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A Characterization of the Critical Value for Kalman Filtering with Intermittent Observations

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***Abstract*— In [1], Sinopoli et al. analyzed the problem of optimal estimation for linear Gaussian systems where packets containing observations are dropped according to an i.i.d. Bernoulli process, modeling a memoryless erasure channel. In this case the authors showed that the Kalman Filter is still the optimal estimator, although boundedness of the error depends directly upon the channel arrival probability, p. In particular they also proved the existence of a critical value, p_c, for such probability, below which the Kalman filter will diverge. The authors were not able to compute the actual value of this critical probability for general linear systems, but provided upper and lower bounds. They were able to show that for special cases, i.e. C invertible, such critical value coincides with the lower bound. This paper computes the value of the critical arrival probability, under minimally restrictive conditions on the matrices A and C. This paper also gives an example to illustrate that the lower bound is not always tight.**

I. Introduction

A large wealth of applications demand wireless communication among small embedded devices. Wireless Sensor Network (WSN) technology provides the architectural paradigm to implement systems with a high degree of temporal and spatial granularity. Applications of sensor networks are becoming ubiquitous, ranging from environmental monitoring and control to building automation, surveillance and many others. Given their low power nature and the requirement of long lasting deployment, communication between devices is power constrained and therefore limited in range and reliability. Changes in the environment, such as the simple relocation of a large metal object in a room or the presence of people, will inevitably affect the propagation properties of the wireless medium. Channels will be time-varying and unreliable. Spurred by this consideration, our effort concentrates on the design and analysis of estimation and control algorithms over unreliable networks. A substantial body of literature has been devoted to such issues in the past few years. In this paper we want to revisit the paper of Sinopoli et al. [1]. In that paper, the authors analyzed the problem of optimal state estimation for discrete-time linear Gaussian systems, under the assumption that observations are sent to the estimator via a memoryless erasure channel. This implies the existence of a non-unitary arrival probability associated with each packet. Consequently some observations will inevitably be lost. In this case although the Kalman Filter is still the optimal estimator, the boundedness of its error depends on the arrival probabilities of the observation packets. In particular the authors proved the existence of a critical arrival value p_c, below which the Kalman filter will diverge. The authors were not able to compute the actual value of this critical probability for general linear systems, but provided upper and lower bounds. They were able to show that for special cases such critical value coincides with the lower bound.

A fair amount of research effort has been made toward finding the critical value. In [1], the author proved that the critical value coincides with the lower bound in a special case when the system observation matrix C is invertible. The condition was further weakened by Plarre and Bullo [2] to C only invertible on the observable subspace. Other schemes were also considered. In [3], the authors introduced smart sensors, which send the local Kalman estimation instead of raw observation. In [4], a similar scenario was discussed where the sensor sends a linear combination of the current and previous measurement. A Markovian packet dropping model was introduced in [5] and a stability criterion was given.

However, in all the above work, the critical value was derived under the condition that C is either invertible or invertible on the observable subspace and the critical value always coincides with the lower bound. In this paper, we characterize the critical value under more general conditions showing that it meets the lower bound in most cases. We also provide an example where the lower bound is not tight.

The paper is organized in the following manner. Section II states the problem; Section III derives the critical value under restrictive assumptions on the system structure; Section IV relaxes these assumptions to take into account a larger class of systems. Section V gives an example to show that in general the critical value is not the lower bound. Finally Section VI concludes the paper.

II. Problem Formulation

Consider the following linear system

$$\begin{aligned} x_{k+1} &= Ax_k + w_k, \\ y_k &= Cx_k + v_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^m$ is the output vector, $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are Gaussian random vectors with zero mean and covariance matrices $Q > 0$ and $R > 0$, respectively. Assume that the initial state, x_0 is also a

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Gaussian vector of mean Ex_0 and covariance matrix $\Sigma_0 > 0$. Let w_i, v_i, x_0 be mutually independent. Note that we assume the covariance matrices of w_i, v_i, x_0 to be strictly positive definite. Define $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ as the eigenvalues of A.

Consider the case where observations are sent to the estimator via a memoryless erasure channel, i.e. where the arrival of the observations is modeled by a Bernoulli independent process γ_k. According to this model, the measurement y_k sent at time k reaches its destination if $\gamma_k = 1$; it is lost otherwise. Let γ_k be independent of w_k, v_k, x_0, i.e. the communication channel is independent of both process and measurement noises and let $P(\gamma_k = 1) = p$.

