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Global synchronization of complex dynamical networks with non-identical nodes

David J.Hill and Jun Zhao

***Abstract*— This paper addresses the problem of synchronization for complex dynamical networks with non-identical nodes. Neither an equilibrium for each node nor a synchronization manifold is assumed to exist. A criterion of global synchronization in the sense of boundedness of the maximum state deviation between nodes is proposed by introducing the average dynamics of all nodes. An explicit bound of the maximum state deviation between nodes is obtained by the maximum difference between each node dynamics and the average dynamics. The proposed criterion is an extension of the related synchronization criteria for the case of identical nodes to the case of non-identical nodes.**

I. INTRODUCTION

A complex dynamical network is a collection of dynamic systems, called nodes, connected by links that exhibit complex topological properties. Complex dynamical networks have been widely exploited to model many complex systems in sciences, engineering and society, and have attracted tremendous attention in recent years (see [2] and the references therein). As the major collective behavior, synchronization is one of the key issues that have been extensively addressed. A vast number of papers on the topic have appeared. Some recent overviews have recently appeared in [13], [23], [27], [30]. This topic has been mainly explored mathematically in the physics community with some recent papers in circuits and systems [8], [28], [29] and automatic control [15], [31] journals and conferences. The topic is not unrelated to the study of consensus problems in swarms [1], [18], [22], [24] which can be seen as a kind of time-varying network.

For topological structure of links of a network, a constant, symmetric and irreducible coupling configuration matrix can always give rise to local synchronization criteria that only need to check simultaneous stability of several lower dimensional dynamical systems [2]. This technique has also been extended to deal with networks with time-delays [6], [12], [19], and time-varying and switching topologies [9], [15], [18], [21].

The behavior of a network is determined by two main features: the dynamics of the isolated nodes, and the coupling configuration between the nodes. Most efforts have been put on the study of the latter–the coupling configuration– by assuming that all the node dynamics are identical. Adoption of the assumption that all the node dynamics are identical makes it much easier to analyze the network, especially for the synchronizability problem. However, this assumption of identical nodes is a highly unlikely circumstance for technological networks in the real-world. This assumption has its origins in physical connections in biology, physics and social science [17]. Indeed, almost all complex dynamical networks in engineering have different nodes. Taking a power system as an example [8], the generators (power sources) and loads (power sinks) are connected to buses which are interconnected by transmission lines in a network structure. Therefore, the power system can be viewed as a dynamical network where the nodes consist of generators and (dynamical) loads. Since individual generators usually have different physical parameters, the generator models result in different dynamics and the power system is obviously a dynamical network with non-identical nodes.

The behavior of dynamical networks with non-identical nodes is much more complicated than the identical-node case. Usually, no common equilibrium for all nodes exists even if each isolated node has an equilibrium, neither does a synchronization manifold exist in the classical sense. Synchronization of a complex dynamical network with identical nodes is usually described in terms of (asymptotically) identical dynamical evolution of state variables of every node in the network, which is easy to understand. However, this collective behavior, called complete or identical synchronization no longer exists in networks with non-identical nodes due to the difference between the dynamics of the nodes. Yet, a network with non-identical nodes may still exhibit some kind of synchronization behaviors which are far from being fully understood. Certain reasonable and satisfactory boundedness of state motion errors between different nodes can be taken as useful synchronization properties. In general, this needs to be systematically described and addressed. In the special case of power systems, the property of transient stability is essentially a synchronization in this sense and well-understood as a stability property [7], [11].

The study of synchronization of dynamical networks with non-identical nodes is very hard and very few results have been reported by now. A simple case where all non-identical nodes have the same equilibrium was considered in [31] and a synchronization criterion was given by constructing a common Lyapunov function for all the nodes. Several collective properties for coupled non-identical chaotic systems were respectively discussed in [4], [5], [25], [26]. A simulation study for non-identical Kuramoto oscillators was carried out in [3]. Controlled synchronization was considered for the

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D.J.Hill and J.Zhao are with Department of Information Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra ACT 0200, Australia
David.Hill@anu.edu.au jun.zhao@anu.edu.au

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case that each node has a normal form with a linear main part [20], and distributed controllers were designed to achieve synchronization.

