Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Quantized H_∞ Filtering for Continuous-Time Systems with Quantizer Ranges Consideration

Wei-Wei Che and Guang-Hong Yang

Abstract— **This paper studies the quantized H_∞ filtering problem for continuous-time systems with a type of dynamic quantizers, which are conjuncted with static quantizers via dynamic scalings. The static quantizer ranges are fully considered here for practical transmission channels requirements. A quantized H_∞ filter design strategy is proposed, where a convex optimization method is developed to minimize static quantizer ranges. The resulting design guarantees that the quantized augmented system is asymptotically stable and with a prescribed H_∞ performance bound. The effectiveness of the proposed filter design method is demonstrated by a numerical example.**

I. INTRODUCTION

The need for representing signals by a finite number of bits implies that quantization noise is present in almost all digital signal processing systems and networked control systems. The distortion error, or quantization noise, consists of the difference between the input to the quantizer and the discrete output signal. Filters are the most essential building blocks of signals processing. And the problem of filtering with quantized signals has been considered by several researchers [16], [20], ect.

There has been a significant amount of works examining the impact of quantization on linear systems. These quantization policies can be mainly categorized depending on whether the quantizer is static or dynamic. A static quantizer is a memoryless nonlinear function, whereas a dynamic quantizer uses memory and thus can be much more complex and potentially more powerful.

There are many works using static quantizers, such as [3]-[5], [7], [10], [14] and [21]. The main drawback of static policies is that they require an infinite number of quantization bits to ensure asymptotic stability [3]. In contrast, quantization policies with memory (dynamic quantization policies) have been shown to achieve asymptotic stability with a finite number of quantization levels. Noticeable work along this line includes [1], [8], [11]-[13], [15], [17]-[19]. [1] established sufficient conditions for asymptotic stability that were later tightened in [12]. Due to the superiority of the dynamic quantization policies, in this paper, we focus on the quantized systems with dynamic quantizers.

But noting that all the above works with dynamic quantizers deal with only stability/stabilization problems, there is no work for considering the H_∞ control problem. As pointed out in [22] that the control strategies of updating the quantizer's parameter are dependent on time in the existing works [1] and [12], and such control strategies cannot be applied for the case of H_∞ control systems since we do not know the value of the disturbance inputs and thus cannot drive the state into an invariant region, as done in [12]. In contrast, in [22], a state or output dependent control strategy is proposed, and by using which the quantized continuous-time H_∞ control problem is solved. Similarly, the latest work [2] studies the networked-based H_∞ control problem with dynamic quantizers. But these works do not consider the minimum number of quantization levels required to assure the H_∞ performance requirements for quantized systems. However, in digital signal processing systems and networked control systems, a major question about the quantized systems concerns the minimum number of quantization levels required to assure closed loop system stability and performance, which may have many benefits for networked systems that include lower cost, higher reliability, and easier maintenance.

Due to the above reasons, this paper considers the quantized filtering problem for continuous-time linear systems with a type of dynamic quantizers, which are conjuncted with static quantizers via dynamic scalings. A quantized H_∞ filter design strategy is proposed, where a convex optimization method is developed to minimize static quantizer ranges (minimize the required quantization levels for fixed quantization sensitivity). The resulting design guarantees that the quantized augmented system is asymptotically stable and with the same H_∞ performance bound as that for the case without quantization. The effectiveness of the proposed filter design method is demonstrated by a numerical example. Here it should be mentioned that the quantized filtering problem for discrete-time systems has been considered in our another paper.

The organization of this paper is as follows. Section II presents the problem under consideration and some preliminaries. Section III gives design methods of quantized H_∞ filtering strategies. In Section IV, an example is presented to illustrate the effectiveness of the proposed methods. Finally, Section V gives some concluding remarks.

Notation: Given a matrix E, E^T and E^{-1} denote its transpose, and inverse when it exists, respectively. The symbol $*$ within a matrix represents the symmetric entries. For a vector $x \in R^k$, the 2-norm of x is defined as $|x| := (x^T x)^{\frac{1}{2}}$, and for a matrix $Q \in R^{m\times n}$, $\|Q\|$, $\lambda_{max}(Q)$ and $\lambda_{min}(Q)$ is defined as the largest singular value, the maximum eigenvalue and

Wei-Wei Che is with the College of Information Science and Engineering, Northeastern University, Shenyang, 110004, P.R. China. Fax: (86)24-83681939; email: cwwemail1980@126.com

Guang-Hong Yang is with the College of Information Science and Engineering, Northeastern University, Shenyang, China 110004. Corresponding author. Fax: (86) 24-83681939; e-mail: yangguanghong@ise.neu.edu.cn

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

the minimum eigenvalue of matrix Q, respectively.