The Kalman Filter equations for this system were derived in [1] and take the following form:

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + \gamma_k K_k (y_k - C\hat{x}_{k|k-1}),$$
$$P_{k|k} = P_{k|k-1} - \gamma_k K_k C P_{k|k-1},$$

where

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k}, \quad P_{k+1|k} = AP_{k|k}A^T + Q,$$
$$K_k = P_{k|k-1}C^T(CP_{k|k-1}C^T + R)^{-1},$$
$$\hat{x}_{0|-1} = Ex_0, P_{0|-1} = \Sigma_0.$$

In the hope to improve the legibility of the paper we will slightly abuse the notation, by substituting $P_{k|k-1}$ with P_k. The equation for the error covariance of the one-step predictor is the following:

$$P_{k+1} = AP_kA^T + Q - \gamma_k AP_kC^T(CP_kC^T + R)^{-1}CP_kA^T. \tag{2}$$

If γ_ks are i.i.d. Bernoulli random variables, the following theorem holds [1]:

Theorem 1: If $(A, Q^{\frac{1}{2}})$ is controllable, (A, C) is detectable, and A is unstable, then there exists a $p_c \in [0, 1)$ such that [1] [2]

$$\lim_{t\to\infty} EP_k = +\infty \quad \text{for } 0 \leq p \leq p_c \text{ and } \exists P_0 \geq 0 \tag{3}$$
$$EP_k \leq M_{P_0} \ \ \forall t \quad \text{for } p_c < p \leq 1 \text{ and } \forall P_0 \geq 0, \tag{4}$$

where $M_{P_0} > 0$ depends on the initial condition $P_0 \geq 0$.

For simplicity, we will say that EP_k is unbounded if $\lim_{t\to\infty} EP_k = +\infty$ or EP_k is bounded if there exists a uniform bound independent of k.

It is also quite simple to prove the following theorem.

Theorem 2: If $R, \Sigma_0, Q > 0$, then the critical value of a system is a function of just A, C, which is independent of R, Q, Σ_0.

Proof: Since $R, \Sigma_0, Q > 0$, we can find uniform upper and lower bounds $\alpha, \beta > 0$, such that $\alpha I_m \leq R \leq \beta I_m$, $\alpha I_n \leq \Sigma_0 \leq \beta I_n$ and $\alpha I_n \leq Q \leq \beta I_n$. Let $P_k, \underline{P}_k, \overline{P}_k, P_k^*$ be the error covariance matrices of systems (A, C, R, Σ_0, Q), $(A, C, \alpha I_m, \alpha I_n, \alpha I_n)$, $(A, C, \beta I_m, \beta I_n, \beta I_n)$ and (A, C, I_m, I_n, I_n) at time k respectively[3]. Since $P_{k+1} = AP_kA^T + Q - \gamma_k AP_kC^T(CP_kC^T + R)^{-1}CP_kA^T$ is monotonically increasing with respect to R, Q, P_k, we know that

$$\underline{P}_k \leq P_k \leq \overline{P}_k. \tag{5}$$

Since the initial condition satisfies $\underline{P}_0 = \alpha I_n = \alpha P_0^*$, and if $\underline{P}_k = \alpha P_k^*$, then

$$A\underline{P}_kA^T + \alpha I_n - \gamma_k A\underline{P}_kC^T(C\underline{P}_kC^T + \alpha I_m)^{-1}C\underline{P}_kA^T$$
$$= \alpha[AP_k^*A^T + I_n - \gamma_k AP_K^*C^T(CP_k^*C^T + I_m)^{-1}CP_k^*A^T]$$
$$= \underline{P}_{k+1} = \alpha P_{k+1}^*.$$

Thus, by induction we can conclude that $\underline{P}_k = \alpha P_k^*$ and $\overline{P}_k = \beta P_k^*, \forall k$. By inequality (5), $\alpha P_k^* \leq P_k \leq \beta P_k^*$, which shows that EP_k is bounded if and only if EP_k^* is bounded. Thus, we have proved that the critical value is independent of R, Q, Σ_0 if they are all positive definite. ∎

Since we have already assumed that R, Q, Σ_0 are strictly positive definite, by Theorem 2, we can let $R = I_m, Q = I_n, \Sigma_0 = I_n$ for convenience.

III. Critical Value under Restrictive Conditions

In this section we will show that

$$p_c = 1 - \frac{1}{|\lambda_1|^2}, \tag{6}$$

if the system of equations (1) satisfies the following conditions:

1) (C, A) is detectable,
2) $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 1$.

In Section IV we will be able to relax these conditions to include a wider class of systems. Since all the eigenvalues of A are different, without loss of generality, we will restrict our analysis to systems with diagonal A. Also by Theorem 2, we will assume that $R = I_m, Q = \Sigma_0 = I_n$.

A. Equivalent Conditions for Boundedness

This subsection is devoted to the derivation of equivalent conditions for boundedness of the Kalman Estimation. We will use the fact that the Kalman filter is the optimal unbiased filter for (1).

