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The Polynomial Extended Kalman Filter as an Exponential Observer for Nonlinear Discrete-Time Systems

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Abstract— **This paper presents some results on the local exponential convergence of the Polynomial Extended Kalman Filter (PEKF, see [14]) used as a state observer for deterministic nonlinear discrete-time systems (Polynomial Extended Kalman Observer, PEKO). A new compact formalism is introduced for the representation of the so called *Carleman linearization* of nonlinear discrete time systems, that allows for the derivation of the observation error dynamics in a concise form, similar to the one of the classical Extended Kalman Filter. The stability analysis performed in this paper is also important in the stochastic framework, in that the exponential stability of the error dynamics can be used to prove that the moments of the estimation error, up to a given order, remain bounded over time (stability of the PEKF).**

I. INTRODUCTION

This paper considers the state observation problem for nonlinear discrete-time systems of the type

$$x_{t+1} = f(x_t, u_t), \tag{1}$$

$$y_t = h(x_t, u_t), \tag{2}$$

where $x_t \in \mathbb{R}^n$ is the system state, $y_t \in \mathbb{R}^q$ is the measured output, $u_t \in \mathbb{R}^p$ is the sequence of known inputs. $f(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are analytic functions of the first argument (the state). Many approaches have been explored in the literature for the derivation of asymptotic state observers, and many types of solutions exist for classes of systems. An approach widely investigated is to find a nonlinear change of coordinates and, if necessary, an output transformation, that transform the system into some canonical form suitable for the observer design using linear methodologies. Some papers on the subject are [9],[21],[22], [26],[27], for autonomous systems, and [6] for nonautonomous systems. These papers study the conditions for the existence of the coordinate transformation that allows the observer design with linearizable error dynamics. The drawback of this approach is that in general the computation of the coordinate transformation, when existing, is a very difficult task. Another approach consists in designing observers in the original coordinates, finding iterative algorithms, typically based on the Newton method, that asymptotically solve a suitable extension of the state-output map, [10], [11], [23]. Sufficient conditions of local convergence are provided, in general, under the assumption of Lipschitz nonlinearities. Many authors restrict the attention to the class of systems characterized by nonlinear dynamics and linear output map (see, e.g. [1], [5], [28]). The cases of bilinear dynamics and polynomial and rational output maps are considered in [15],[17].

Although so many different approaches have been studied, the most popular algorithm of state estimation for nonlinear system is the Extended Kalman Filter (EKF), see e.g. [2]. The main reasons of the popularity of the EKF is its simplicity of implementation and its good behavior in most applications. The use of the Extended Kalman Filter as a local observer in the deterministic framework has been investigated [3], [4], [24] and [25].

This paper aims to extend the existing results of local convergence of the EKF to the Polynomial Extended Kalman Filter (PEKF) presented in [14]. The same approach used in [24] for the study of the exponential error convergence of the EKF has been used in this paper.

The paper is organized as follows. In section II the EKF equations and the convergence theorem of [24] are briefly recalled. The PEKO is presented in section III and the convergence property is discussed in section IV. Some elements of Kronecher algebra, used throughout the paper, are briefly reported in the Appendix.

II. THE EKF AS AN OBSERVER

Before to proceed with the construction of a PEKO (Polynomial Extended Kalman Observer), let us briefly recall the standard form of the EKF and its use as an Observer (EKO: Extended Kalman Observer), and discuss the convergence properties following the approach of [24]. From now on, the following more compact notation will be used for the system (1)–(2):

$$x_{t+1} = f_{u_t}(x_t), \tag{3}$$

$$y_t = h_{u_t}(x_t). \tag{4}$$

Global or local assumptions on the uniform boundedness of the derivatives of the functions $f_u(x)$ and $h_u(x)$ can be made. Local assumptions are sufficient in the proof of convergence of the EKO if the system (3)–(4) is input-state stable. For this reason the following assumption is made:

Assumption $\mathbf{A_0}$. There exist a compact set $\bar{U} \subset \mathbb{R}^p$ and bounded open sets Ω_0 and Ω, with $\Omega_0 \subset \Omega \subset \mathbb{R}^n$, such that for any input sequence with $u_t \in \bar{U}$, $\forall t \geq t_0$, if $x_{t_0} \in \Omega_0$ then $x_t \in \Omega$, $\forall t \geq t_0$. Moreover, $\forall u \in \bar{U}$, $f_u(x)$ and $h_u(x)$ are analytical functions in Ω.

Let $\bar{\Omega}$ denote the closure of Ω.