This paper addresses the issue of synchronization for complex dynamical networks with non-identical nodes. We consider the general case where neither an equilibrium for each isolated node nor a synchronization manifold exists. A global synchronization criterion is proposed, which exploits the average node dynamics. The results cover the related existing criteria of asymptotical synchronization for complex dynamical networks with identical nodes as a special case.

II. Preliminaries

We study a complex dynamical network modeled as:

$$\dot{x}_i = f_i(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j, \qquad i = 1,\ldots,N, \tag{1}$$

where $x_i = (x_{i1},\ldots,x_{in})^T \in R^n$ is the state of the i-th node. Suppose the matrix $A = (a_{ij})_{N\times N}$ is symmetric and irreducible and $\sum_{j=1}^{N} a_{ij} = 0$, $i = 1,\ldots,N$, f_i are continuously differentiable with Jacobian Df_i.

The average dynamics of all node dynamics is defined by the vector field

$$\bar{f}(x) = \frac{1}{N}\sum_{k=1}^{N} f_k(x).$$

The average state trajectory is

$$s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t).$$

Obviously, the deviations $e_i = x_i - s(t)$ satisfy $\sum_{i=1}^{N} e_i = 0$. We can easily have

$$\begin{aligned}
\dot{e}_i \;=\;& f_i(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j \\
& -\frac{1}{N}\sum_{k=1}^{N}(f_k(x_k) + c\sum_{j=1}^{N} a_{kj}\Gamma x_j) \\
=\;& f_i(x_i) - \frac{1}{N}\sum_{k=1}^{N} f_k(x_k) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j \\
=\;& f_i(s+e_i) - \frac{1}{N}\sum_{k=1}^{N} f_k(s+e_k) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j \\
=\;& f_i(s) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j + \int_0^1 Df_i(s+\tau e_i)e_i d\tau - \\
& \frac{1}{N}\sum_{k=1}^{N}(f_k(s) + \int_0^1 Df_k(s+\tau e_k)e_k d\tau) \\
=\;& D\bar{f}(s)e_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j \\
& +\int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))e_i d\tau \\
& -\frac{1}{N}\sum_{k=1}^{N}\int_0^1 Df_k(s+\tau e_k)e_k d\tau + f_i(s) - \bar{f}(s).
\end{aligned} \tag{2}$$

If we consider the linearized network model of (1), we have

$$\begin{aligned}
\dot{e}_i \;=\;& D\bar{f}(s)e_i + c\sum_{j=1}^{N} a_{ij}\Gamma e_j + (Df_i(s) - D\bar{f}(s))e_i \\
& -\frac{1}{N}\sum_{k=1}^{N} Df_k(s)e_k + f_i(s) - \bar{f}(s).
\end{aligned} \tag{3}$$

Let $e = (e_1^T,\ldots,e_N^T)^T$. Then (2) becomes

$$\begin{aligned}
\dot{e} =& \\
& (I_N \otimes D\bar{f}(s) + cA\otimes\Gamma)e \\
& +\text{diag}\left\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau\right. \\
& \left.\cdots\int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\right\} e - \frac{1}{N}\times \\
& \begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \end{pmatrix} e \\
& + \begin{pmatrix} f_1(s) - \bar{f}(s) \\ \vdots \\ f_N(s) - \bar{f}(s) \end{pmatrix}.
\end{aligned} \tag{4}$$

Since A is symmetric and irreducible, there exists a unitary matrix $\Phi = (\varphi_{ij})_{N\times N} = (\Phi_1,\ldots,\Phi_N)$, such that

$$\Phi^T A \Phi = \Lambda = \text{diag}\{\lambda_1,\lambda_2,\ldots,\lambda_N\}, \tag{5}$$

where Φ_i is the ith column of Φ with $\Phi_1 = (\frac{1}{\sqrt{N}},\ldots,\frac{1}{\sqrt{N}})^T$ and $0 = \lambda_1 > \lambda_2 \ge \cdots \ge \lambda_N$.