First consider the case where no packets are lost. We can write the estimation problem in the following form:

$$\begin{bmatrix} y_k \\ \vdots \\ y_1 \\ Ex_0 \end{bmatrix} = \begin{bmatrix} CA^{-1} \\ \vdots \\ CA^{-k} \\ A^{-k-1} \end{bmatrix} x_{k+1} + \begin{bmatrix} v_k \\ \vdots \\ v_1 \\ Ex_0 - x_0 \end{bmatrix} - \begin{bmatrix} CA^{-1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ CA^{-k} & \cdots & CA^{-1} & 0 \\ A^{-k-1} & \cdots & A^{-2} & A^{-1} \end{bmatrix} \begin{bmatrix} w_k \\ \vdots \\ w_1 \\ w_0 \end{bmatrix}. \tag{7}$$

[1] We use the notation $\lim_{t\to\infty} A_k = +\infty$ when the sequence $A_k \geq 0$ is not bounded; i.e., there is no matrix $M \geq 0$ such that $A_k \leq M, \forall t$.

[2] Note that all the comparisons between matrices in this paper are in the sense of positive definite if without further notice

[3] $P_k, \overline{P}_k, \underline{P}_k, P_k^*$ are random matrices which depend on the random variables $\gamma_0, \ldots, \gamma_{k-1}$

Let us define the following quantities:

$$F_k \triangleq \begin{bmatrix} A^{-1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ A^{-k} & \cdots & A^{-1} & 0 \\ A^{-k-1} & \cdots & A^{-2} & A^{-1} \end{bmatrix} \in \mathbb{R}^{n(k+1)\times n(k+1)}, \quad (8)$$

$$G_k \triangleq diag(\underbrace{C, C, \ldots, C}_{k}, I) \in \mathbb{R}^{(n+mk)\times n(k+1)}, \quad (9)$$

$$e_k \triangleq -G_k F_k \begin{bmatrix} w_k \\ \vdots \\ w_1 \\ w_0 \end{bmatrix} + \begin{bmatrix} v_k \\ \vdots \\ v_1 \\ Ex_0 - x_0 \end{bmatrix} \in \mathbb{R}^{n+mk}, \quad (10)$$

$$T_k \triangleq \begin{bmatrix} CA^{-1} \\ \vdots \\ CA^{-k} \\ A^{-k-1} \end{bmatrix} \in \mathbb{R}^{(n+mk)\times n}, \quad (11)$$

$$Y_k \triangleq \begin{bmatrix} y_k \\ \vdots \\ y_1 \\ Ex_0 \end{bmatrix} \in \mathbb{R}^{n+mk}. \quad (12)$$

Equation (7) can be written in a more compact form as

$$Y_k = T_k x_{k+1} + e_k. \quad (13)$$

When the observation packets travel through a lossy network, equation (7) will be modified in the following way:

$$\begin{bmatrix} \gamma_k y_k \\ \vdots \\ \gamma_1 y_1 \\ Ex_0 \end{bmatrix} = \begin{bmatrix} \gamma_k CA^{-1} \\ \vdots \\ \gamma_1 CA^{-k} \\ A^{-k-1} \end{bmatrix} x_{k+1} + \begin{bmatrix} \gamma_k v_k \\ \vdots \\ \gamma_1 v_1 \\ Ex_0 - x_0 \end{bmatrix} - \begin{bmatrix} \gamma_k CA^{-1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_1 CA^{-k} & \cdots & \gamma_1 CA^{-1} & 0 \\ A^{-k-1} & \cdots & A^{-2} & A^{-1} \end{bmatrix} \begin{bmatrix} w_k \\ \vdots \\ w_1 \\ w_0 \end{bmatrix}. \quad (14)$$

The rows where γ_is are zero can be deleted, since they do not contribute any information to improve the estimate of x_{k+1}.

Define Γ_k as the matrix of all non zero rows of $diag(\gamma_k I_m, \ldots, \gamma_1 I_m, I_n)$. Thus Γ_k is $(m\sum_{i=1}^{k}\gamma_i + n)$ by $(n+mk)$ matrix. Also define

$$\widetilde{Y}_k \triangleq \Gamma_k Y_k, \widetilde{G}_k \triangleq \Gamma_k G_k, \widetilde{T}_k \triangleq \widetilde{G}_k F_k, \widetilde{e}_k \triangleq \Gamma_k e_k.$$

$\widetilde{Y}_k, \widetilde{T}_k, \widetilde{e}_k$ are now stochastic matrices as they are functions of $\gamma_k, \gamma_{k-1}, \ldots, \gamma_1$.

We can rewrite (13) as

$$\widetilde{Y}_k = \widetilde{T}_k x_{k+1} + \widetilde{e}_k. \quad (15)$$

We are now ready to prove the following theorem to bound the error covariance of the Kalman filter.

Theorem 3: The error covariance matrix of the Kalman Filter is bounded by

$$\underline{\alpha}(\widetilde{T}_k^T \widetilde{T}_k)^{-1} \le P_{k+1} \le \overline{\alpha}(\widetilde{T}_k^T \widetilde{T}_k)^{-1}, \quad (16)$$

where $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}$ are constant and independent of both γ_i and k.

