewed as a graph is connected. Moreover, (ii) ensures that it is well defined.

III. Control Law and Communication Topology

The control law proposed in [2] is a one step ahead optimization law using the cost function

$$J(k) = [Ax(k+1) - b]^T [Ax(k+1) - b] + u^T(k) Q u(k) \tag{8}$$

Where $Q = Q^T > 0$ penalizes the input. The key step in achieving the control law with the desired characteristics described in the introduction is to appropriately select Q.

Since $x(k+1)$ is dependent on $u(k)$ we begin by substituting (2) into the cost function defined in (8). Taking the partial derivative of the resultant expression with respect to $u(k)$, we obtain:

$$\left[\Gamma^T A^T A \Gamma + Q\right] u(k) = \Gamma^T A^T [b - A\Phi x(k)] \tag{9}$$

Setting:

$$Q = \alpha I - \Gamma^T A^T A \Gamma, \tag{10}$$

with α greater than the largest eigenvalue of $\Gamma^T A^T A \Gamma$, Q is invertible and positive definite. Further by making α

arbitrarily large one can penalize the input to an arbitrary degree. The control law becomes.

$$u(k) = \frac{1}{\alpha}\Gamma^T A^T b - \frac{1}{\alpha}\Gamma^T A^T A\Phi x(k) \tag{11}$$

The Theorem below proved in [2], shows that this law has all the desirable attributes noted in the introduction.

Theorem 3.1: Consider (11). Then finding $u_{2i-1}(k)$ and $u_{2i}(k)$ requires: (A) The states of agent l only if there is an arc between agents l and i in the formation topology. (B) The l-th row of A_p only if for some $j \in \{i, i+N\}$ $a_{lj} \neq 0$. (C) The l-th element of b only if for some $j \in \{i, i+N\}$ $a_{lj} \neq 0$.

IV. A Kalman Decomposition

In this section we use a state transformation to allow a more concise presentation of the main result of the paper.

First note that with α chosen to ensure that Q in (10) to be positive definite, one has that

$$I - \frac{1}{\alpha}\Gamma^T A^T A\Gamma > 0. \tag{12}$$

With

$$F = \Phi - \frac{1}{\alpha}\Gamma\Gamma^T A^T A\Phi \tag{13}$$

and

$$G = \frac{1}{\alpha}\Gamma\Gamma^T A^T b \tag{14}$$

the closed loop control law becomes

$$x(k+1) = Fx(k) + G. \tag{15}$$

Define

$$y(k) = Ax(k) - b. \tag{16}$$

Now consider a singular value decomposition (SVD) of A_p:

$$A_p = UDV \tag{17}$$

In view of (7) and Assumption 2.1, U is an $2L \times 2L$ unitary matrix, V is an $2N \times 2N$ unitary matrix and D is as defined below:

$$D = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \tag{18}$$

with $n_1 = 2N-2$, Δ diagonal, $n_1 \times n_1$, real positive definite. Under (12),

$$0 < \Delta < 1. \tag{19}$$

Observe that the double integrator dynamics of the agents, coupled with the lack of velocity constraints and the fact that rank[A]=$2N-2$, ensures that $2N-2$ eigenvalues of F are guaranteed to be 1. The point of the Kalman decomposition developed in this section is to: (a) demonstrate that these eigenvalues are not the poles of $A(zI-F)^{-1}$, and (b) to isolate the poles that can be made stable. Define

$$S = \begin{bmatrix} V & 0 \\ 0 & V \end{bmatrix} \tag{20}$$

Then consider the state transformation described in the theorem below.

Lemma 4.1: With Φ, Γ, A, b, F, G, U, V, D and S defined in (3), (4), (6), (13), (14), and (17 - 20) define:

$$\widehat{A} = \begin{bmatrix} U\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \tag{21}$$

$$\widehat{F} = \Phi - \frac{1}{\alpha}\Gamma\Gamma^T \widehat{A}^T \widehat{A}\Phi \tag{22}$$

$$\widehat{G} = \frac{1}{\alpha}\Gamma\Gamma^T \widehat{A}^T b \tag{23}$$

$$\widehat{x}(k) = Sx(k) \tag{24}$$

Then one has that

$$\widehat{x}(k+1) = \widehat{F}\widehat{x}(k) + \widehat{G} \tag{25}$$

$$y(k) = \widehat{A}\widehat{x}(k) - b \tag{26}$$

and

$$\widehat{A} = AS^{-1}. \tag{27}$$

Proof: First note that

$$\begin{aligned} AS^{-1} &= \begin{bmatrix} A_p & 0 \end{bmatrix} \begin{bmatrix} V^H & 0 \\ 0 & V^H \end{bmatrix} \\ &= \begin{bmatrix} A_p V^H & 0 \end{bmatrix} \\ &= \begin{bmatrix} U\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \\ &= \widehat{A} \end{aligned} \tag{28}$$

Further in view of (20) and (28)

$$\begin{aligned} SFS^{-1} &= S\left(I - \frac{1}{\alpha}\Gamma\Gamma^T A^T A\right)\Phi S^{-1} \\ &= \widehat{F} \end{aligned}$$

and similarly,

$$\begin{aligned} SG &= \frac{1}{\alpha}S\Gamma\Gamma^T A^T b \\ &= \widehat{G} \end{aligned}$$

■

The next lemma whose prove is trivial shows that a condition comparable to (12) holds.