II. Problem statement and preliminaries

A. Problem statement

Consider an LTI model described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1\omega(t), \\ z(t) &= C_1x(t), \\ y(t) &= C_2x(t), \end{aligned} \quad (1)$$

where $x(t) \in R^n$ is the state, $y(t) \in R^p$ is the measured output and $z(t) \in R^q$ is the regulated output, respectively. A, B_1, C_1 and C_2 are known constant matrices of appropriate dimensions.

To formulate the filtering problem, we consider a filter of the following form:

$$\begin{aligned} \dot{\xi}(t) &= A_F\xi(t) + B_Fy(t), \\ z_F(t) &= C_F\xi(t) + D_Fy(t), \end{aligned} \quad (2)$$

where $\xi(t) \in R^n$ is the filter state, $z_F(t)$ is the estimation of $z(t)$, and the constant matrices A_F, B_F, C_F and D_F are filter matrices to be designed.

Applying filter (2) to system (1), the following augmented system is obtained:

$$\begin{aligned} \dot{x}_e(t) &= A_ex_e(t) + B_e\omega(t) \\ z_e(t) &= C_ex_e(t), \end{aligned} \quad (3)$$

where $x_e(t) = [x(t)^T, \xi(t)^T]^T$, $z_e(t) = z(t) - z_F(t)$ is the estimation error, and

$$A_e = \begin{bmatrix} A & 0 \\ B_FC_2 & A_F \end{bmatrix},$$

$$B_e = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, C_e = \begin{bmatrix} C_1 - D_FC_2 & -C_F \end{bmatrix}.$$

The transfer function matrix of the augmented system (3) from ω to z_e is give by

$$T_{z_e\omega} = C_e(sI - A_e)^{-1}B_e.$$

Let A_F, B_F, C_F and D_F be given, and such that

$$\|T_{z_e\omega}\| = \|C_e(sI - A_e)^{-1}B_e\| < \gamma. \quad (4)$$

Due to the well known Bounded Real Lemma, without loss of generality, we may assume that A_F, B_F, C_F and D_F satisfy the following assumption:

Assumption 1: There exist two positive definite matrices P and Q satisfying the following LMI

$$A_e^TP + PA_e + \gamma^{-2}PB_eB_e^TP + C_e^TC_e + Q < 0. \quad (5)$$

However, in practical network transmission and digital signal processing systems, signals are quantized before they are communicated, and a quantizer is with different form.

In this paper, the definition of a quantizer is given with general form as in [12]. Let $\xi \in R^l$ be the variable being quantized. By a *quantizer* we mean a piecewise constant function $q: R^l \to \mathbb{D}$, where $\mathbb{D}$ is a finite subset of R^l. This leads to a partition of R^l into a finite number of quantization regions of the form $\{\xi \in R^l : q(\xi) = i\}, i \in \mathbb{D}$. These quantization regions are not assumed to have any particular shapes. When ξ does not belong to the union of quantization regions of finite size, the quantizer saturates. More precisely, we assume that there exist positive real numbers M and Δ such that the following two conditions hold:

$$|q(\xi) - \xi| \le \Delta, \quad \text{if } |\xi| \le M. \quad (6)$$

$$|q(\xi) - \xi| > \Delta, \quad \text{if } |\xi| > M. \quad (7)$$

Condition (6) gives a bound on the quantization error when the quantizer does not saturate. Condition (7) provides a way to detect the possibility of saturation. M and Δ represent the range and the quantization error bound of the quantizer $q(\cdot)$, respectively. Assume that $q(\xi) = 0$ for ξ in some neighborhood of the origin, i.e., the origin lies in the interior of the set $\{\xi : q(\xi) = 0\}$. In the filtering strategy to be developed below, we consider the one-parameter family of quantizers

$$q_\mu(\xi) = \mu q(\frac{\xi}{\mu}), \quad \mu > 0, \quad (8)$$

where μ is the quantizer's parameter. The range and the error of this quantizer is $M\mu$ and $\Delta\mu$, respectively. μ can be seen as the "zoom" variable: increasing μ corresponds to zooming out and essentially obtaining a new quantizer with larger range and larger quantization error such that any signals can be adequately measured, while decreasing μ corresponds to zooming in and obtaining a quantizer with smaller range but also smaller quantization error such that the signals can be driven to 0.

With the quantizer defined above, a model for the quantized filter (2) is given as

$$\begin{aligned} \dot{\xi}(t) &= A_F\xi + B_Fq_{\mu_1}(y) = A_F\xi + B_F\mu_1q_1(\tfrac{y}{\mu_1}), \\ z_F &= C_F\xi + D_Fq_{\mu_1}(y) = C_F\xi + D_F\mu_1q_1(\tfrac{y}{\mu_1}), \end{aligned} \quad (9)$$

where $q_{\mu_1}(\cdot)$ is a dynamic quantizer defined by (8) and is composed of dynamic scaling μ_1 and static quantizer $q_1(\cdot)$ defined by (6) with range M_1 and error Δ_1.