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In order to apply the standard Kalman Filter to the nonlinear system (3)–(4), it is useful to represent the state transition map $f_u(x)$ and the output map $h_u(x)$ using a first order Taylor expansion around the best estimates available: the observation $\hat{x}_t$ for $f_u(x)$ and the prediction $\tilde{x}_t$ for $h_u(x)$:

$$x_{t+1} = f_{u_t}(\hat{x}_t) + A_t(x_t - \hat{x}_t) + \varphi_f(x_t, \hat{x}_t), \quad (5)$$
$$y_t = h_{u_t}(\tilde{x}_t) + C_t(x_t - \tilde{x}_t) + \varphi_h(x_t, \tilde{x}_t), \quad (6)$$

where C_t and A_t are the Jacobians of $h_{u_t}(x)$ and $f_{u_t}(x)$ computed at the predicted and estimated state, rspectively. Using the notation introduced in the Appendix, eq. (88),

$$A_t = \nabla_x \otimes f_{u_t}(x)|_{\hat{x}_t}, \quad C_t = \nabla_x \otimes h_{u_t}(x)|_{\tilde{x}_t}. \quad (7)$$

The remainders φ_f and φ_h are such that there exist positive ϵ_f, ϵ_h, γ_f, γ_h such that

$$\begin{aligned} \|\varphi_f(x, \bar{x})\| &\le \gamma_f \|x - \bar{x}\|^2, \quad \forall \|x - \bar{x}\| \le \epsilon_f, \\ \|\varphi_h(x, \bar{x})\| &\le \gamma_h \|x - \bar{x}\|^2, \quad \forall \|x - \bar{x}\| \le \epsilon_h. \end{aligned} \quad (8)$$

(In [24] the inequalities (8) are assumed to hold for all $\bar{x} \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$.) The EKO approach consists in neglecting the remainders in the representation (5)–(6), so that it appears as a linear system with known forcing terms, and in applying the standard Kalman Filter equations. An initial state estimation $\bar{x}_0$ is needed as a starting value of the EKO. Also a positive definite (PD) matrix $\overline{P}_0$ is needed to inizialize the Riccati equations that provide the Kalman gain K_t. The EKO algorithm is reported below. The symbol I_n denotes the identity matrix of dimension n.

Extended Kalman Observer (EKO)

Starting values: $\tilde{x}_{t_0} = \bar{x}_0$, $\widetilde{P}_{t_0} = \overline{P}_0$, $t = t_0$,

$$\tilde{y}_t = h_{u_t}(\tilde{x}_t), \qquad \text{output prediction} \quad (9)$$
$$C_t = \nabla_x \otimes h_{u_t}\big|_{\tilde{x}_t}, \quad (10)$$
$$K_t = \widetilde{P}_t C_t^T (C_t \widetilde{P}_t C_t^T + R_t)^{-1}, \quad (11)$$
$$\hat{x}_t = \tilde{x}_t + K_t(y_t - \tilde{y}_t), \qquad \text{state observation} \quad (12)$$
$$P_t = (I_n - K_t C_t)\widetilde{P}_t, \quad (13)$$
$$\tilde{x}_{t+1} = f_{u_t}(\hat{x}_t), \qquad \text{state prediction} \quad (14)$$
$$A_t = \nabla_x \otimes f_{u_t}\big|_{\hat{x}_t}, \quad (15)$$
$$\widetilde{P}_{t+1} = \alpha^2 A_t P_t A_t^T + Q_t. \quad (16)$$

Q_t and R_t are known sequences of PD matrices that act as forcing terms in the Riccati equations, and must be chosen uniformly upper and lower bounded over $t \in \mathbb{Z}$ (in fact, in [24] they are chosen constant). In the EKO such sequences are free design parameters, while in the stochastic framework (EKF) they are the covariances of the state and output noises. The constant coefficient $\alpha \ge 1$ has the meaning of a *forgetting factor* and provides exponential data weighting when $\alpha > 1$.

The convergence analysis of the EKO in [24] has been pursued by studying the stability of the recursive equation that governs the prediction error $x_t - \tilde{x}_t$. This equation is obtained subtracting the prediction $\tilde{x}_{t+1}$ given by (14) from x_{t+1} as given by (5):

$$x_{t+1} - \tilde{x}_{t+1} = A_t(x_t - \hat{x}_t) + \varphi_f(x_t, \hat{x}_t), \quad (17)$$

and then finding a suitable expression for the estimation error $x_t - \hat{x}_t$. Considering the two identities below, obtained using (12) and (9),

$$x_t - \hat{x}_t = x_t - \tilde{x}_t - K_t(y_t - \tilde{y}_t), \quad (18)$$
$$y_t - \tilde{y}_t = C_t(x_t - \tilde{x}_t) + \varphi_h(x_t, \tilde{x}_t), \quad (19)$$

the following recursion can be easily obtained

$$x_{t+1} - \tilde{x}_{t+1} = A_t(I_n - K_t C_t)(x_t - \tilde{x}_t) + \varphi_0, \quad (20)$$
$$\text{where} \quad \varphi_0 = \varphi_f(x_t, \hat{x}_t) - K_t \varphi_h(x_t, \tilde{x}_t). \quad (21)$$

The convergence result in [24] is based on the proof of asymptotic stability of equation (20), and is summarized below:

Theorem 1. *Consider the EKO equations* (9)–(16)*, and let the following assumptions hold*

i) *There are positive numbers $\bar{a}, \bar{c}, \bar{p}, \underline{p}$ such that for all $t \ge t_0$*
$$\|A_t\| \le \bar{a}, \qquad \|C_t\| \le \bar{c}, \quad (22)$$
$$\underline{p} I_n \le \widetilde{P}_t \le \bar{p} I_n \qquad \underline{p} I_n \le \widetilde{P}_t \le \bar{p} I_n. \quad (23)$$

ii) *A_t is nonsingular $\forall t \ge t_0$.*

iii) *There are positive real numbers ϵ_f, ϵ_h, γ_f, γ_h, such that inequalities* (8) *hold.*

Then, there exist positive real numbers η, ϵ_0, θ, with $\theta > \alpha$, such that, if $\|x_{t_0} - \tilde{x}_{t_0}\| < \epsilon_0$, then

$$\|x_t - \tilde{x}_t\| \le \eta \|x_{t_0} - \tilde{x}_{t_0}\| \theta^{-(t-t_0)}, \quad (24)$$

that means that the EKO is a local exponential observer.

Remark 1. The parameter $\alpha \ge 1$, that appears in equation (16), can be tuned to assign the error convergence rate. However, in the proof of Theorem 1 in [24] it appears that the larger is chosen α, the smaller is the convergence region ϵ_0.

Remark 2. The existence of lower and upper bounds for the PD matrices $\widetilde{P}_t$ and P_t can only checked on line, and is ensured if the pair (A_t, C_t) satisfies a uniform observability condition (see e.g. [12]).

Remark 3. It is important to stress that the proof reported in [24], based on the bounds (8) on the norms of the remainders φ_f and φ_h, can be easily modified to deal with bounds of higher order, i.e. of the type

$$\|\varphi_f\| \le \gamma_f \|x - \bar{x}\|^k, \quad \|\varphi_h\| \le \gamma_h \|x - \bar{x}\|^k, \quad (25)$$

for $k > 2$.

Remark 4. The bounds (8) assumed in Theorem 1, in [24] are formulated in a global form, i.e. the inequalities are assumed to hold for all $\bar{x} \in \mathbb{R}^n$ and for all $u \in \mathbb{R}^p$. This can be a too strong assumption. However, the proof of Theorem 1 can be suitably modified when inequalities (8) hold only on bounded sets. In this case the additional assumption $\mathbf{A_0}$ is required, so that it is sufficient that properties (8) on the remainders are true for all $\bar{x} \in \Omega$.

III. POLYNOMIAL EXTENDED KALMAN OBSERVER

The PEKF algorithm presented in [14] is based on the polynomial approximation of the state transition map $f_u(x)$ and of the output map $h_u(x)$ of the system (3)–(4). The use of the filter in [14] as an Observer for deterministic systems is denoted PEKO in this paper.

The formalism of the Kronecker algebra is used in this paper for the efficient manipulation of multivariate polynomials. The definition of the Kronecker product $\otimes$ of matrices is given in the Appendix, together with other relevant definitions and properties used throughout this work. The symbol $v^{[k]}$ denotes the Kronecker power of a vector $v \in \mathbb{R}^n$. For the kind of computations carried out in this paper, it is extremely useful the definition of a symbol for the vector that collects all the Kronecker powers of a given vector from 1 up to a given degree m. The symbol chosen is $[\,\cdot\,]^m$, and operates on vectors $v \in \mathbb{R}^n$ as follows

$$[v]^m = \begin{bmatrix} v \\ v^{[2]} \\ \vdots \\ v^{[m]} \end{bmatrix}, \qquad \begin{array}{c} v^{[0]} = 1, \\ v^{[k+1]} = v^{[k]} \otimes v. \end{array} \tag{26}$$

Recalling that $v^{[k]} \in \mathbb{R}^{n^k}$, then $[v]^m \in \mathbb{R}^{n_m}$, with $n_m = \sum_{k=1}^{m} n^k$. A property of the symbol $[\cdot]^m$ repeatedly used throughout the paper is the following (see Lemma (3)) in the Appendix)

$$[v - \bar{v}]^m = \mathcal{I}_m(-\bar{v})\big([v]^m - [\bar{v}]^m\big), \quad \forall v, \bar{v} \in \mathbb{R}^n, \tag{27}$$

where the matrix $\mathcal{I}_m(-\bar{v})$ is defined in the Appendix, eq. (81).