Let $\omega = (\Phi^T \otimes I_n)e$. Then,

$$\begin{aligned}
\dot{\omega} =& (\Phi^T \otimes I_n)\dot{e} \\
=& (\Phi^T \otimes I_n)[I_N \otimes D\bar{f}(s) + cA\otimes\Gamma](\Phi\otimes I_n)\omega \\
& + (\Phi^T \otimes I_n)\times \\
& \text{diag}\left\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau\right. \\
& \left.\cdots\int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\right\}\times \\
& (\Phi\otimes I_n)\omega - \frac{1}{N}(\Phi^T \otimes I_n)\times \\
& \begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \end{pmatrix} \\
& \times(\Phi\otimes I_n)\omega \\
& + (\Phi^T \otimes I_n)\begin{pmatrix} f_1(s) - \bar{f}(s) \\ \vdots \\ f_N(s) - \bar{f}(s) \end{pmatrix}
\end{aligned} \tag{6}$$

Note that

$$\begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \end{pmatrix}$$
$$\begin{aligned} =&\sqrt{N}[(\Phi_1,\mathbf{0},\dots,\mathbf{0})\otimes\int_0^1 Df_1(s+\tau e_1)d\tau] \\ &+\sqrt{N}[(\mathbf{0},\Phi_1,\dots,\mathbf{0})\otimes\int_0^1 Df_2(s+\tau e_2)d\tau] \\ &+\cdots+\sqrt{N}[(\mathbf{0},\mathbf{0},\dots,\Phi_1)\otimes\int_0^1 Df_N(s+\tau e_N)d\tau] \end{aligned} \tag{7}$$

and Φ is a unitary matrix, it turns out that

$$\frac{1}{N}(\Phi^T\otimes I_n)\times$$
$$\begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \end{pmatrix}$$
$$\times(\Phi\otimes I_n)$$
$$\begin{aligned} =&\frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\ &\otimes\int_0^1 Df_1(s+\tau e_1)d\tau(\Phi\otimes I_n) \\ &+\frac{1}{\sqrt{N}}\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\ &\otimes\int_0^1 Df_2(s+\tau e_2)d\tau(\Phi\otimes I_n) \\ &+\cdots+\frac{1}{\sqrt{N}}\begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\ &\otimes\int_0^1 Df_N(s+\tau e_N)d\tau(\Phi\otimes I_n). \end{aligned} \tag{8}$$

Thus, a simple calculation gives

$$\frac{1}{N}(\Phi^T\otimes I_n)\times$$
$$\begin{pmatrix} \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \\ \vdots & \ddots & \vdots \\ \int_0^1 Df_1(s+\tau e_1)d\tau & \cdots & \int_0^1 Df_N(s+\tau e_N)d\tau \end{pmatrix}$$
$$\times(\Phi\otimes I_n)$$
$$\begin{aligned} =&\frac{1}{\sqrt{N}}\begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1N} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\ &\otimes\int_0^1 Df_1(s+\tau e_1)d\tau \\ &+\frac{1}{\sqrt{N}}\begin{pmatrix} \varphi_{21} & \varphi_{22} & \cdots & \varphi_{2N} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\ &\otimes\int_0^1 Df_2(s+\tau e_2)d\tau \\ &+\cdots+\frac{1}{\sqrt{N}}\begin{pmatrix} \varphi_{N1} & \varphi_{N2} & \cdots & \varphi_{NN} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\ &\otimes\int_0^1 Df_N(s+\tau e_N)d\tau. \end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned} \dot{\omega}=&(I_N\otimes D\bar{f}(s)+c\Lambda\otimes\Gamma)\omega+(\Phi^T\otimes I_n)\times \\ &\mathrm{diag}\left\{\int_0^1(Df_1(s+\tau e_1)-D\bar{f}(s))d\tau\right. \\ &\left.\cdots\int_0^1(Df_N(s+\tau e_N)-D\bar{f}(s))d\tau\right\}\times(\Phi\otimes I_n)\omega \\ &-\begin{pmatrix} * & * & \cdots & * \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\omega \\ &+(\Phi^T\otimes I_n)\begin{pmatrix} f_1(s)-\bar{f}(s) \\ \vdots \\ f_N(s)-\bar{f}(s) \end{pmatrix}. \end{aligned} \tag{10}$$

Since $\omega_1 \equiv 0$, we only need to consider $\omega_2, \ldots, \omega_N$. Rewriting (10) in the component form we have

$$\begin{aligned}\dot{\omega}_i =&(D\bar{f}(s) + c\lambda_i\Gamma)\omega_i + (\Phi_i^T \otimes I_n)\times \\ &\text{diag}\left\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau, \ldots,\right. \\ &\left.\int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\right\}(\Phi \otimes I_n)\omega \\ &+ (\Phi_i^T \otimes I_n)\begin{pmatrix} f_1(s) - \bar{f}(s) \\ \vdots \\ f_N(s) - \bar{f}(s)\end{pmatrix}, \\ &i = 2, \ldots, N,\end{aligned} \tag{11}$$

which is a key expression for the study of synchronization. We now conclude this section by introducing some notations.