Applying (9) to (1), the following quantized augmented system is obtained as

$$\begin{aligned} \dot{x}_e(t) &= A_ex_e(t) + B_e\omega(t) + \bar{B}_1\bar{e}, \\ z_e(t) &= C_ex_e(t) + D_e\bar{e}, \end{aligned} \quad (10)$$

where A_e, B_e, C_e are the same as in (3) and $\bar{B}_1 = \begin{pmatrix} 0 \\ B_F \end{pmatrix}$, $D_e = -D_F$, $\bar{e} = \mu_1 e$ with $e = q_1(\frac{y}{\mu_1}) - \frac{y}{\mu_1}$.

Now, due to the existence of quantization errors, the following problem is proposed.

Problem 1: Develop a method to optimize the static quantizer range M_1, and then design a quantized H_∞ filtering strategy with the minimized quantizer range such that the augmented system (10) is asymptotically stable and $\|T_{z_e\omega}\| < \gamma$.

B. Preliminaries

The following lemmas are presented, which will be used in this paper.

Lemma 1: [23] Let $T_{az\omega} = C_a(sI - A_a)^{-1}B_a$, then A_a is Hurwitz and $\|T_{az\omega}\| < \gamma$ for some constant $\gamma > 0$ if and only if there exists a symmetric matrix $X > 0$ such that

$$A_a^T X + XA_a + \frac{1}{\gamma^2} XB_aB_a^T X + C_a^T C_a < 0.$$

Lemma 2: Let $\gamma > 0$ be a given constant. Then the following statements are equivalent:
(i) A_e is Hurwitz, and $\|T_{z_e\omega}\| < \gamma$;
(ii) there exist symmetric positive matrices $X > 0$ and $Q_a = \begin{bmatrix} Q_{a11} & Q_{a12} \\ Q_{a12}^T & Q_{a22} \end{bmatrix} > 0$ such that

$$A_e^T X + XA_e + \frac{1}{\gamma^2} XB_eB_e^T X + C_e^T C_e + Q_a < 0. \quad (11)$$

(iii) there exist a nonsingular matrix T, a symmetric matrix $P > 0$ with

$$P = \begin{bmatrix} Y & N \\ N & -N \end{bmatrix}, \quad (12)$$

and $Q_b = \begin{bmatrix} Q_{b11} & Q_{b12} \\ Q_{b12}^T & Q_{b22} \end{bmatrix} > 0$ such that

$$A_{ea}^T P + PA_{ea} + \frac{1}{\gamma^2} PB_{ea}B_{ea}^T P + C_{ea}^T C_{ea} + Q_b < 0, \quad (13)$$

where

$$A_{ea} = \begin{bmatrix} A & 0 \\ B_{Fa}C_2 & A_{Fa} \end{bmatrix}, \quad B_{ea} = \begin{bmatrix} B_1 \\ B_{Fa}D_{21} \end{bmatrix},$$
$$C_{ea} = \begin{bmatrix} C_1 - D_{Fa}C_2 & -C_{Fa} \end{bmatrix}.$$

and

$$\begin{aligned} &A_{Fa} = T^{-1}A_F T, \quad B_{Fa} = T^{-1}B_F, \\ &C_{Fa} = C_F T, \quad D_{Fa} = D_F \end{aligned} \quad (14)$$

Proof: Due to the limit of the space, it is omitted. ■

III. QUANTIZED H_∞ FILTERING STRATEGY DESIGN

In this section, we first give a quantized H_∞ filtering strategy design method without the consideration of optimizing static quantizer ranges in subsection A, which is an extension of the existing method given in [22]. Then, in subsection B, a convex optimization method is developed to optimize static quantizer ranges, and a quantized H_∞ filtering strategy is proposed to solve Problem 1.

A. Quantized filter design without considering static quantizer ranges

Firstly, to facilitate to present Theorem 1, we give an algorithm to design the filter gains and the matrix variables P, Q satisfying (5) without the consideration of optimizing static quantizer ranges.

Algorithm 1:
Step 1. Design the standard H_∞ filter with gains A_F, B_F, C_F and D_F, correspondingly, the optimal H_∞ performance bound is denoted as γ_{opt}.
Step 2. By using A_F, B_F, C_F, D_F obtained in Step 1, according to (5), compute matrix variables P and Q.
Step 3. Compute the value of $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}$, where $\eta = \phi + \sqrt{\phi^2 + \varphi\lambda_{min}(Q)}$ with $\phi = \|P\bar{B}_1 + C_e^T D_e\|$ and $\varphi = \|D_e^T D_e\|$.
Then, based on Algorithm 1, the following theorem presents a quantized H_∞ filtering strategy without the consideration of optimizing static quantizer ranges, which guarantees that system (10) is global asymptotic stability and with the H_∞ performance attenuation level γ_{opt}.