Before we can prove this theorem, we need the following lemmas.

Lemma 1: If A is invertible, then the Kalman filter satisfies the following equation:

$$P_{k+1} = (\widetilde{T}_k^T Cov(\widetilde{e}_k|\Gamma_k)^{-1}\widetilde{T}_k)^{-1}. \quad (17)$$

Due to space constraints, we refer the interested readers to [6].

Lemma 2: If $A = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, where $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n| > 1$, then $F_k F_k^T$ is bounded by

$$\frac{1}{(|\lambda_1|+1)^2} I_{n(k+1)} \le F_k F_k^T \le \frac{1}{(|\lambda_n|-1)^2} I_{n(k+1)}, \quad (18)$$

where F_k is defined in (8).

Proof: Notice that

$$F_k^{-1} = \begin{bmatrix} A & & & \\ -I & \ddots & & \\ & \ddots & A & \\ & & -I & A \end{bmatrix}.$$

Therefore,

$$(F_k F_k^T)^{-1} = \begin{bmatrix} AA^T + I & -A & & \\ -A^T & \ddots & \ddots & \\ & \ddots & AA^T + I & -A \\ & & -A^T & AA^T \end{bmatrix}.$$

By Gershgorin's Circle Theorem [7], we know that all the eigenvalues of $(F_k F_k^T)^{-1}$ are located inside one of the following circles: $|\zeta - |\lambda_i|^2 - 1| = |\lambda_i|, |\zeta - |\lambda_i|^2 - 1| = 2|\lambda_i|, |\zeta - |\lambda_i|^2| = |\lambda_i|$, where ζs are the eigenvalues of $(F_k F_k^T)^{-1}$.

Since $|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_n| > 1$, we know that for each eigenvalue of $(F_k F_k^T)^{-1}$ the following holds:

$$\zeta \ge min\{|\lambda_i|^2+1-|\lambda_i|, |\lambda_i|^2+1-2|\lambda_i|, |\lambda_i|^2-|\lambda_i|\}, \quad (19)$$

and

$$\zeta \le max\{|\lambda_i|^2+1+|\lambda_i|, |\lambda_i|^2+1+2|\lambda_i|, |\lambda_i|^2+|\lambda_i|\}. \quad (20)$$

Thus, $(|\lambda_n|-1)^2 \le \zeta \le (|\lambda_1|+1)^2$, which in turn gives

$$\frac{1}{(|\lambda_1|+1)^2} I_{n(k+1)} \le F_k F_k^T \le \frac{1}{(|\lambda_n|-1)^2} I_{n(k+1)}.$$

■

We are now ready to prove Theorem 3.

Proof: Since w_i, v_j, x_0 are mutually independent,

$$Cov(e_k) = Cov(G_kF_k[w_k,\ldots,w_0]^T) + Cov([v_k,\ldots,v_1,x_0]^T) = G_kF_k diag(Q,Q,\ldots,Q)F_k^TG_k^T + diag(R,R,\ldots,R,\Sigma_0).$$

Since we assume that $Q = I_m, R = I_n, \Sigma_0 = I_n$, it is easy to show that:

$$G_kF_kF_k^TG_k^T + I_{n+mk} \leq Cov(e_k) = G_kF_k diag(Q,Q,\ldots,Q)F_k^TG_k^T + diag(R,R,\ldots,R,\Sigma_0) \leq G_kF_kF_k^TG_k^T + I_{n+mk}.$$

Using Lemma 2,

$$G_kF_kF_k^TG_k^T + I_{n+mk} \leq \frac{1}{(|\lambda_n|-1)^2}G_kG_k^T + I_{n+mk},$$

and

$$G_kF_kF_k^TG_k^T + I_{n+mk} \geq \frac{1}{(|\lambda_1|+1)^2}G_kG_k^T + I_{n+mk}.$$

Since $G_kG_k^T = diag(CC^T,\ldots,CC^T,I_n)$, we define $\underline{n}_G \triangleq \min(\lambda_{min}(CC^T),1)$ and $\overline{n}_G \triangleq \max(\lambda_{max}(CC^T),1)$. We know that $\underline{n}_G I_{n+mk} \leq G_kG_k^T \leq \overline{n}_G I_{n+mk}$, which gives

$$\underline{\alpha}\Gamma_k\Gamma_k^T \leq Cov(\widetilde{e}_k|\Gamma_k) = \Gamma_k Cov(e_k)\Gamma_k^T \leq \overline{\alpha}\Gamma_k\Gamma_k^T,$$

where $\underline{\alpha} = \frac{\underline{n}_G}{(|\lambda_1|+1)^2} + 1, \overline{\alpha} = \frac{\overline{n}_G}{(|\lambda_n|-1)^2} + 1$. Notice that $\Gamma_k\Gamma_k^T = I$. Therefore,

$$\underline{\alpha}I \leq Cov(\widetilde{e}_k|\Gamma_k) \leq \overline{\alpha}I.$$