Let $\mathcal{PC}_{n\times n}$ ($\mathcal{PC}^1_{n\times n}$) be the linear space of the uniformly bounded continuous (continuously differentiable) real matrix-valued functions defined on $[0, \infty)$. For any $P \in \mathcal{PC}_{n\times n}$, the norm of P is defined by

$$\|P\| = \sup_{0\le t<\infty}\{\|P(t)\|\}.$$

III. Synchronization Criterion

We consider the problem of synchronization for the network (1). For a network with identical nodes, synchronization means $x_i - x_j \to 0, \forall i, j$. While for the network (1), this property cannot be expected to hold. In this case, the synchronization property can be described in terms of certain boundedness of $x_i - x_j, \forall i, j$. In order to derive a synchronization criterion, we first present a lemma which shows how the state trajectory of a dynamical system converges to a set.

Lemma 3.1 Let $g(t)$ be a non-negative bounded function defined on $[0, \infty)$ and

$$\Omega = \{x \in R^n | \|x\| \le \overline{\lim}_{t\to\infty} g(t)\}. \tag{12}$$

Suppose there exist a strictly positive definite matrix $P(t) \in \mathcal{PC}^1_{n\times n}$ and a constant $\delta > 0$ such that the derivative of $V(x,t) = x^T P(t)x$ along the trajectory of the system

$$\dot{x} = f(x,t), \qquad x \in R^n,\ t \in [0, \infty) \tag{13}$$

satisfies

$$\dot{V} \le -\delta\|x\|^2 \ \text{ if } \ \|x\| \ge g(t).$$

For any $t > 0$, let

$$Q_t = \left\{x | V(x,t) \le \sup_{y\in\Omega, s\ge 0}\{V(y,s)\}\right\} \tag{14}$$

and

$$c = \overline{\lim}_{t\to\infty} (\max\{\| x \| \,| x \in Q_t\}) \tag{15}$$

Then, $x(t)$ converges to the set

$$M = \{x | \|x\| \le c\}. \tag{16}$$

Proof. Omitted due to the space limitation.

Now we are in the position to give the criterion of synchronization.

Theorem 3.2. Suppose there exist positive definite matrices $P_i(t) \in \mathcal{PC}^1_{n\times n}$ and constants $\alpha > 0$, $\gamma \ge 0$ such that

$$a\|x\|^2 \le x^T P_i(t)x \le b\|x\|^2, \qquad \forall t \in R_+,\ x \in R^n, \quad i = 2, \ldots, N, \tag{17}$$

$$\dot{P}_i(t) + P_i(t)(D\bar{f}(s) + c\lambda_i\Gamma) + (D\bar{f}(s) + c\lambda_i\Gamma)^T P_i(t) + \alpha I \le 0,\ i = 1, \ldots, N, \tag{18}$$

$$\| \int_0^1 (Df_i(s+\tau e_i) - D\bar{f}(s))d\tau\| \le \gamma, \qquad i = 1, \ldots, N. \tag{19}$$

Let

$$\mu(t) = \left\| \begin{pmatrix} f_1(s) - \bar{f}(s) \\ \vdots \\ f_N(s) - \bar{f}(s)\end{pmatrix} \right\| \tag{20}$$

be bounded and

$$\beta = (\sum_{i=2}^N \|P_i\|^2)^{\frac{1}{2}}. \tag{21}$$

If $\alpha > 2\gamma\beta$, then the network (4) synchronizes to the set

$$M = \{e | \|e\| \le \frac{2b}{a}\frac{\beta\, \overline{\lim}_{t\to\infty}\mu(t)}{\alpha - 2\gamma\beta - \delta}\}, \tag{22}$$

namely, $e(t) = x_i(t) - \frac{1}{N}\sum_{k=1}^N x_k(t) \to \Omega$ as $t \to \infty$, where $\delta > 0$ is any constant satisfying $\delta < \alpha - 2\gamma\beta$.