Theorem 1: Consider system (1), assume that M_1 is chosen large enough such that

$$M_1 > \frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}. \quad (15)$$

Then, filtering strategy (9) with the designed filter gains A_F, B_F, C_F, D_F and with the dynamic scaling

$$\mu_1 = \frac{2|y|}{M_1 + \frac{\eta\Delta_1\|C_2\|}{\lambda_{min}(Q)}}, \quad (16)$$

renders the augmented system (10) asymptotically stable and achieves the H_∞ disturbance attenuation level γ_{opt}.

Proof: By using the properties of (6) for the quantizer $q_1(\cdot)$, it is easy to check that whenever $|y| \le M_1\mu_1$,

$$|\bar{e}| = \mu_1|q_1(\frac{y}{\mu}) - \frac{y}{\mu_1}| \le \mu_1\Delta_1. \quad (17)$$

Then consider the Lyapunov function candidate $V(t) = x_e^T(t)Px_e(t)$ for the quantized augmented system (10), and by using (5), the derivative of $V(t)$ along solutions of (10) is computed as

$$\begin{aligned} \dot{V}(t) &= 2x_e^T(t)P(A_e x_e(t) + B_e\omega + \bar{B}_1\bar{e}) \\ &\le -x_e^T(Q + \gamma^{-2}PB_eB_e^T P + C_e^T C_e)x_e \\ &\quad + 2x_e^T PB_e\omega + 2x_e^T P\bar{B}_1\bar{e} \\ &\le -z_e^T z_e + \gamma^2\omega^T\omega - x_e^T Qx_e + 2x_e^T P\bar{B}_1\bar{e} \\ &\quad + 2x_e^T C_e^T D_e\bar{e} + \bar{e}^T D_e^T D_e\bar{e} \\ &\le -z_e^T z_e + \gamma^2\omega^T\omega - \lambda_{min}(Q)|x_e|^2 \\ &\quad + 2|x_e|(\|P\bar{B}_1 + C_e^T D_e\|)|\bar{e}| + |\bar{e}|^2\|D_e^T D_e\|, \\ &= -z_e^T z_e + \gamma^2\omega^T\omega - \lambda_{min}(Q)(|x_e| - \tfrac{\eta|\bar{e}|}{\lambda_{min}(Q)}) \\ &\quad \times (|x_e| - \tfrac{(\phi - \sqrt{\phi^2 + \varphi\lambda_{min}(Q)})}{\lambda_{min}(Q)}|\bar{e}|). \end{aligned} \quad (18)$$

On one hand, according to (36), we can always find a scalar $\varepsilon \in (0, 1)$ such that

$$M_1 > \frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)(1 - \varepsilon)}. \quad (19)$$

On the other hand, according to (16), for any nonzero y, there always exists a positive scalar μ_1 such that

$$|y| = (M_1 + \frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)})\mu_1/2. \quad (20)$$

Then, by using (19) and (20), there exists a sufficient small $\varepsilon \in (0, 1)$ such that the following holds

$$\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)(1-\varepsilon)}\mu_1 \le |y| = (M_1 + \frac{\eta\Delta_1\|C_2\|}{\lambda_{min}(Q)})\mu_1/2 \le M_1\mu_1. \quad (21)$$

This is also true in the case of $y = 0$, where we set $\mu_1 = 0$ as an extreme case and consider the outputs of the quantizer $q_1(\cdot)$ as zeros.
In other words, if we always choose μ_1 satisfying (21), then (18) holds and thus

$$\dot{V}(t) \le -z_e^T z_e + \gamma^2\omega^T\omega - \varepsilon\frac{\lambda_{min}(Q)}{\|C_2\|}|x_e||y|. \quad (22)$$

By setting $\omega = 0$, obviously, $\dot{V}(t) < 0$, i.e., the system is asymptotically stable.
In addition, for any $t > t_0$, we can obtain

$$V(t) - V(t_0) \le -\int_{t_0}^{t}(z_e^T(\tau)z_e(\tau) - \gamma^2\omega^T(\tau)\omega(\tau))d\tau. \quad (23)$$

Using $V(t) \ge 0$ and zero initial condition, the following is obtained

$$\|z_e(t)\|_2^2 < \gamma^2\|\omega(t)\|_2^2, \quad (24)$$

which implies that the H_∞ disturbance attention level γ is achieved. ■

Remark 1: Theorem 1 presents a quantized H_∞ filtering strategy without the consideration of optimizing static quantizer ranges, which is motivated by the result in [22]. The difference between them is that in Theorem 1, a concrete filtering strategy with dynamic scaling (16) is given, whereas [22] only gives the existence condition for a quantized control strategy.