The PEKF in [14] is based on the Taylor polynomial approximation of a chosen degree $m > 1$ of both maps $f_{u_t}(x)$ and $h_{u_t}(x)$. The convergence of the PEKF used as an observer (PEKO) requires the assumption $\mathbf{A_0}$, in order to ensure the existence of uniform upper bounds on the norms of the remainders of the Taylor approximation. The Taylor expansion of degree m of the output map $h_{u_t}(x_t)$ around the prediction $\tilde{x}_t$ is

$$\begin{aligned} h_{u_t}(x_t) = \sum_{j=0}^{m} \frac{1}{j!}\big(\nabla_x^{[j]} \otimes h_{u_t}(x)\big)\big|_{\tilde{x}_t}(x_t - \tilde{x}_t)^{[j]} \\ + \varphi_h(x_t, \tilde{x}_t). \end{aligned} \tag{28}$$

where the differential symbol $\nabla_x^{[j]}\otimes$ is defined in the Appendix, eq. (88). Based on the Lagrange remainder formula, the following bound can be given, for all $(x, \bar{x}, u) \in \Omega \times \Omega \times \overline{U}$

$$\|\varphi_h(x, \bar{x})\| \leq \gamma_h \|x - \bar{x}\|^{m+1}, \tag{29}$$

$$\text{where} \qquad \gamma_h = \sup_{(x,u)\in\overline{\Omega}\times\overline{U}} \frac{\big\|\nabla_x^{[m+1]} \otimes h_u(x)\big\|}{(m+1)!}. \tag{30}$$

Using the symbol $[\,\cdot\,]^m$, the Taylor formula (28) can be written in the compact form

$$h_{u_t}(x_t) = h_{u_t}(\tilde{x}_t) + \mathcal{H}_m(\tilde{x}_t)[x_t - \tilde{x}_t]^m + \varphi_h, \tag{31}$$

where matrix $\mathcal{H}_m(\tilde{x}_t) \in \mathbb{R}^{q\times n_m}$ has a row-block structure

$$\mathcal{H}_m(x) = \big[\big[\mathcal{H}_m(x)\big]_1 \quad \cdots \quad \big[\mathcal{H}_m(x)\big]_m\big] \tag{32}$$

$$\text{where} \quad \big[\mathcal{H}_m(x)\big]_j = \frac{1}{j!}\big(\nabla_x^{[j]} \otimes h_{u_t}(x)\big), \tag{33}$$

(for a simpler notation, the dependence of $\mathcal{H}_m$ on u_t is not shown). Using the identity (27), equation (31) can be written as

$$h_{u_t}(x_t) = h_{u_t}(\tilde{x}_t) + C_t\big([x_t]^m - [\tilde{x}_t]^m\big) + \varphi_h, \tag{34}$$

$$\text{where} \qquad C_t = \mathcal{H}_m(\tilde{x}_t)\mathcal{I}_m(-\tilde{x}_t). \tag{35}$$

Now define the *polynomial extended state* $X_t \in \mathbb{R}^{n_m}$ and the *selection matrix* $\Sigma \in \mathbb{R}^{n\times n_m}$ as follows

$$X_t = [x_t]^m, \quad \Sigma = \big[I_n \quad 0_{n\times(n_m - n)}\big], \tag{36}$$

so that $x = \Sigma[x]^m$, $\forall x \in \mathbb{R}^n$, and in particular

$$x_t = \Sigma X_t. \tag{37}$$

The use of X_t in (34), allows to write the output equation (4) as

$$y_t = h_{u_t}(\tilde{x}_t) + C_t\big(X_t - [\tilde{x}_t]^m\big) + \varphi_h, \tag{38}$$

where $\varphi_h = \varphi_h(\Sigma X_t, \tilde{x}_t)$. In this form the output equation depends linearly on the extended state X_t. In order to use this linear form in a linear filter, a linear transition function is needed for the extended state. Consider the transition map of the extended state

$$X_{t+1} = [x_{t+1}]^m = [f_{u_t}(x_t)]^m = \begin{bmatrix} f_{u_t}(x_t) \\ \vdots \\ f_{u_t}^{[m]}(x_t) \end{bmatrix}. \tag{39}$$

The Taylor formula for the component $f_{u_t}^{[k]}(x_t)$ around the current estimate $\hat{x}_t$, is the following

$$\begin{aligned} f_{u_t}^{[k]}(x_t) = \sum_{j=0}^{m} \frac{1}{j!}\big(\nabla_x^{[j]} \otimes f_{u_t}^{[k]}(x)\big)\big|_{\hat{x}_t}(x_t - \hat{x}_t)^{[j]} \\ + \varphi_{f,k}(x_t, \hat{x}_t). \end{aligned} \tag{40}$$

The remainder is such that, $\forall (x, \bar{x}, u) \in \Omega \times \Omega \times \overline{U}$,

$$\|\varphi_{f,k}(x, \bar{x})\| \leq \gamma_{f,k}\|x - \bar{x}\|^{m+1}. \tag{41}$$

$$\text{where} \quad \gamma_{f,k} = \sup_{(x,u)\in\overline{\Omega}\times\overline{U}} \frac{\big\|\nabla_x^{[m+1]} \otimes f_u^{[k]}(x)\big\|}{(m+1)!}. \tag{42}$$