Proof. Differentiating $V_i(\omega_i, t) = \omega_i^T P_i(t)\omega_i$ along the trajectory of (11) gives

$$\begin{aligned}\dot{V}_i =&\omega_i^T(\dot{P}_i(t) + P_i(t)(D\bar{f}(s) + c\lambda_i\Gamma) \\ &+ (D\bar{f}(s) + c\lambda_i\Gamma)^T P_i(t))\omega_i + 2\omega_i^T P_i(t)(\Phi_i^T \otimes I_n) \\ &\times \text{diag}\left\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau, \ldots,\right. \\ &\left.\int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\right\}(\Phi \otimes I_n)\omega \\ &+ 2\omega_i^T P_i(t)(\Phi_i^T \otimes I_n)\begin{pmatrix} f_1(s) - \bar{f}(s) \\ \vdots \\ f_N(s) - \bar{f}(s)\end{pmatrix}.\end{aligned} \tag{23}$$

Condition (18) implies that the first term on the right hand side of (23) satisfies

$$\begin{aligned}&\omega_i^T(\dot{P}_i(t) + P_i(t)(D\bar{f}(s) + c\lambda_i\Gamma) \\ &+(D\bar{f}(s) + c\lambda_i\Gamma)^T P_i(t))\omega_i \\ \le\ & -\alpha\|\omega_i\|^2.\end{aligned} \tag{24}$$

Applying the condition (19) we know that the second term on the right hand side of (23) satisfies

$$\begin{aligned}&2\omega_i^T P_i(t)(\Phi_i^T \otimes I_n)\times\\ &\operatorname{diag}\left\{\int_0^1 (Df_1(s+\tau e_1) - D\bar{f}(s))d\tau,\right.\\ &\left.\cdots, \int_0^1 (Df_N(s+\tau e_N) - D\bar{f}(s))d\tau\right\}(\Phi\otimes I_n)\omega\\ &\le 2\gamma\|P_i\|\|\omega_i\|\|\omega\|,\end{aligned} \tag{25}$$

while the third term of (23) satisfies

$$\begin{aligned}&2\omega_i^T P_i(t)(\varphi_i^T \otimes I_n)\begin{pmatrix} f_1(s)-\bar{f}(s)\\ \vdots \\ f_N(s)-\bar{f}(s)\end{pmatrix}\\ \le& 2\|P_i\|\|\omega_i\|\left\|\begin{pmatrix} f_1(s)-\bar{f}(s)\\ \vdots \\ f_N(s)-\bar{f}(s)\end{pmatrix}\right\|\\ =&2\|P_i\|\|\omega_i\|\mu(t)\end{aligned} \tag{26}$$

Let $V(\omega,t)=\sum_{i=2}^{N} V_i(\omega_i,t)$. Then, we have

$$\begin{aligned}\dot{V} =& \sum_{i=2}^{N}\dot{V}_i(\omega_i,t)\\ \le& \sum_{i=2}^{N}(-\alpha\|\omega_i\|^2 + 2\gamma\|P_i\|\|\omega_i\|\|\omega\| + 2\|P_i\|\|\omega_i\|\mu(t))\\ =& -\alpha\|\omega\|^2 + 2(\gamma\|\omega\| + \mu(t))\sum_{i=2}^{N}\|\omega_i\|\|P_i\|\\ \le& -\alpha\|\omega\|^2 + 2(\gamma\|\omega\| + \mu(t))\|\omega\|(\sum_{i=2}^{N}\|P_i\|^2)^{\frac{1}{2}}\\ =&\|\omega\|((2\gamma\beta-\alpha)\|\omega\| + 2\beta\mu(t)).\end{aligned} \tag{27}$$

Thus when

$$\|\omega\| \ge \frac{2\beta\mu(t)}{\alpha - 2\gamma\beta - \delta},$$

we have

$$\dot{V} \le -\delta\|\omega\|^2. \tag{28}$$

Applying Lemma 3.1 completes the proof.

Corollary 3.3. When

$$\overline{\lim}_{t\to\infty}\mu(t) = 0,$$

we have asymptotic synchronization in the classical sense. In particular, when $f_i = f$, that is, all nodes are identical, we have $\mu(t)\equiv 0$. In this case, applying Theorem 3.2 to the linearized network (3), which is equivalent to letting $\gamma = 0$ in (19), immediately gives the well-known synchronization criterion in the literature [2]. Therefore, Theorem 3.2 covers the existing criteria of networks with identical nodes as special cases.