The above bound is independent of k and γ_i, which proves

$$\underline{\alpha}(\widetilde{T}_k^T\widetilde{T}_k)^{-1} \leq P_{k+1} = (\widetilde{T}_k^T Cov(\widetilde{e}_k|\Gamma_k)^{-1}\widetilde{T}_k)^{-1} \leq \overline{\alpha}(\widetilde{T}_k^T\widetilde{T}_k)^{-1}.$$

■

B. Derivation of the Critical Value

In this subsection we will derive the actual critical value for systems satisfying the assumptions mentioned before on A, C. We will first find a uniform upper bound for $E(\widetilde{T}_k^T\widetilde{T}_k)^{-1}$.

Theorem 4: If $E(\sum_{i=1}^{\infty}\mu_i A^{-iT}C^TCA^{-i})^{-1}$ exists, then the following inequality holds:

$$E(\widetilde{T}_k^T\widetilde{T}_k)^{-1} \leq \beta E(\sum_{i=1}^{\infty}\mu_i A^{-iT}C^TCA^{-i})^{-1}, \tag{21}$$

where μ_is are i.i.d. Bernoulli random variables with the same distribution as γ_is, and β is a constant.

Proof: Rewrite $\widetilde{T}_k^T\widetilde{T}_k$ as

$$\widetilde{T}_k^T\widetilde{T}_k = \sum_{i=1}^{k}\gamma_{k+1-i}A^{-iT}C^TCA^{-i} + A^{-(k+1)T}A^{-(k+1)}.$$

We know that $\widetilde{T}_k^T\widetilde{T}_k$ will have the same distribution as $\sum_{i=1}^{k}\mu_i A^{-iT}C^TCA^{-i} + A^{-(k+1)T}A^{-(k+1)}$, because $\mu_1,\ldots,\mu_k$ have the same distribution as $\gamma_k,\ldots,\gamma_1$.

Since all the eigenvalues of A are unstable, $\sum_{i=0}^{\infty}A^{-iT}C^TCA^{-i} \leq \alpha I$, where $\alpha > 0$ is a constant. Thus

$$A^{-(k+1)T}A^{-(k+1)} \geq A^{-(k+1)T}\left(\alpha^{-1}\sum_{i=0}^{\infty}A^{-iT}C^TCA^{-i}\right)A^{-(k+1)} = \alpha^{-1}\sum_{i=k+1}^{\infty}A^{-iT}C^TCA^{-i} \geq \alpha^{-1}\sum_{i=k+1}^{\infty}\mu_iA^{-iT}C^TCA^{-i}.$$

Also,

$$\sum_{i=1}^{k}\mu_iA^{-iT}C^TCA^{-i} + A^{-(k+1)T}A^{-(k+1)} \geq \sum_{i=1}^{k}\mu_iA^{-iT}C^TCA^{-i} + \alpha^{-1}\sum_{i=k+1}^{\infty}\mu_iA^{-iT}C^TCA^{-i} \geq min(1,\alpha^{-1})\sum_{i=1}^{\infty}\mu_iA^{-iT}C^TCA^{-i},$$

and

$$E(\widetilde{T}_k^T\widetilde{T}_k)^{-1} \leq max(1,\alpha)E(\sum_{i=1}^{\infty}\mu_iA^{-iT}C^TCA^{-i})^{-1}$$

for all k, which proves the inequality (21). ■

The upper bound for $E(\widetilde{T}_k^T\widetilde{T}_k)^{-1}$ we derived is an expectation of an infinite sum. In the following part we will use only n terms of the infinite summ. We will show that by careful choice of these n terms, we can derive an upper bound of EP_k whose critical arrival possibility meets the lower bound in [1].

Lemma 3: Define $\Delta i_1 = i_1$ and $\Delta i_j = i_j - i_{j-1}, j = 2,3,\ldots,n$. If $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ and $l_{i,i} = 1, i = 1,\ldots,n$, then the determinant

$$D \triangleq \begin{vmatrix} l_{1,1}\lambda_1^{-i_1} & l_{1,2}\lambda_1^{-i_2} & \cdots & l_{1,n}\lambda_1^{-i_n} \\ l_{2,1}\lambda_2^{-i_1} & l_{2,2}\lambda_2^{-i_2} & \cdots & l_{2,n}\lambda_2^{-i_n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n,1}\lambda_n^{-i_1} & l_{n,2}\lambda_n^{-i_2} & \cdots & l_{n,n}\lambda_n^{-i_n} \end{vmatrix}$$

is asymptotic to $\prod_{k=1}^{n}\lambda_k^{-i_k}$, i.e.

$$\lim_{\Delta i_1,\Delta i_2,\ldots,\Delta i_n\to\infty}\frac{D}{\prod_{k=1}^{n}\lambda_k^{-i_k}} = 1 \tag{22}$$

Proof: The determinant is the summation of $n!$ terms. It is easy to compute the limit for each term and find that (22) holds. For a more rigorous proof please refer to [6]. ■

We are now ready to establish the main result.