Remark 2: In Theorem 1, the static quantizer range M_1 for the existence condition of the quantized H_∞ filtering strategy is given based on Algorithm 1. But the static quantizer range achieved by this method may be very large and does not accord with practical communication channel requirements. In the following subsection, a method will be developed to minimize the static quantizer ranges.

B. Quantized filter design with considering static quantizer ranges

In this subsection, we will develop a convex optimization method to optimize M_1, and further, give a quantized H_∞ filtering strategy to solve Problem 1.
From (36), it can be seen that in order to minimize M_1, one should minimize the value of $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}$. However, $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}$ is complicated because it depends on design parameters $\lambda_{min}(Q), \|P\bar{B}_1 + C_e^T D_e\|$ and $\|D_e^T D_e\|$. In the sequel, we aim to optimize these parameters, and consequently minimize M_1 indirectly.
Let $\beta > 0, \alpha > 0, \delta > 0$ and $\varepsilon > 0$ be scalars, then we can optimize the values of $\|P\bar{B}_1\|, \|C_e^T\|, \|D_e\|$ and $\|D_e^T D_e\|$ by optimizing scalars β, α, δ and ε, respectively, according to inequalities $\|P\bar{B}_1\| < \beta, \|C_e^T\| < \alpha, \|D_e\| < \delta, \|D_e^T D_e\| < \varepsilon$. Obviously, they are, respectively, equivalent to

$$\begin{bmatrix} -\beta^2 I & (P\bar{B}_1)^T \\ * & -I \end{bmatrix} < 0, \quad (25)$$

$$\begin{bmatrix} -\alpha^2 I & C_e \\ * & -I \end{bmatrix} < 0, \quad (26)$$

$$\begin{bmatrix} -\delta^2 I & D_e^T \\ * & -I \end{bmatrix} < 0, \quad (27)$$

$$\begin{bmatrix} -\varepsilon^2 I & D_e^T D_e \\ * & -I \end{bmatrix} < 0, \quad (28)$$

where C_e is defined in (3), $\bar{B}_1$ and D_e are defined in (10). Now, in order to solve Problem 1, we need solve inequalities (25)- (28) combined with (5). However, inequalities (25), (28) and (5) are not convex and cannot be solved directly. Thus, the following lemma is presented to convert them to convex ones.
Denote

$$A_F = N^{-1}F_A, \; B_F = N^{-1}F_B, \quad (29)$$

then, we have

Lemma 3: Let $\gamma > 0$ be a given scalar, for scalars $\beta > 0, \alpha > 0, \delta > 0$ and $\varepsilon > 0$, matrix variables F_A, F_B, C_F, $S > 0$, $N < 0$ and

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} > 0, \quad (30)$$

the following statements hold:
(i) (5) holds if and only if the following LMI holds

$$\begin{bmatrix} \Sigma_1 & \Sigma_2 & SB_1 & C_1^T - C_2^T D_F^T - C_F^T \\ * & \Sigma_3 & (S-N)B_1 & C_1^T - C_2^T D_F^T \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \quad (31)$$

where

$$\Sigma_1 = SA + A^T S + Q_{11} + Q_{12} + Q_{12}^T + Q_{22},$$
$$\Sigma_2 = SA + F_A^T + C_2^T F_B^T + A^T(S-N) + Q_{11} + Q_{12}^T,$$
$$\Sigma_3 = (S-N)A + A^T(S-N) + F_B C_2 + C_2^T F_B^T + Q_{11}.$$

(ii) (25) holds if and only if the following LMI holds

$$\begin{bmatrix} -\beta^2 I & 0 & F_B^T \\ * & -2I & -I \\ * & * & -I \end{bmatrix} < 0 \quad (32)$$

(iii) (28) holds if the following LMI holds

$$\begin{bmatrix} -\varepsilon^2 I & 0 & D_F^T & 0 \\ * & -I & 0 & D_F^T \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \quad (33)$$

Proof: Due to the limit of the space, it is omitted. ■

Let $\rho = c_1\beta^2 + c_2\alpha^2 + c_3\delta^2 + c_4\varepsilon^2$, where c_1, c_2, c_3 and c_4 are constants to be chosen. Then, based on Lemma 3, by optimizing ρ, the following algorithm is presented to give a convex optimization method to design the filter gain matrices and matrix variables P, Q with the consideration of system performance and quantizer ranges.

Denote

$$Q(\xi) = \begin{bmatrix} Q_{11} - \xi I & Q_{12} \\ * & Q_{22} - \xi I \end{bmatrix} > 0. \quad (34)$$

Algorithm 2:

Step 1. Solving the following optimization problem

$$\min \quad \rho$$
$$\text{subject to } (26),\ (27),\ (31)-(33),\ (34)$$
$$\text{w. r. t} \quad F_A, F_B, C_F, D_F, S, N, \beta, \alpha, \delta, \varepsilon, \xi, Q_{11}, Q_{12}, Q_{22}$$

Output the optimal solutions as $N = N_{opt},\ S = S_{opt},\ F_A = F_{Aopt},\ F_B = F_{Bopt},\ C_F = C_{Fopt},\ Q_{11} = Q_{11opt},\ Q_{12} = Q_{12opt},\ Q_{22} = Q_{22opt}$.