As another special case, consider

$$f_i = f + \theta_i g \tag{29}$$

with constant vector fields f and g, and constant parameters θ_i. In this case,

$$\begin{aligned}&f_i - \bar{f} = (\theta_i - \frac{1}{N}\sum_{j=1}^{N}\theta_j)g,\\ &\gamma = 0,\\ &\mu(t) = \left(\sum_{i=1}^{N}\theta_i^2 - \frac{1}{N}(\sum_{j=1}^{N}\theta_j)^2\right)^{\frac{1}{2}}\|g\|\end{aligned} \tag{30}$$

IV. EXAMPLE

Consider the the following dynamical network with 3 non-identical nodes

$$\dot{x}_i = B_i x_i + g(x_i) + \sum_{j=1}^{N} a_{ij}\Gamma x_j, \qquad i = 1,2,3, \tag{31}$$

where

$$g(x_i) = (-9.5\sin(\frac{\pi x_{i1}}{3.2}+\pi),0,0)^T,$$

$$\Gamma = \operatorname{diag}\{2,2,2\},$$

,

$$x_0 = (1,0.5,-1,2,1,-2,-1,1.5,1)^T$$

and

$$A = \begin{pmatrix} -2 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{pmatrix},$$

$$B_1 = \begin{pmatrix} -10 & 10 & 0\\ 1 & -1 & 1\\ 0 & -15 & 0\end{pmatrix},$$

$$B_2 = \begin{pmatrix} -9.5 & 9.5 & 0\\ 1 & -1 & 1\\ 0 & -15.5 & 0\end{pmatrix},$$

$$B_3 = \begin{pmatrix} -10.5 & 10.5 & 0\\ 1 & -1 & 1\\ 0 & -14.5 & 0\end{pmatrix}.$$

Applying Theorem 3.2 we know synchronization in the sense of boundedness is achieved. Simulation results are depicted in Fig.1

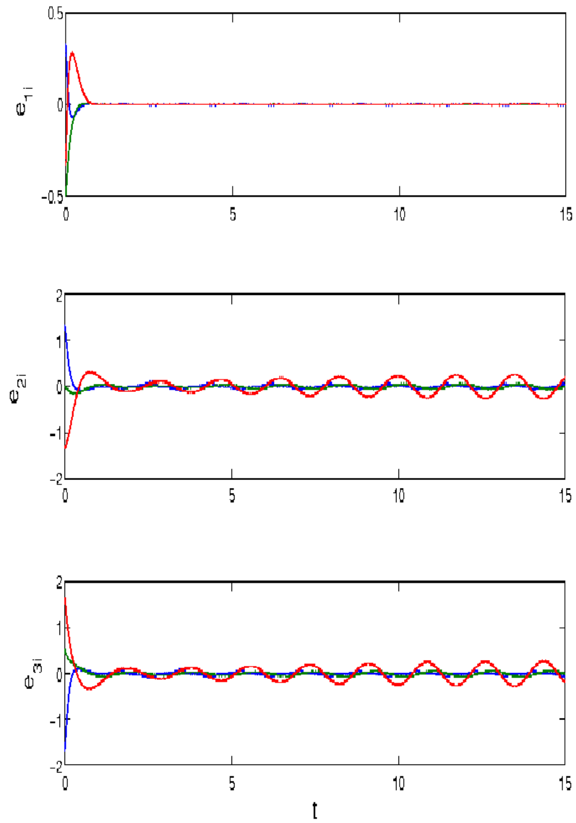

Fig. 1. The synchronization errors of the network.

V. CONCLUSIONS

We have studied the synchronization problem for a complex dynamical network with non-identical nodes. Deviation equations are established by introducing the average dynamics of all nodes. Based on these deviation equations a synchronization criterion in the sense of boundedness is proposed with an explicit bound given. This result extends the relevant asymptotic synchronization criteria to the case of non-identical nodes.

Unlike for networks with identical nodes, we have known little about behaviors of networks with non-identical nodes. Efficient techniques are need to analyze the networks. The method of the average dynamics of all nodes seems to be useful as demonstrated by the established results of this paper, but deserves further study.

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