Using the symbol $[\,\cdot\,]^m$, the extended transition map $[f_{u_t}(x_t)]^m$ can be written in the compact form

$$[f_{u_t}(x_t)]^m = \big[f_{u_t}(\hat{x}_t)\big]^m + \mathcal{F}_m(\hat{x}_t)[x_t - \hat{x}_t]^m + \varphi_f(x_t, \hat{x}_t), \tag{43}$$

where the matrix $\mathcal{F}_m(\hat{x}_t) \in \mathbb{R}^{n_m\times n_m}$ is made of $m \times m$ blocks, defined, for $k = 1, \dots, m,\ j = 1, \dots, m$, as

$$\big[\mathcal{F}_m(x)\big]_{k,j} = \frac{1}{j!}\big(\nabla_x^{[j]} \otimes f_{u_t}^{[k]}(x)\big) \in \mathbb{R}^{n^k\times n^j} \tag{44}$$

(the dependence of $\mathcal{F}_m$ on u_t is not shown). For $(x, \bar{x}, u) \in \Omega \times \Omega \times \bar{U}$, the remainder φ_f obeys the inequality

$$\|\varphi_f(x, \bar{x})\| \leq \gamma_f \|x - \bar{x}\|^{m+1}, \tag{45}$$

$$\text{where} \quad \gamma_f = \sup_{(x,u)\in\bar{\Omega}\times\bar{U}} \frac{\left\|\nabla_x^{[m+1]} \otimes [f_u(x)]^m\right\|}{(m+1)!}. \tag{46}$$

Now the extended state transition step $X_{t+1} = [f_{u_t}(x_t)]^m$ can be written using the representation (43) with the substitutions

$$\begin{aligned}\left[x_t - \hat{x}_t\right]^m &= \mathcal{I}_m(-\hat{x}_t)\big([x_t]^m - [\hat{x}_t]^m\big) \\ &= \mathcal{I}_m(-\hat{x}_t)\big(X_t - [\hat{x}_t]^m\big),\end{aligned} \tag{47}$$

obtaining

$$X_{t+1} = \left[f_{u_t}(\hat{x}_t)\right]^m + A_t\big(X_t - [\hat{x}_t]^m\big) + \varphi_f(\Sigma X_t, \hat{x}_t), \tag{48}$$

$$\text{where} \quad A_t = \mathcal{F}_m(\hat{x}_t)\mathcal{I}_m(-\hat{x}_t). \tag{49}$$

The equations (48) and (38) describing system (3)–(4), can be written as

$$X_{t+1} = A_t X_t + \left[f_{u_t}(\hat{x}_t)\right]^m - A_t[\hat{x}_t]^m + \varphi_f, \tag{50}$$

$$y_t = C_t X_t + h_{u_t}(\tilde{x}_t) - C_t[\tilde{x}_t]^m + \varphi_h, \tag{51}$$

$$\text{where} \quad \varphi_f = \varphi_f(\Sigma X_t, \hat{x}_t), \quad \varphi_h = \varphi_h(\Sigma X_t, \tilde{x}_t). \tag{52}$$

System (3)–(4) is said to be *immersed* into the system (50)–(51), which has higher dimension, in the sense that if at a given t_0 it is $X_{t_0} = [x_{t_0}]^m$, then $X_t = [x_t]^m$, for all $t \geq t_0$, and therefore $x_t = \Sigma X_t$.

Note that in the derivation of (50)–(51) the sequences $\hat{x}_t$ and $\tilde{x}_t$ can be any sequences in Ω (they may even be constant).

The Carleman linearization of (3)–(4) around the sequences $\hat{x}_t$ and $\tilde{x}_t$ consists in neglecting the remainders φ_f and φ_h in equations (50)–(51), and in replacing the polynomial extended state $X_t = [x_t]^m$ with an approximating vector $\mathcal{X}_t \in \mathbb{R}^{n_m}$, to obtain

$$\mathcal{X}_{t+1} = A_t\mathcal{X}_t + \left[f_{u_t}(\hat{x}_t)\right]^m - A_t[\hat{x}_t]^m, \tag{53}$$

$$y'_t = A_t\mathcal{X}_t + h_{u_t}(\tilde{x}_t) - C_t[\tilde{x}_t]^m. \tag{54}$$

The Carleman linearization (53)–(54) is an approximation of system (3)–(4) if the differences $[x_t]^m - \mathcal{X}_t$ and $y_t - y'_t$ can be made as small as desired, at least in a finite time interval, by increasing the degree m, provided that both x_t and $\mathcal{X}_t$ are consistently initialized at time t_0 (i.e., $\mathcal{X}_{t_0} = [x_{t_0}]^m$) and both systems are forced by the same input sequence.

Remark 5. Note that, by definition, X_t evolves on the consistency manifold $\mathcal{M}_m$, defined in Appendix, eq. (80), while, in general, $\mathcal{X}_t$ is not consistent (i.e., $\mathcal{X}_t \notin \mathcal{M}_m$).