Theorem 5: The critical value of the arrival probability for system (1) is

$$p_c = 1 - \frac{1}{|\lambda_1|^2},$$

if the system satisfies the following properties:

1) (C, A) is detectable.

2) $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 1$.

If the arrival probability $p > p_c$, then for all initial conditions, EP_k will be bounded for all k. Else if $p < p_c$, for some initial conditions, EP_k is unbounded.

Proof: Since A is a diagonal matrix with all unstable eigenvalues and (C, A) is detectable, we know that for each column of C, there exists at least one element which is not zero. Thus, there exist row vectors $L_1, L_2, \cdots, L_n$, such that $L_iC = [l_{i,1}, \ldots, l_{i,n}]$ is a row vector with $l_{i,i} = 1$. For example, if

$$C = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

we could then choose $L_1 = [0.5, 0], L_2 = [0, 1], L_3 = [1, 0]$. Also define matrix

$$U \triangleq \begin{bmatrix} l_{1,1}\lambda_1^{-i_1} & l_{1,2}\lambda_2^{-i_1} & \cdots & l_{1,n}\lambda_n^{-i_1} \\ l_{2,1}\lambda_1^{-i_2} & l_{2,2}\lambda_2^{-i_2} & \cdots & l_{2,n}\lambda_n^{-i_2} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n,1}\lambda_1^{-i_n} & l_{n,2}\lambda_2^{-i_n} & \cdots & l_{n,n}\lambda_n^{-i_n} \end{bmatrix}. \quad (23)$$

By Lemma 3, we know that

$$\lim_{\Delta i_1, \Delta i_2, \ldots, \Delta i_n \to \infty} \frac{\det(U)}{\prod_{k=1}^n \lambda_k^{-i_k}} = 1.$$

Thus, there exists $\xi_i > 0$, such that if $\Delta i_j \geq \xi_j, j = 1, 2, \ldots, n$, then $|det(U)| \geq 0.5|\prod_{k=1}^n \lambda_k^{-i_k}| > 0$. Define the stopping time $i_1 \triangleq \inf\{i \geq \xi_1 | \mu_i = 1\}$ and $i_j \triangleq \inf\{i \geq \xi_j + i_{j-1} | \mu_i = 1\}, j = 2, 3, \ldots, n$. Basically i_j is the index of the first occurrence of $\mu_i = 1$ after $i_{j-1} + \xi_i$.

From the definition of i_j, it is easy to show that:

$$\sum_{i=1}^{\infty} \mu_i A^{-iT} C^T C A^{-i} \geq \sum_{j=1}^{n} A^{-i_j T} C^T C A^{-i_j}. \quad (24)$$

Define $\overline{n}_l \triangleq \max(\lambda_{max}(L_1^T L_1), \ldots, \lambda_{max}(L_n^T L_n))$. Thus, $L_i^T L_i \leq \overline{n}_l I_m$, and

$$\sum_{j=1}^{n} A^{-i_j T} C^T C A^{-i_j} \geq \sum_{j=1}^{n} \frac{1}{\overline{n}_l} A^{-i_j T} C^T L_j^T L_j C A^{-i_j}$$
$$= \frac{1}{\overline{n}_l} \begin{bmatrix} A^{-i_1 T} C^T L_1^T & \cdots & A^{-i_n T} C^T L_1^T \end{bmatrix} \begin{bmatrix} L_1 C A^{-i_1} \\ \vdots \\ L_n C A^{-i_n} \end{bmatrix}$$
$$= \frac{1}{\overline{n}_l} U^T U.$$

Define $O \triangleq U^{-1}$. We know that

$$\left(\sum_{i=1}^{\infty} \mu_i A^{-iT} C^T C A^{-i} \right)^{-1} \leq \overline{n}_l O O^T, \quad (25)$$

which implies that if $E(OO^T)$ is bounded, then EP_k is bounded. We also know that

$$trace(OO^T) = \sum_{i,j} O_{i,j} (O^T)_{j,i} = \sum_{i,j} O_{i,j}^2.$$

Since the boundedness of a positive semidefinite matrix is equivalent to the boundedness of the trace, we only need to check whether $E(\sum_{i,j} O_{i,j}^2)$ is bounded.