Lemma 4.2: With $\widehat{A}$ as defined in (21)

$$I - \frac{1}{\alpha}\Gamma^T \widehat{A}^T \widehat{A}\Gamma > 0 \tag{29}$$

Denoting 0_p to be the $p\times p$, 0 matrix, and I_p to be the $p\times p$, identity matrix, we observe from (18) and (21) that

$$\begin{aligned}\widehat{F} &= SFS^{-1}\\ &= S\begin{bmatrix} I-\frac{1}{\alpha}A_p^TA_p & I-\frac{1}{\alpha}A_p^TA_p\\ -\frac{2}{\alpha}A_p^TA_p & I-\frac{2}{\alpha}A_p^TA_p\end{bmatrix}S^{-1}\\ &= \begin{bmatrix} I-\frac{1}{\alpha}VA_p^TA_pV^H & I-\frac{1}{\alpha}VA_p^TA_pV^H\\ -\frac{2}{\alpha}VA_p^TA_pV^H & I-\frac{2}{\alpha}VA_p^TA_pV^H\end{bmatrix}\\ &= \begin{bmatrix} I-\frac{1}{\alpha}\Delta^2 & 0 & I-\frac{1}{\alpha}\Delta^2 & 0\\ 0 & I_2 & 0 & I_2\\ -\frac{2}{\alpha}\Delta^2 & 0 & I-\frac{2}{\alpha}\Delta^2 & 0\\ 0 & 0_2 & 0 & I_2\end{bmatrix}\end{aligned} \tag{30}$$

Then the following lemma goes toward a Kalman like decomposition.

Lemma 4.3: Under (17-26), and $n_1 = 2N-2$,

$$\widehat{A}\left(zI-\widehat{F}\right)^{-1} = \begin{bmatrix} H(z) & 0_{L\times 2(n-n_1)}\end{bmatrix} \tag{31}$$

where

$$H(z) = C\left(zI-\Upsilon\right)^{-1}, \tag{32}$$

$$\Upsilon = \begin{bmatrix} I-\frac{1}{\alpha}\Delta^2 & I-\frac{1}{\alpha}\Delta^2\\ -\frac{2}{\alpha}\Delta^2 & I-\frac{2}{\alpha}\Delta^2\end{bmatrix} \tag{33}$$

$$C = \begin{bmatrix} U\begin{bmatrix}\Delta\\0\end{bmatrix} & 0_{L\times n_1}\end{bmatrix} \tag{34}$$

and

$$\Pi = \begin{bmatrix} I_{n_1} & 0 & 0 & 0\\ 0 & 0 & I_{n_1} & 0\\ 0 & I_{n-n_1} & 0 & 0\\ 0 & 0 & 0 & I_{n-n_1}\end{bmatrix} \tag{35}$$

Proof: Note

$$\Pi^T\Pi = I. \tag{36}$$

Hence

$$\widehat{A}\left(zI-\widehat{F}\right)^{-1} = \widehat{A}\Pi^T\left[zI-\Pi\widehat{F}\Pi^T\right]^{-1}\Pi. \tag{37}$$

Now,

$$\begin{aligned}\widehat{A}\Pi^T &= \begin{bmatrix} U\begin{bmatrix}\Delta\\0\end{bmatrix} & 0_{L\times n-n_1} & 0_{L\times n}\end{bmatrix}\\ &\quad\begin{bmatrix} I_{n_1} & 0 & 0 & 0\\ 0 & 0 & I_{n-n_1} & 0\\ 0 & I_{n_1} & 0 & 0\\ 0 & 0 & 0 & I_{n-n_1}\end{bmatrix}\\ &= \begin{bmatrix} U\begin{bmatrix}\Delta\\0\end{bmatrix} & 0_{L\times n_1} & 0_{L\times 2(n-n_1)}\end{bmatrix}\\ &= \begin{bmatrix} C & 0_{L\times 2(n-n_1)}\end{bmatrix}\end{aligned}$$