Equations (53)–(54) have the appearance of a time-varing linear system, with known system matrices and forcing terms. Thus, the construction of an observer with the standard Kalman Filter structure is straightforward. Such an observer, denoted here PEKO, has the same prediction-correction structure of the EKO, where the correction gain K_t is the output of Riccati equations forced by two sequences of PD matrices, $Q_t \in \mathbb{R}^{n_m \times n_m}$ and $R_t \in \mathbb{R}^{q\times q}$, uniformly lower and upper bounded over $t \in \mathbb{Z}$. The equations of the PEKO need to be initializated using an a priori state estimate $\bar{x}_0 \in \mathbb{R}^n$ at time t_0. The Riccati equations require a PD matrix $\bar{P}_0 \in \mathbb{R}^{n_m\times n_m}$ for the initialization, representing the uncertainty on the initial estimate $\bar{x}_0$.

The sequences of state observations $\hat{x}_t$ and predictions $\tilde{x}_t$ in the PEKO are obtained as subvectors of the extended state observations $\widehat{X}_t$ and predictions $\widetilde{X}_t$ produced by the algorithm.

Polynomial Extended Kalman Observer (PEKO)

Starting values: $\tilde{x}_{t_0} = \bar{x}_0$, $\widetilde{X}_{t_0} = [\tilde{x}_{t_0}]^m$, $\widetilde{P}_{t_0} = \bar{P}_0$, $t = t_0$.

$$C_t = \mathcal{H}_m(\tilde{x}_t)\mathcal{I}_m(-\tilde{x}_t), \tag{55}$$

$$\tilde{y}_t = h_{u_t}(\tilde{x}_t) - C_t\big(\widetilde{X}_t - [\tilde{x}_t]^m\big), \qquad \textit{output pred.} \tag{56}$$

$$K_t = \widetilde{P}_t C_t^T \big(C_t\widetilde{P}_t C_t^T + R_t\big)^{-1}, \tag{57}$$

$$\widehat{X}_t = \widetilde{X}_t + K_t(y_t - \tilde{y}_t), \qquad \textit{ext. state estim.} \tag{58}$$

$$P_t = \big(I_n - K_t C_t\big)\widetilde{P}_t, \tag{59}$$

$$\hat{x}_t = \Sigma\widehat{X}_t, \qquad \textit{state estim.} \tag{60}$$

$$A_t = \mathcal{F}_m(\hat{x}_t)\mathcal{I}_m(-\hat{x}_t), \tag{61}$$

$$\widetilde{P}_{t+1} = \alpha^2 A_t P_t A_t^T + Q_t, \tag{62}$$

$$\widetilde{X}_{t+1} = [f_{u_t}(\hat{x}_t)]^m - A_t\big(\widehat{X}_t - [\hat{x}_t]^m\big), \qquad \textit{ext. state pred.} \tag{63}$$

$$\tilde{x}_{t+1} = \Sigma\widetilde{X}_{t+1}. \qquad \textit{state prediction} \tag{64}$$

Remark 6. As in the EKO, a constant coefficient $\alpha \geq 1$ (forgetting factor) has been considered in equation (62), so that exponential data weighting is achieved when $\alpha > 1$.

Remark 7. The vector $\widetilde{X}_t$ is an estimate of $[\tilde{x}_t]^m$, and, by construction, see eq. (64), it is such that $\Sigma\big(\widetilde{X}_t - [\tilde{x}_t]^m\big) = 0$, although in general $\widetilde{X}_t \neq [\tilde{x}_t]^m$. Stated in other words, $\widetilde{X}_t$ in general do not belong to the consistency manifold $\mathcal{M}_m \subset \mathbb{R}^{n_m}$. The same considerations can be made for $\widehat{X}_t$, that is an estimate of $[\hat{x}]^m$ such that $\Sigma\big(\widehat{X}_t - [\hat{x}_t]^m\big) = 0$, but in general $\widehat{X}_t - [\hat{x}_t]^m \neq 0$. Note that the difference $\widetilde{X}_t - [\tilde{x}_t]^m$ appear as a forcing term in the output prediction equation (56), while the mismatch $\widehat{X}_t - [\hat{x}_t]^m$ appear as a forcing term in the extended state prediction (63).

IV. Convergence Analysis of the PEKO

Following the approach in [24], the convergence analysis of the PEKO is addressed in this section by deriving and studying the recursive equation that governs the dynamics of the prediction error. Note that the PEKO provides a sequence of estimates and predictions of the extended state $X_t = [x_t]^m$. The following relationship exists between the state prediction error $x_t - \tilde{x}_t$ and the extended state prediction error $X_t - \widetilde{X}_t$:

$$\|x_t - \tilde{x}_t\| = \|\Sigma(X_t - \widetilde{X}_t)\| \leq \|X_t - \widetilde{X}_t\|. \tag{65}$$

This inequality implies that the convergence of the extended prediction implies the convergence of the state prediction (if $\|X_t - \widetilde{X}_t\| \to 0$, then $\|x_t - \tilde{x}_t\| \to 0$).

