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What Can Linear State Feedback Accomplish for Nonlinear Systems?

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***Abstract*— The purpose of this paper is to address the important question of when an uncertain system with higher-order nonlinearities can be effectively controlled by linear state feedback. In particular, for a family of uncertain nonlinear systems whose linearization is usually uncontrollable or, even worse, has uncontrollable modes associated with eigenvalues on the right-half plane, there is no linear or smooth state feedback to achieve global asymptotic stabilization (GAS). However, we show that if a less aggressive control objective such as *semi-global asymptotic stabilization* (SGAS) or *semi-global practical stabilization* (SGPS) is sought, *linear* controllers would be sufficient and meet the control goal. Several examples are provided to illustrate the effectiveness of the proposed robust linear state feedback control laws.**

I. INTRODUCTION

In this paper, we consider a family of uncertain nonlinear systems of the form

$$\begin{aligned} \dot{x}_1 &= x_2^{p_1} + f_1(t,x,u) \\ &\vdots \\ \dot{x}_{n-1} &= x_n^{p_{n-1}} + f_{n-1}(t,x,u) \\ \dot{x}_n &= u + f_n(t,x,u) \end{aligned} \tag{1.1}$$

where $u \in \mathbb{R}$, $x = (x_1, \cdots, x_n)^T$ are the system input and state, respectively. For $i = 1, \cdots, n-1$, $p_i \geq 1$ is an *odd* positive integer, and $f_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, is a C^0 mapping with $f_i(t, 0, \cdots, 0) = 0$, $\forall t$, which involve uncertainty and may not be known precisely.

Because system (1.1) with appropriate $f_i(\cdot)$ represents a generalized *normal form* of the affine system $\dot{\xi} = f(\xi) + g(\xi)u$ when exact feedback linearization is not possible [4], [19], it becomes a natural starting point when investigating various control problems for nonlinear systems whose first approximation does not provide any useful information.

In the literature, a number of researchers have considered the nonlinear system (1.1) and obtained many important results. For example, the papers [1], [9], [10], [11], [12], [15], [16] gave the local stabilization results by state feedback for a class of two- or three-dimensional nonlinear systems. In the higher-dimensional case, the local stabilization results were reported in [5], [6], [8], [22], while the global stabilization of the nonlinear system (1.1) was established in the work [17], [18]. All of these results rely heavily on the homogeneous systems theory, in particular, the idea of homogeneous approximation, the notion of homogeneity with respect to a family of dilations [12], [15], [16], and the robustness of homogeneous systems [12], [20].

The state feedback control laws derived in the aforementioned work are either smooth (see, e.g., [17]) or non-smooth but Hölder continuous. In particular, they are in general nonlinear functions of the system state and complicated, making them difficult to be implemented. Since linear controllers are the most simple ones that have been widely employed in practice, we are interested in the fundamental question that to what extent linear state feedback is sufficient for the control of uncertain nonlinear systems, such as the one in the form (1.1).

The purpose of this paper is to address such an important issue. Specifically, we intend to provide some answers to the questions such as

1) Is the nonlinear system (1.1) stabilizable by *linear* state feedback?

2) What is a reasonable control objective in the context of linear feedback?

3) How are linear controllers systematically designed to meet such a goal if possible?

It turns out that if one aims at a little less aggressive control objective, namely, *semi-global asymptotic stabilization* (SGAS) or *semi-global practical stabilization* (SGPS) instead of global asymptotic stabilization (GAS), then linear controllers suffice and still work for the highly nonlinear system (1.1) under appropriate conditions.

Following the work [2], [13], [21], we formulate the two control problems for the nonlinear system (1.1) below.

(i) **SGAS by linear feedback:** Given an upper bound $M > 0$, find, if possible, a *linear* controller $u = L_M x$ with the gain L_M depending on M, such that all the trajectories of the closed-loop system starting from the compact set $B_M \triangleq [-M, M]^n \subset \mathbb{R}^n$ converge uniformly to the origin.

(ii) **SGPS by linear feedback:** Given an upper bound $M > 0$ and a lower bound $\varepsilon > 0$, find, if possible, a *linear* controller $u = Lx$ with the gain L depending on M and ε, such that all the trajectories of the closed-loop system starting from the compact set B_M are driven into the neighborhood B_ε in a finite time T and stay in B_ε for all $t \geq T$.

Since the upper bound of the initial condition of a controlled plant can be estimated and the tolerance of the stabilization can be preset, achieving SGAS or SGPS rather than GAS may be good enough in many practical applications.