Now by using Lemma 3, we can compute the cofactor matrix of U and hence $O = U^{-1}$. Define the minor $M_{i,j}$ of U as the $(n-1) \times (n-1)$ matrix that results from deleting row i and column j. Thus

$$O_{i,j} = \frac{(-1)^{i+j} \det(M_{j,i})}{\det(U)}.$$

From the definition of the random variable i_j, we know that $\Delta i_j \geq \xi_i$, which shows that $|\det(U)| \geq 0.5 \prod_{k=1}^n |\lambda_k^{-i_k}|$. And since $M_{i,j}$ has the same structure as U, it is easy to show that $|M_{i,j}|$ is always bounded by $|\rho_{i,j}| \prod_{k=2}^n |\lambda_k^{-i_{k-1}}|$, where $\rho_{i,j}$ is a constant.

Thus,

$$|O_{i,j}|^2 \leq |2\rho_{i,j} \frac{\prod_{k=2}^n \lambda_k^{-i_{k-1}}}{\prod_{k=1}^n \lambda_k^{-i_k}}|^2 = 4\rho_{i,j}^2 |\prod_{k=1}^n \lambda_k^{2\Delta i_k}|. \quad (26)$$

Therefore, $E[\sum_{i,j} O_{i,j}^2]$ is bounded if $E \prod_{k=1}^n |\lambda_k^{2\Delta i_k}|$ is bounded. From the definition of random variable i_j, we know that Δi_js are independent of each other. Also

$$P(\mu_{i_{j-1}+\xi_j} = 0, \ldots, \mu_{i_{j-1}+\xi_j+k-1} = 0, \mu_{i_{j-1}+\xi_j+k} = 1)$$
$$= P(\Delta i_j = k) = (1-p)^{k-\xi_j-1} p, \quad k \geq \xi_i.$$

Now we can compute the expectation

$$E \prod_{j=1}^n |\lambda_j^{2\Delta i_j}| = \prod_{j=1}^n E|\lambda_j|^{2\Delta i_j} = \prod_{j=1}^n \sum_{k=\xi_j}^{\infty} |\lambda_j|^{2k} P(\Delta i_j = k)$$
$$= \prod_{j=1}^n \sum_{k=\xi_j}^{\infty} |\lambda_j|^{2k} (1-p)^{k-\xi_j-1} p,$$

which is bounded if and only if

$$|\lambda_j|^2 (1-p) < 1, \quad j = 1, 2, \ldots, n$$

Since the boundedness of $E \prod_{k=1}^n |\lambda_k^{2\Delta i_k}|$ implies the boundedness of $E(\widetilde{T}_k^T \widetilde{T}_k)^{-1}$, we immediately know that the upper bound for the critical value is $1 - |\lambda_1|^{-2}$. Combining such bound with the lower bound given in [1], we complete the proof. ■

IV. Boundedness under Weaker Conditions

In this section we give a theorem to show that the result in Section IV holds even if A has stable eigenvalues. Due to space constrains, we have to omit the proof. Please refer to [6] for the proof.

Theorem 6: For systems satisfying:

1) (C, A) detectable,
2) $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_i| \geq 1 > |\lambda_{i+1}| \geq \cdots \geq |\lambda_n|$,
3) A can be diagonalized,

the critical value of the system is $1 - |\lambda_1|^{-2}$.

V. Example

In this section we want to give an example to show that, if some unstable eigenvalues of A have the same absolute value, then in general the critical value is not $1-|\lambda_1|^{-2}$.

Consider the following system:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} x_k + w_k \\ y_k &= \begin{bmatrix} 1 & 1 \end{bmatrix} x_k + v_k. \end{aligned} \tag{27}$$

By Theorem 3, we know that the Kalman Estimation Error will be bounded if and only if $E(\widetilde{T}_k^T\widetilde{T}_k)^{-1}$ is bounded. We can rewrite $\widetilde{T}_k^T\widetilde{T}_k$ as

$$\begin{aligned} \widetilde{T}_k^T\widetilde{T}_k &= \sum_{i=1}^{k}\gamma_{k+1-i}4^{-i}\begin{bmatrix} 1 & (-1)^i \\ (-1)^i & 1 \end{bmatrix} + 4^{-(k+1)}I_2 \\ &= \begin{bmatrix} \alpha+\beta+\delta & \beta-\alpha \\ \beta-\alpha & \alpha+\beta+\delta \end{bmatrix}, \end{aligned}$$

where $\delta = 4^{-(k+1)}, \alpha = 4\sum_{j=1}^{\lceil k/2\rceil}\gamma_{k+2-2j}16^{-j}, \beta = \sum_{j=1}^{\lfloor k/2\rfloor}\gamma_{k+1-2j}16^{-j}$, with $\lceil x\rceil = \inf\{z\in\mathbb{Z}|z\geq x\}$ and $\lfloor x\rfloor = \sup\{z\in\mathbb{Z}|z\leq x\}$.