Step 2. Compute

$$\begin{aligned} A_{F_{opt}} &= N_{opt}^{-1} F_{Aopt},\ B_{F_{opt}} = N_{opt}^{-1} F_{Bopt}, \\ C_{F_{opt}} &= C_{Fopt}, D_{Fopt} = -D_{eopt}, \\ P_{opt} &= \begin{bmatrix} S_{opt} - N_{opt} & N_{opt} \\ N_{opt} & -N_{opt} \end{bmatrix}, \\ Q_{opt} &= \begin{bmatrix} Q_{11opt} & Q_{12opt} \\ Q_{12opt}^T & Q_{22opt} \end{bmatrix}, \bar{B}_{1opt} = \begin{bmatrix} 0 \\ B_{Fopt} \end{bmatrix}. \end{aligned} \quad (35)$$

The resulting $A_{Fopt}, B_{Fopt}, C_{Fopt}$ and D_{Fopt} will form the optimized filter gains, and P_{opt}, Q_{opt} are optimized matrices.

Step 3. Compute the value of $\frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})}$, where $\eta_{opt} = \phi_{opt} + \sqrt{\phi_{opt}^2 + \varphi_{opt}\lambda_{min}(Q_{opt})}$ with $\phi_{opt} = \|P_{opt}\bar{B}_{1opt}\| + \|C_{eopt}^T\|\|D_{eopt}\|$ and $\varphi_{opt} = \|D_{eopt}^T D_{eopt}\|$.

Remark 3: Because $\lambda_{min}(Q)$ has a significant effect on the value of $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}$, condition (34) is introduced to restrict the value of $\lambda_{min}(Q)$, such that $\lambda_{min}(Q) \geq \xi$.

Remark 4: Algorithm 2 presents a convex optimization method to design the filter gains A_F, B_F, C_F, D_F and matrix variables P, Q with the consideration of optimizing the value of $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)}$ (realized by conditions (26), (27), (32), (33), (34)) as well as guaranteeing the H_∞ performance (realized by condition (31)). Obviously, the designed $A_{Fopt}, B_{Fopt}, C_{Fopt}, D_{Fopt}, P_{opt}, Q_{opt}$ satisfy Assumption 1, and at the same time, they may result in a minimized value of $\frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})}$.

Now, based on Algorithm 2, the following corollary similar to Theorem 1 can be obtained:

Corollary 1: Consider system (1), assume that M_{1min} is chosen large enough such that

$$M_{1min} > \frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})}, \quad (36)$$

Then, filtering strategy (9) with the designed filter gains $A_{Fopt}, B_{Fopt}, C_{Fopt}, D_{Fopt}$ and with the dynamic scaling

$$\mu_1 = \frac{2|y|}{M_{1min} + \frac{\eta_{opt}\Delta_1\|C_2\|}{\lambda_{min}(Q_{opt})}}, \quad (37)$$

solves Problem 1.

IV. Example

To illustrate the effectiveness of the proposed optimized H_∞ filtering strategy, an example is given to provide a comparison between our design method with the consideration of optimizing M_1 and the design method without the consideration of optimizing M_1.

Example 1: Consider a linear system of form (1) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -2.5 & -1 \end{bmatrix}, B_1 = \begin{bmatrix} -1.5 & 0 \\ 1.5 & 0 \\ 2 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}, C_2 = \begin{bmatrix} -1.5 & 0.5 & 1 \end{bmatrix}.$$

A. Quantized filter design by Algorithm 1

In this part, we design a quantized H_∞ filtering strategy based on Algorithm 1 without the consideration of optimizing M_1, which is an extension of the existing method.

A standard H_∞ filter is designed with gains as

$$\begin{aligned} A_F &= \begin{bmatrix} -6.1451 & 2.8083 & 3.8566 \\ -281.4187 & 95.4706 & 191.7865 \\ 391.6585 & -134.7995 & -266.0323 \end{bmatrix}, \\ B_F &= \begin{bmatrix} -4.1569 & -189.3829 & 263.9612 \end{bmatrix}^T, \\ C_F &= \begin{bmatrix} 0.1424 & -0.0475 & 3.9050 \end{bmatrix}, \\ D_F &= -1.9050. \end{aligned} \quad (38)$$

and the optimal H_∞ performance index is obtained as $\gamma_{opt} = 6.0000$.