Thus, the convergence analysis can proceed by deriving a recursive equation for the extended state prediction error $X_t - \widetilde{X}_t$. Subtracting (63) from (50) yields

$$X_{t+1} - \widetilde{X}_{t+1} = A_t\big(X_t - \widehat{X}_t\big) + \varphi_f. \tag{66}$$

The estimation error of the extended state $X_t - \widehat{X}_t$ is computed subtracting (58) from X_t

$$X_t - \widehat{X}_t = X_t - \widetilde{X}_t - K_t(y_t - \tilde{y}_t). \tag{67}$$

The output prediction error $y_t - \tilde{y}_t$ is computed by subtracting (56) from (51)

$$y_t - \tilde{y}_t = C_t\big(X_t - \widetilde{X}_t\big) + \varphi_h. \tag{68}$$

Substitution of this into (67) yields

$$X_t - \widehat{X}_t = \big(I_{n_m} - K_t C_t\big)(X_t - \widetilde{X}_t) - K_t\varphi_h. \tag{69}$$

Substitution of (69) into (66) gives

$$X_{t+1} - \widetilde{X}_{t+1} = A_t\big(I_{n_m} - K_t C_t\big)(X_t - \widetilde{X}_t) + \varphi_I, \tag{70}$$

where $\varphi_I = \varphi_f(x_t, \hat{x}_t) - A_t K_t \varphi_h(x_t, \tilde{x}_t)$. (71)

Consider the bounds (45) and (29) on φ_f and φ_h. Using the inequalities $\|x_t - \tilde{x}_t\| \le \|X_t - \widetilde{X}_t\|$ and $\|x_t - \hat{x}_t\| \le \|X_t - \widehat{X}_t\|$, it follows

$$\begin{aligned} \|\varphi_f(x_t, \hat{x}_t)\| &\le \gamma_f \|X_t - \widehat{X}_t\|^{m+1}, \\ \|\varphi_h(x_t, \tilde{x}_t)\| &\le \gamma_h \|X_t - \widetilde{X}_t\|^{m+1}. \end{aligned} \tag{72}$$

The following theorem can be proved following the same lines of Theorem 1:

Theorem 2. *Consider system* (3)–(4)*, with assumption* $\mathbf{A_0}$*, and the PEKO equations* (56)–(64)*, for a given degree* m*, and let the following assumptions hold*

i) *There exist positive numbers* $\bar{a}, \bar{c}, \bar{p}, \underline{p}$ *such that for all* $t \ge t_0$

$$\|A_t\| \le \bar{a}, \qquad \|C_t\| \le \bar{c}, \tag{73}$$

$$\underline{p} I_n \le \widetilde{P}_t \le \bar{p} I_n \qquad \underline{p} I_n \le \widetilde{P}_t \le \bar{p} I_n. \tag{74}$$

ii) A_t *is nonsingular* $\forall t \ge t_0$.

iii) *There exist positive real numbers* γ_f, γ_h, *such that inequalities* (72) *hold, for* $(x_t, \hat{x}_t, u_t) \in \Omega \times \Omega \times \overline{U}$.

Then, there exist positive real numbers η, ϵ_0, θ, *with* $\theta > \alpha$, *such that, if* $\|[x_{t_0}]^m - [\tilde{x}_{t_0}]^m\| < \epsilon_0$, *then*

$$\|[x_t]^m - [\tilde{x}_t]^m\| \le \eta \|[x_{t_0}]^m - [\tilde{x}_{t_0}]^m\| \theta^{-(t-t_0)}, \tag{75}$$

that means that the PEKO is a local exponential observer (recall that $\|x_t - \tilde{x}_t\| \le \|[x_t]^m - [\tilde{x}_t]^m\|$*, see* (65)*).*

V. Final Remarks and Conclusions

The local stability of the Polynomial Extended Kalman Filter used as an asymptotic state observer (PEKO, Polynomial Extended Kalman Observer) has been investigated. The analysis is performed following the approach used in [24] to study the convergence properties of the Extended Kalman Filter used as an observer. A new compact formalism is introduced for the representation of the *Carleman linearization* of nonlinear discrete time systems, that allows for the derivation of the state prediction error dynamics in a form similar to the one developed in [24] for the classical Extended Kalman Filter. It follows that the conditions that ensure the exponential convergence of the observation error of the PEKO are formally similar to those given in [24].

The stability analysis performed in this paper is also important in the stochastic framework, when both state and output noises are present. In this case the Polynomial Extended Kalman Filter [14] should be applied, where the sequences of matrices Q_t and R_t in the Riccati equations are not free design parameters. The conditions of exponential stability of the error dynamics in the deterministic setting ensure that in the stochastic setting the moments of the estimation error, up to a given order, remain bounded over time (stability of the PEKF).

An interesting issue to investigate in future work will be whether higher order PEKO's provide better convergence properties than lower order ones, in terms of basin of attraction and rate of convergence.