Further, from (30)

$$\begin{aligned}\Pi\widehat{F}\Pi^T &= \begin{bmatrix} I-\frac{1}{\alpha}\Delta^2 & I-\frac{1}{\alpha}\Delta^2 & 0 & 0\\ -\frac{2}{\alpha}\Delta^2 & I-\frac{2}{\alpha}\Delta^2 & 0 & 0\\ 0 & 0 & I_{n-n_1} & I_{n-n_1}\\ 0 & 0 & 0 & I_{n-n_1}\end{bmatrix}\\ &= \begin{bmatrix} \Upsilon & 0\\ 0 & \begin{bmatrix} I_{n-n_1} & I_{n-n_1}\\ 0 & I_{n-n_1}\end{bmatrix}\end{bmatrix}.\end{aligned} \tag{38}$$

Then the result follows. ■

Taken together, the results of this section show that the poles of $A(zI-F)^{-1}$ are in fact the eigenvalues of Υ. The next section shows that (a) these can be made stable, and (b) that their magnitudes determine the rate of convergence.

V. Rates of Convergence

To ease notation call

$$B = \frac{1}{\alpha}\Delta^2 \tag{39}$$

and note that because of (19),

$$0 < \lambda(B) < 1 \tag{40}$$

Then:

$$\Upsilon = \begin{bmatrix} I-B & I-B\\ -2B & I-2B\end{bmatrix} \tag{41}$$

Recall that the poles of $A(zI-F)^{-1}$ are in fact the eigenvalues of Υ. Assume for the moment that all the eigenvalues of Υ are in the open unit disc. Further under assumption 2.1 there exists an x such that

$$A_px = b.$$

Thus using (3), (4), (13) and (14) and using the fact that

$$\Phi\begin{bmatrix}x\\0\end{bmatrix} = \begin{bmatrix}x\\0\end{bmatrix},$$

we obtain

$$\begin{aligned}\lim_{k\to\infty}(Ax(k)-b) &= \lim_{z\to 1}(z-1)\frac{A(zI-F)^{-1}G-b}{z-1}\\ &= \lim_{z\to 1}(A(zI-F)^{-1}G-b)\\ &= \lim_{z\to 1}\left(A(zI-F)^{-1}\frac{\Gamma\Gamma'A'A}{\alpha}\begin{bmatrix}x\\0\end{bmatrix}\right.\\ &\quad\left.-A\begin{bmatrix}x\\0\end{bmatrix}\right)\\ &= \lim_{z\to 1}A\left((zI-F)^{-1}\frac{\Gamma\Gamma'A'A}{\alpha}-I\right)\\ &\quad\begin{bmatrix}x\\0\end{bmatrix}\\ &= \lim_{z\to 1}A(zI-F)^{-1}\\ &\quad\left(\frac{\Gamma\Gamma'A'A}{\alpha}-I+F\right)\begin{bmatrix}x\\0\end{bmatrix}\\ &= 0.\end{aligned}$$

This analysis reveals two facts. First should all eigenvalues of Υ be inside the unit circle then the formation is attained.

Second the deeper inside the unit circle these eigenvalues are, the faster the rate of convergence. In the sequel we tie the magnitude of these eigenvalues to the redundancy in the network. In particular, observe that the edges in the formation topology completely determine the matrix A, and hence Υ. For a formation topology described by the undirected graph $G = (V, E)$, we will define the corresponding A matrix as $A(G)$ and the B in (39) as $B(G)$. Then the following lemma is crucial.

Lemma 5.1: Consider the formation topologies with associated undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Suppose $E_1 \subset E_2$. Then

$$B(G_1) \leq B(G_2).$$

Proof: Associate with an undirected edge between i and j the vector e_{ij} the $2N$-vector all but i and j-th elements of which are zero. One of the remaining elements is 1, the other -1. Then,

$$A(G_2)^T A(G_2) = A(G_1)^T A(G_1) + \sum_{\{i,j\} \in E_2 - E_1} e_{ij} e_{ij}^T \quad (42)$$

Thus

$$A(G_2)^T A(G_2) \geq A(G_1)^T A(G_1).$$

Thus the result follows from (17) and the definition $B(G_i)$. ∎

Now observe that with $B =$ diag b_i, within a symmetric perturbation one can express

$$\Upsilon = \bigoplus_{i=1}^{2N-2} \Upsilon_{ii} \quad (43)$$

where

$$\Upsilon_{ii} = \begin{bmatrix} 1 - b_i & 1 - b_i \\ -2b_i & 1 - 2b_i \end{bmatrix}. \quad (44)$$

The characteristic polynomial of each Υ_{ii} is

$$\lambda^2 - (2 - 3b_i)\lambda + (1 - b_i).$$

Observe that as long as

$$0 < b_i < 8/9, \quad (45)$$

both eigenvalues of Υ_{ii} are complex with magnitude $1 - b_i$. Thus as long as (45) holds increasing b_i forces the eigenvalues further inside the unit circle. On the other hand in the range $b_i \in [8/9, 1)$, observe from (**??**), $0 < b_i < 1$, as b_i increases while one eigenvalue of Υ_{ii} approaches 0, the other approaches 1. This leads us to the main result of this paper.