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However, such a tradeoff in the control goal dramatically improves the implementability of the controllers.

We conclude this section by introducing some useful lemmas to be used frequently in the rest of the paper. Their proofs can be found, for instance, in [17], [18].

Lemma 1.1: Given real numbers $x, y, m, n, a, b > 0$, the following inequality holds:

$$ax^m y^n \leq bx^{m+n} + \frac{n}{m+n}(\frac{m+n}{m})^{-\frac{m}{n}} a^{\frac{m+n}{n}} b^{-\frac{m}{n}} y^{m+n}.$$

Lemma 1.2: Let $x_1, \cdots, x_n, p > 0$ be real numbers. Then,

$$(x_1 + \cdots + x_n)^p \leq \max(n^{p-1}, 1)(x_1^p + \cdots + x_n^p).$$

Lemma 1.3: For all $x, y \in \mathbb{R}$, $\varepsilon > 0$ and any odd positive integer p, the following inequality holds:

$$|x^p - y^p| \leq \varepsilon |y|^p + C|x - y|^p,$$

where C is a constant, which depends only on p and ε.

Throughout this paper, K represents a fixed generic positive real number for which we use the convention $K + K = K$ and $K \times K = K$, etc.

II. SGAS BY LINEAR STATE FEEDBACK

To address the SGAS problem for the nonlinear system (1.1) by linear state feedback, it is helpful to recall how the global stabilization was achieved by nonlinear state feedback. In [17], it was shown that GAS of system (1.1) is achievable by *smooth state* feedback under the assumptions:

Assumption 2.1: $p_1 \geq p_2 \geq \cdots \geq p_{n-1} \geq 1$.

Assumption 2.2: There is a C^0 function $\rho_i(x_1, \cdots, x_i) \geq 0$, such that $\forall (t, x, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ $(i = 1, \cdots, n)$,

$$|f_i(\cdot)| \leq (|x_1|^{p_i} + \cdots + |x_i|^{p_i})\rho_i(x_1, \cdots, x_i). \tag{2.1}$$

As pointed out in [17], Assumptions 2.1-2.2 are automatically satisfied when $p_1 = \cdots = p_{n-1} = 1$ and f_i is smooth. Moreover, they are somewhat necessary for global stabilization by *smooth* state feedback. Indeed, if either one of the conditions fails to be fulfilled, counter-examples exist illustrating the non-existence of smooth controllers [17]. These observations suggest that in the context of *linear state* feedback (which is a special case of smooth feedback), Assumptions 2.1-2.2 are still needed and may not be relaxed when the problem of GAS by linear state feedback is under consideration.

In contrast to the GAS result established in [17] by *nonlinear* state feedback, the following theorem shows that under the same conditions as in [17], i.e., Assumptions 2.1-2.2, SGAS of the nonlinear system (1.1) is possible by *linear* state feedback. In other words, by taking a trade-off of the control goals, or, by pursuing a less ambitious control objective, namely, SGAS instead of GAS, the structure of the controller can be dramatically simplified and a linear controller would do the job, even for a class of nonlinear systems of the form (1.1).

Theorem 2.3: Under Assumptions 2.1 and 2.2, a *linear* state feedback controller can be explicitly constructed to semi-globally asymptotically stabilize the nonlinear system (1.1).

Proof. The proof is based on a Lyapunov method, a series of subtle constructions of the level sets, and the domination design philosophy proposed in [17]. For the convenience of the reader, we break down the proof into several steps.

Step 1. Without the loss of generality, we assume the prescribed upper bound $M > 1$. Then, choose the Lyapunov function $V_1 = \frac{x_1^2}{2}$. By Assumption 2.2 and Lemmas 1.1 and 1.3, the time derivative V_1 along the trajectories of system (1.1) can be estimated as

$$\begin{aligned}
\dot{V}_1 &\leq x_1[x_2^{*p_1} + (x_2^{p_1} - x_2^{*p_1})] + x_1^{p_1+1}\rho_1(x_1) \\
&\leq x_1 x_2^{*p_1} + \frac{1}{2}|x_1||x_2^{*p_1}| + |x_1||x_2 - x_2^*|^{p_1} \\
&\quad + x_1^{p_1+1}\rho_1(x_1) \\
&\leq x_1 x_2^{*p_1} + \frac{1}{2}|x_1||x_2^{*p_1}| + K|x_2 - x_2^*|^{p_1+1} \\
&\quad + x_1^{p_1+1}[\rho_1(x_1) + 1],
\end{aligned}$$

where x_2^* is a virtual controller to be determined later and $K > 0$ is a generic constant.