Let the quantization error $\Delta_1 = 0.1$. By Algorithm 1 with filter gains (38) and $\xi = 0.1$ for $\gamma = 6.1, \gamma = 6.4$ and $\gamma = 7.0$, respectively, we obtain $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)} = 77196$, $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)} = 42023$ and $\frac{\eta\|C_2\|\Delta_1}{\lambda_{min}(Q)} = 25807$. For these three cases, by Theorem 1, let $M_1 = 77196.1 > 77196$, $M_1 = 42023.1 > 42023$ and $M_1 = 25807.1 > 25807$, respectively, the filtering strategy (9) with filter gains (38) and dynamic scaling (16) guarantees the H_∞ filtering objective.

B. Quantized filter design by Algorithm 2

In this subsection, we design a quantized H_∞ filtering strategy based on Algorithm 2 with optimizing M_1 consideration .

Let the quantization error $\Delta_1 = 0.1$ and $c_1 = 1, c_2 = 10000, c_3 = 100000, c_4 = 100000$. For one case, let $\gamma = 6.1$, by Algorithm 2 with $\xi = 0.1$ we obtain

$$\begin{aligned} A_{Fopt} &= \begin{bmatrix} -96.9741 & 22.7699 & -3.3318 \\ -141.9869 & 32.2035 & -4.2321 \\ 395.8639 & -91.9370 & 12.8568 \end{bmatrix}, \\ B_{Fopt} &= \begin{bmatrix} -35.7029 & -51.6374 & 145.3041 \end{bmatrix}^T, \\ C_{Fopt} &= \begin{bmatrix} -0.1155 & -0.5311 & 0.7352 \end{bmatrix}, \\ D_{Fopt} &= -0.6529. \end{aligned} \quad (39)$$

and matrices P_{opt}, Q_{opt}.

It is easy to compute $\frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})} = 14.3393$. By Theorem 1, let $M_{1min} = 14.5 > 14.3393$, then, filtering strategy (9) with the optimized filter gains (39) and dynamic scaling (37) solves Problem 1.

For the second case, let $\gamma = 6.4$, we obtain

$$\begin{aligned} A_{Fopt} &= \begin{bmatrix} -41.9558 & 13.1852 & -0.8868 \\ -55.5593 & 16.1668 & -0.6638 \\ 160.8423 & -49.5721 & 2.7435 \end{bmatrix}, \\ B_{Fopt} &= \begin{bmatrix} -16.9850 & -21.7258 & 64.6265 \end{bmatrix}^T, \\ C_{Fopt} &= \begin{bmatrix} -0.1100 & -0.4843 & 0.6817 \end{bmatrix}, \\ D_{Fopt} &= -0.6635. \end{aligned} \tag{40}$$

and matrices P_{opt}, Q_{opt}. It is easy to compute $\frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})} = 13.8698$. For this case, by Theorem 1, let $M_{1min} = 14.0 > 13.8698$, then filtering strategy (9) with the optimized filter gains (40) and dynamic scaling (37) solves Problem 1.
Finally, for $\gamma = 7.0$, we obtain

$$\begin{aligned} A_{Fopt} &= \begin{bmatrix} -13.3011 & 8.5567 & 1.0946 \\ -16.4856 & 8.3206 & 1.3663 \\ 48.3882 & -29.6938 & -4.3364 \end{bmatrix}, \\ B_{Fopt} &= \begin{bmatrix} -7.6299 & -8.0965 & 26.9504 \end{bmatrix}^T, \\ C_{Fopt} &= \begin{bmatrix} -0.0975 & -0.3956 & 0.5755 \end{bmatrix}, \\ D_{Fopt} &= -0.6859. \end{aligned} \tag{41}$$

and matrices P_{opt}, Q_{opt}. Similarly, we can compute $\frac{\eta_{opt}\|C_2\|\Delta_1}{\lambda_{min}(Q_{opt})} = 13.4073$. For this case, by Theorem 1, let $M_{1min} = 13.6 > 13.4073$, filtering strategy (9) with the optimized filter gains (41) and dynamic scaling (37) solves Problem 1.

C. Comparison

Table 1 is given to compare the quantizer ranges obtained based on Algorithm 1 and the optimization method given in Algorithm 2.

	Algorithm 1	Algorithm 2
$M_1(\gamma = 6.1)$	77196.1	14.5
$M_1(\gamma = 6.4)$	42023.1	14.0
$M_1(\gamma = 7.0)$	25807.1	13.6

TABLE I
COMPARISON OF THE QUANTIZER RANGES

From this table, we can see that compared with the quantizer ranges obtained based on Algorithm 1, the optimized quantizer ranges obtained based on Algorithm 2 are much more improved. This phenomenon shows the effectiveness of our optimization method. On the other hand, we can see that the tighter the H_∞ performance bound to the optimal H_∞ performance bound γ_{opt} is, the larger quantizer range is needed.