It is easy to see that

$$\begin{aligned} tr(\widetilde{T}_k^T\widetilde{T}_k) &= 2(\alpha+\beta+\delta) \\ \det(\widetilde{T}_k^T\widetilde{T}_k) &= 4\alpha\beta+2(\alpha+\beta)\delta+\delta^2, \end{aligned}$$

Since the boundedness of a positive definite matrix is equivalent to the boundedness of its trace, we can just study the trace.

$$\begin{aligned} tr[(\widetilde{T}_k^T\widetilde{T}_k)^{-1}] &= \sigma_1^{-1}+\sigma_2^{-1} = \frac{\sigma_1+\sigma_2}{\sigma_1\sigma_2} = \frac{tr(\widetilde{T}_k^T\widetilde{T}_k)}{\det(\widetilde{T}_k^T\widetilde{T}_k)} \\ &= \frac{2(\alpha+\beta+\delta)}{4\alpha\beta+2(\alpha+\beta)\delta+\delta^2} = \frac{1}{2\alpha+\delta}+\frac{1}{2\beta+\delta}, \end{aligned}$$

where σ_1,σ_2 are eigenvalues of $\widetilde{T}_k^T\widetilde{T}_k$.

Since α,β have the same structure, we just need to study the expectation of the first term. Define $\alpha^* \triangleq 4\sum_{i=1}^{\infty}\mu_i 16^{-i}$, where μ_i are i.i.d. Bernoulli random variables with the same distribution as γ_i. Thus [4]

$$\begin{aligned} \sup_k E(2\alpha+\delta)^{-1} &= \sup_k E(2\sum_{i=1}^{\lceil k/2\rceil}\mu_i 16^{-i}+\delta)^{-1} \\ &\geq \sup_k E(2\alpha^*+\delta)^{-1} = \lim_{k\to\infty}E(2\alpha^*+\delta)^{-1} \\ &= E(\lim_{k\to\infty}(2\alpha^*+\delta)^{-1}) = E(2\alpha^*)^{-1}. \end{aligned}$$

The above inequality indicates that $E(2\alpha^*)^{-1}$ is a lower bound for $E(2\alpha+\delta)^{-1}$. Define stopping time $t_s = \inf\{i>0|\mu_i = 1\}$. We know that

$$16^{-t_s} \leq \sum_{i=0}^{\infty}\mu_i 16^{-i} = \frac{1}{4}\alpha^* \leq \sum_{i=t_s}^{\infty}16^{-i} = \frac{16}{15}16^{-t_s}.$$

[4]By Monotone Convergence Theorem, we can exchange the limit and expectation since $(2\alpha^*+\delta)^{-1}$ is monotonically increasing with respect to k.

Thus, $E(2\alpha^*)^{-1}$ will be bounded if and only if $E(16^{-t_s})^{-1} = E(16^{t_s})$ is bounded. From the definition of t_s, it is easy to show that the arrival probability p needs to be greater than $15/16$ in order to make $E(16^{t_s})$ bounded. Thus, $15/16$ will be a lower bound for critical value because $E(2\alpha^*)^{-1}$ is a lower bound for $E(2\alpha+\delta)^{-1}$. Since we already know from [1] that $1-|2\times(-2)|^{-2} = 15/16$ is an upper bound, we can conclude that $15/16$ is the critical value, which is different from the lower bound $1-2^{-2} = 3/4$ given in [1].

VI. Conclusions and Future Work

In this paper we address the problem of state estimation for a discrete-time linear Gaussian system where observations are sent to the estimator via a memoryless erasure channel. Following the work of Sinopoli et al. [1], we were able to compute the critical arrival value for a very general class of linear systems. The boundedness analysis in this paper can be easily generalized to general Markovian packet loss models and to the boundedness of higher moments of the error covariance. Future work will attempt at determining the complete statistics of the error covariance matrix of the Kalman Filter under Bernoulli losses.

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VIII. REFERENCES

[1] B. Sinopoli, L. Schenato, M. Franceschetti, K. Poolla, M. Jordan, and S. Sastry, "Kalman filtering with intermittent observations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1453–1464, September 2004.

[2] K. Plarre and F. Bullo, "On kalman filtering for detectable systems with intermittent observations," *IEEE Transactions on Automatic Control*, May 2007, submitted for publication.

[3] Y. Xu and J. Hespanha, "Estimation under controlled and uncontrolled communications in networked control systems," in *Proceedings of the CDC-ECC*, Sevilla, Spain, December 2005, pp. 842–847.

[4] C. Robinson and P. R. Kumar, "Sending the most recent observation is not optimal in networked control: Linear temporal coding and towards the design of a control specific transport protocol," in *Proc. of IEEE CDC*, Dec 2007, pp. 334–339.

[5] M. Huang and S. Dey, "Stability of kalman filtering with markovian packet losses," *Automatica*, vol. 43, no. 4, pp. 598–607, 2007.

[6] Y. Mo and B. Sinopoli, "A characterization of the critical value for kalman filtering with intermittent observations," *www.ece.cmu.edu/~brunos/Publications/cr.pdf, preprint.*

[7] R. S. Varga, *Geršgorin and His Circles*. New York: Springer, 2004.