Associated with the Lyapunov function $V_1(x_1)$, one can construct the level set

$$\Omega_1 = \Big\{x \in \mathbb{R}^n \Big| V_1(x_1) \leq N\Big\},$$

where

$$N = \frac{M^2}{2} + \frac{(2M)^{p_1 - p_2 + 2}}{p_1 - p_2 + 2} + \cdots + \frac{(nM)^{p_1 - p_n + 2}}{p_1 - p_n + 2}.$$

As we shall see, such a subtle choice of N is crucial in simplifying the synthesis and analysis of SGAS and SGPS throughout the paper.

By construction, it is easy to show that

$$x \in B_M \Rightarrow |x_1| \leq M \Rightarrow V_1(x_1) \leq N.$$

Hence,

$$B_M \subset \Omega_1.$$

Furthermore, the continuous function $\rho_1(x_1)$ is bounded on Ω_1 by a constant depending on M.

Keeping this in mind, we choose the *linear* virtual controller $x_2^* = -\beta_1 x_1$ with a known constant $\beta_1 \geq 1$, such that

$$\dot{V}_1\Big|_{\Omega_1} \leq -(n+1)x_1^{p_1+1} + K|x_2 - x_2^*|^{p_1+1}. \tag{2.2}$$

Step 2. Now, define

$$\xi_2 = \frac{x_2 - x_2^*}{\beta_1} = \frac{x_2}{\beta_1} + x_1 \Leftrightarrow x_2 = \beta_1(\xi_2 - x_1) \tag{2.3}$$

and choose the Lyapunov function

$$V_2 = V_1 + \frac{\xi_2^{p_1 - p_2 + 2}}{p_1 - p_2 + 2}.$$

The associated level set is defined as

$$\Omega_2 = \Big\{ x \in \mathbb{R}^n \Big| V_2(x_1, x_2) \le N \Big\}.$$

Owing to $\beta_1 \ge 1$ and (2.3), the following implications hold.

$$\begin{aligned} x \in B_M \quad &\Rightarrow \quad |x_1| \le M,\ |\xi_2| \le 2M \\ \Rightarrow \quad V_2(x_1, x_2) \le N \quad &\Rightarrow \quad V_1(x_1) \le V_2(x_1, x_2) \le N. \end{aligned}$$

In other words,

$$B_M \subset \Omega_2 \subset \Omega_1.$$

(see the details in Fig. 1)

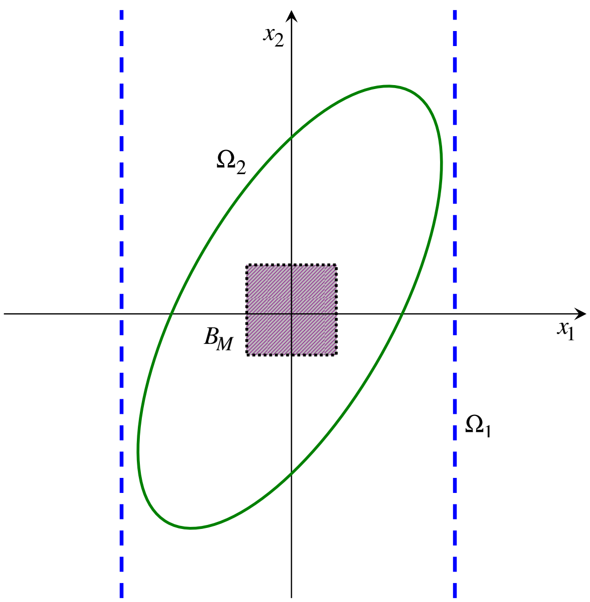

Fig. 1 The level set on (x_1, x_2)-space: SGAS case

As a consequence,

$$\begin{aligned} \dot V_2\Big|_{\Omega_2} \quad \le \quad & -n x_1^{p_1+1} + K\xi_2^{p_1+1} \\ & + \xi_2^{p_1-p_2+1}\Big[\frac{1}{\beta_1}(x_3^{p_2} + f_2(\cdot)) + (x_2^{p_1} + f_1(\cdot))\Big] \\ \le \quad & -(n+1)x_1^{p_1+1} + K\xi_2^{p_1+1} \\ & + \frac{\xi_2^{p_1-p_2+1}}{\beta_1}[x_3^{*p_2} + (x_3^{p_2} - x_3^{*p_2})] \\ & + \xi_2^{p_1-p_2+1}[\frac{f_2(\cdot)}{\beta_1} + (x_2^{p_1} + f_1(\cdot))]. \end{aligned}$$

where x_3^* is a virtual controller to be determined later.