V. CONCLUSION

In this paper, the quantized H_∞ filtering problem of continuous-time LTI systems has been investigated. In particular, the static quantizer ranges are fully considered for their practical importance. A quantized H_∞ filter design strategy is proposed, where a convex optimization method is developed to minimize static quantizer ranges. The resulting design guarantees that the quantized augmented system is asymptotically stable and with a prescribed H_∞ performance bound. A numerical example has been presented to illustrated the effectiveness of the proposed H_∞ filtering strategy.

REFERENCES

[1] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Transactions on Automatic Control,* vol. 45, no. 7, pp. 1279-1289, 2000.
[2] P. Chen and Y. C. Tian, "Networked H_∞ control of linear systems with state quantization," *Information Sciences,* pp. 5763-5774, 2007.
[3] D. F. Delchamps, "Stabilizing a linear system with quantized state feedback," *IEEE Transactions on Automatic Control*, vol. 35, no. 8, pp. 916-924, 1990.
[4] N. Elia and S. K. Mitter, "Stabilization of linear systems with limited information," *IEEE Transactions on Automatic Control,* vol. 46, no. 9, pp. 1384-1400, 2001.
[5] F. Fagnani and S. Zampieri, "Stability analysis and synthesis for scalar linear systems with a quantized feedback," *IEEE Transactions on Automatic Control,* vol. 48, no. 9, pp. 1569-1584, 2003.
[6] F. Fagnani and S. Zampieri, "Quantized stabilization of linear systems: complexity versus performance," *IEEE Transactions on Automatic Control,* vol. 49, no. 9, pp. 1534-1548, 2004.
[7] M. Y. Fu and L. H. Xie, "The sector bound approach to quantized feedback control," *IEEE Transactions on Automatic Control,* vol. 50, no. 11, pp. 1698-1711, 2005.
[8] M. Y. Fu and L. H. Xie, "Finite-level quantized feedback control for linear systems," *Proceedings of the 45th IEEE Conference on Decision and Control*, USA, pp. 1117-1122, 2006.
[9] X. Feng and K. A. Loparo, "Active probing for information in control systems with quantized state measurements: a minimum entropy approach." *IEEE Transactions on Automatic Control*, vol. 42, no. 2, pp. 216-238, 1997.
[10] H. Gao and T. Chen, "A poly-quadratic approach to quantized feedback systems," *Proceedings of the 45th IEEE Conference on Decision and Control,* USA, pp. 5495-5500, 2006.
[11] H. Ishii and B. A. Francis, "Quadratic stabilization of sampled-data systems with quantization," *Automatica*, vol. 39, no. 10, pp. 1793-1800, 2003.
[12] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals", *Automatica,* vol. 39, pp. 1543-1554, 2003.
[13] Q. Ling and M. D. Lemmon, "Stability of quantized control systems under dynamic bit assignment," *IEEE Transactions on Automatic Control,* vol. 50, no. 5, pp. 734-740, 2005.
[14] J. Lunze, B. Nixdorf, and J. Schroder. "Deterministic discrete-event representations of linear continuous-variable systems," *Automatica,* vol. 35, pp. 395-406, 1999.
[15] G. N. Nair and R. J. Evans, "Exponential stabilizability of finite-dimensional linear systems with limited data rates," *Automatica,* vol. 39, pp. 585-593, 2003.
[16] R. L. de Queiroz and P. A. Stein, "LUT filters for quantized processing of signals,"*IEEE Transactions on signal processing,* vol. 52, no. 3, pp. 687-693, 2004.
[17] A. Sahai, "The necessity and sufficiency of anytime capacity for control over a noisy communication link," *Proceedings of the 43th IEEE Conference on Decision and Control,* vol. 2, pp. 1896-1901, Bahamas, 2004.
[18] S. Tatikonda, "Control under Communication Constraints," Ph.D. Thesis, Massachusetts Institute of Technology, Cambridge, MA, 2000.
[19] S. Tatikonda and S. Mitter, "Control over noisy channels,"*IEEE Transactions on Automatic Control,* vol. 49, no. 7, pp. 1196-1201, 2004.
[20] T. Wigren, "Adaptive filtering using quantized output measurements," *IEEE transactions on signal processing,* vol. 46, no. 12, pp. 3423-3426, 1998.
[21] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints II: stabilization with limited information feedback," *IEEE Transactions on Automatic Control,* vol. 44, pp. 1049-1053, 1999.
[22] G. Zhai, X. Chen, J. Imae, and T. Kobayashi, "Analysis and design of H_∞ feedback control systems with two quantized signals," *Proceedings of the 2006 IEEE International Conference on Networking, Sensing and Control,* pp. 346 -350, 2006.
[23] K. Zhou, J. C. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, New Jersey, 1996.