Appendix

Useful formulas of the Kronecker Algebra

The Kronecker product of two matrices M and N of dimensions $p \times q$ and $r \times s$ respectively, is the $(p \cdot r) \times (q \cdot s)$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \dots & m_{1q}N \\ \vdots & \ddots & \vdots \\ m_{p1}N & \dots & m_{pq}N \end{bmatrix}, \tag{76}$$

where the m_{ij} are the entries of M. The Kronecker power of a matrix M is recursively defined as

$$M^{[0]} = 1, \quad M^{[i]} = M \otimes M^{[i-1]}, \quad i \ge 1. \tag{77}$$

Note that if $M \in \mathbb{R}^{p \times q}$, then $M^{[i]} \in \mathbb{R}^{p^i \times q^i}$. A quick survey on the Kronecker algebra can be found in the Appendix of [8]. See [18] for more properties.

The symbol $[x]^k$, with $k \in \mathbb{N}$, defined in equation (26), can also be recursively defined as

$$[x]^1 = x, \qquad [x]^{k+1} = \begin{bmatrix} [x]^k \\ x^{[k+1]} \end{bmatrix}, \quad k \ge 1. \tag{78}$$

Then $x \in \mathbb{R}^n$ implies $[x]^m \in \mathbb{R}^{n_m}$, with $n_m = \sum_{k=1}^m n^k$. Let Σ denote the following matrix in $\mathbb{R}^{n \times n_m}$

$$\Sigma = \begin{bmatrix} I_n & 0_{n \times (n_m - n)} \end{bmatrix}. \tag{79}$$

Σ is called *selection matrix* because it selects the first n component of a vector of dimension n_m. It is such that $x = \Sigma[x]^m$, $\forall x \in \mathbb{R}^n$. Note that the vector $[x]^m$ belongs to a submanifold $\mathcal{M}_m \subset \mathbb{R}^{n_m}$ of dimension n, defined as

$$\mathcal{M}_m = \big\{X \in \mathbb{R}^{n_m} : X = [\Sigma X]^m\big\}. \tag{80}$$

$\mathcal{M}_m$ is called *consistency manifold*, and if $X \in \mathcal{M}_m$, then X is said to be *consistent*, because in this case $x = \Sigma X$ is such that $X = [x]^m$.

Let $\mathcal{I}_m(v)$, where $m \in \mathbb{N}$ be a $n_m \times n_m$ matrix defined as

$$\mathcal{I}_m(v) = I_{n_m} + \mathcal{S}_m(v), \tag{81}$$

where matrix $\mathcal{S}_m(v)$ is a strongly lower block-triangular matrix, whose blocks $S_{h,k} = \left[\mathcal{S}_m(v)\right]_{h,k}$, are defined as

$$\begin{aligned} &S_{h,k} = 0_{n^h \times n^k}, \quad \text{for } h \le k, \ \text{(strongly lower diagonal)} \\ &S_{2,1} = v \otimes I_n + I_n \otimes v, \\ &S_{h,1} = v^{[h-1]} \otimes I_n + S_{h-1,1} \otimes v, \qquad 2 < h \le m \\ &S_{h,k} = S_{h-1,k-1} \otimes I_n + S_{h-1,k} \otimes v, \quad 1 < k < h. \end{aligned} \tag{82}$$

From the definition it easy to see that $\mathcal{S}_m(0) = 0$, and therefore

$$\mathcal{I}_m(0) = I_{n_m}. \tag{83}$$

Lemma 3. *For any given v and $\bar{v}$ in $\mathbb{R}^n$ and $m \in \mathbb{N}$, the following hold*

$$[v + \bar{v}]^m = \mathcal{I}_m(\bar{v})[v]^m + [\bar{v}]^m, \tag{84}$$

$$[v - \bar{v}]^m = \mathcal{I}_m(-\bar{v})\big([v]^m - [\bar{v}]^m\big). \tag{85}$$

$$[v]^m = -\mathcal{I}_m(v)[-v]^m, \tag{86}$$

$$\mathcal{I}_m^{-1}(v) = \mathcal{I}_m(-v), \tag{87}$$

The Kronecker formalism can be used also to represent differential operators. Matrices of derivatives of any order with respect to a vector variable $x \in \mathbb{R}^n$ can be represented defining the operator $\nabla_x^{[i]} \otimes$. Let $\psi : \mathbb{R}^n \mapsto \mathbb{R}^q$ be a differentiable function. The operator $\nabla_x^{[i]} \otimes$ formally acts as a Kronecker product as follows:

$$\begin{aligned} &\nabla_x^{[0]} \otimes \psi = \psi, \\ &\nabla_x^{[i+1]} \otimes \psi = \nabla_x \otimes \left(\nabla_x^{[i]} \otimes \psi\right), \quad i \ge 1. \end{aligned} \tag{88}$$

with $\nabla_x = [\partial/\partial x_1 \ \cdots \ \partial/\partial x_n]$. Note that $\nabla_x \otimes \psi$ is the standard Jacobian of the vector function ψ.

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