Theorem 5.1: Consider two formation topologies associated undirected graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Suppose $E_1 \subset E_2$. Suppose also that for each $i \in \{1, 2\}$,

$$\alpha I - \frac{9A(G_i)A'(G_i)}{8} \geq 0. \quad (46)$$

Then (11) converges for both topologies but at a faster rate for G_2.

Thus depending on the value of α a more redundant network will lead to a faster convergence. The way to interpret the cutoff point of 8/9 is as follows. Too dense a network will cause the positive definiteness of Q to be violated. Thus given an α, the performance improves monotonically upto a clearly demarcated level of redundancy, but degrades thereafter.

VI. Simulation Results

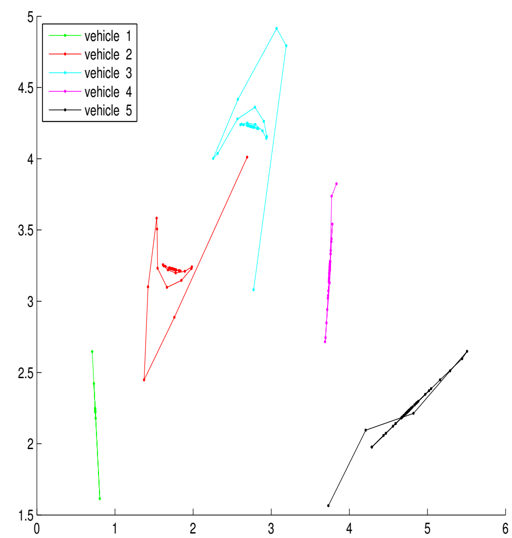

Fig. 3. Non-Redundant Vehicle Trajectory

Here we show some simulations which verify the results of the paper. The trajectories of the agents in a non-redundant formation, as in fig. 1, are shown in fig. 3. The agents initial positions are chosen randomly, but the same random initial positions are used in all of the presented figures. In fig 4 the formation topology is completely connected.

The relative position errors, measured by $\|Ax(k) - b\|_2$, are shown in fig. 5. In this figure simulation 1 represents a non-reduntant formation. Redundancy is added as the simulations numbers increase until we reach the fully connected formation topology in simulation 6. As demonstrated, a more redundant network leads to a faster convergence.

VII. Conclusion

We have examined the cooperative control of a fleet of autonomous agents that achieve arbitrary relative positions. We revisit the control law we have presented in [2] to show that not only does increased connectivity among the agents result in better robustness to loss of craft, but that up to a clearly quantifiable point it also results in faster convergence to the desired formation.

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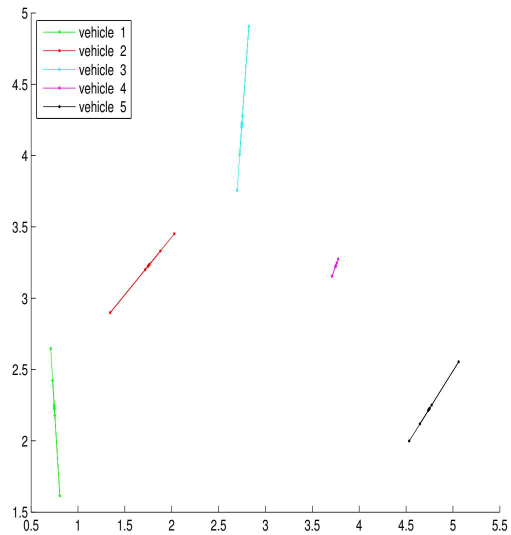

Fig. 4. Fully Connected Vehicle Trajectory

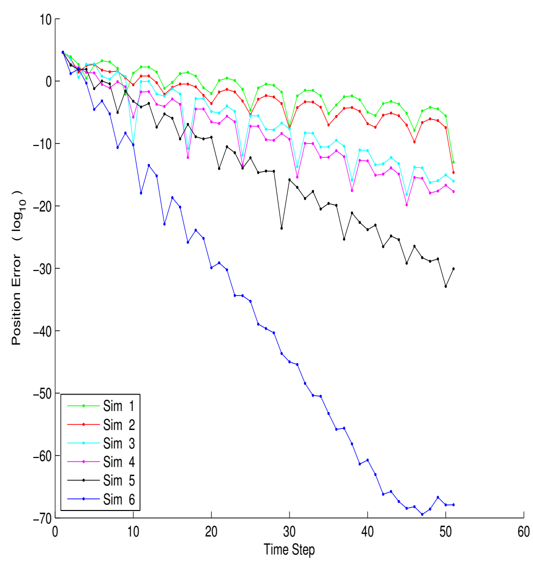

Fig. 5. Convergence Rates

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