Using Assumptions 2.1-2.2 and Lemmas 1.1-1.2, we have

$$\begin{aligned} & |\xi_2^{p_1-p_2+1}||\frac{f_2(\cdot)}{\beta_1} + x_2^{p_1} + f_1(\cdot)| \\ \le \quad & |\xi_2^{p_1-p_2+1}|(|f_2(\cdot)| + |\beta_1(\xi_2 - x_1)|^{p_1} + |f_1(\cdot)|) \\ \le \quad & |\xi_2^{p_1-p_2+1}|(|x_1|^{p_2} + |\xi_2|^{p_2})\tilde\rho_2(x_1, x_2) \\ \le \quad & \frac{1}{2}x_1^{p_1+1} + \xi_2^{p_1+1}\bar\rho_2(x_1, x_2) \end{aligned}$$

where $\tilde\rho_2(x_1, x_2) \ge 0$ and $\bar\rho_2(x_1, x_2) > 0$ are continuous functions.

Since the continuous function $\bar\rho_2(x_1, x_2)$ is bounded on the level set Ω_2, an estimation similar to Step 1 can be obtained as follows:

$$\begin{aligned} \dot V_2\Big|_{\Omega_2} \quad \le \quad & -(n-1)x_1^{p_1+1} + \frac{\xi_2^{p_1-p_2+1}x_3^{*p_2}}{\beta_1} \\ & + \frac{|\xi_2^{p_1-p_2+1}x_3^{*p_2}|}{2\beta_1} + K\xi_2^{p_1+1} + K|x_3 - x_3^*|^{p_1+1}. \end{aligned}$$

Consequently, it is possible to find a *linear* virtual controller $x_3^* = -\beta_2\xi_2$ with a fixed constant $\beta_2 > 1$ such that

$$\dot V_2\Big|_{\Omega_2} \le -(n-1)[x_1^{p_1+1} + \xi_2^{p_1+1}] + K|x_3 - x_3^*|^{p_1+1}.$$

Step n. By induction, one can prove that a similar conclusion holds at the n-th step. As a matter of fact, we can recursively construct a set of

(1) linear virtual controllers:

$$x_1^* = 0,\ x_2^* = -\beta_1\xi_1, \cdots,\ x_n^* = -\beta_{n-1}\xi_{n-1}$$

with fixed gains $\beta_i \ge 1$ $(i = 1, \cdots, n)$;

(2) auxiliary variables:

$$\begin{aligned} \xi_1 \quad &= \quad x_1 - x_1^* = x_1, \quad \xi_2 = \frac{x_2 - x_2^*}{\beta_1} = \frac{x_2}{\beta_1} + x_1, \\ \cdots,\ \xi_n \quad &= \quad \frac{x_n - x_n^*}{\beta_{n-1}} = \frac{x_n}{\beta_{n-1}} + \cdots + \frac{x_2}{\beta_1} + x_1; \end{aligned}$$

(3) Lyapunov functions:

$$\begin{aligned} V_1 \quad &= \quad \frac{\xi_1^2}{2}, \quad V_2 = V_1 + \frac{\xi_2^{p_1-p_2+2}}{p_1 - p_2 + 2}, \\ \cdots,\ V_n \quad &= \quad V_{n-1} + \frac{\xi_n^{p_1-p_n+2}}{p_1 - p_n + 2}, \qquad (p_n = 1) \end{aligned}$$

and the level sets

$$\Omega_i \triangleq \Big\{ x \in \mathbb{R}^n \Big| V_i(x_1, \cdots, x_i) \le N \Big\}$$

satisfying

$$B_M \subset \Omega_n \subset \cdots \subset \Omega_2 \subset \Omega_1;$$

(4) and a *linear* state feedback control law

$$u = -\beta_n\xi_n, \tag{2.4}$$

such that the closed-loop system (1.1)-(2.4) satisfies

$$\dot V_n\Big|_{\Omega_n} \le -\xi_1^{p_1+1} - \cdots - \xi_n^{p_1+1}.$$

This, in turn, implies that system (1.1) is SGAS by linear state feedback. ∎

Remark 2.4: When $p_1 = p_2 = \cdots = p_{n-1} = 1$, the uncertain system (1.1) is of a strict feedback form. In this case, Theorem 2.3 reduces to the SGAS result in [21]. However, the proof presented above is much simpler than the one in [21], thanks to the use of the simple *quadratic* Lyapunov functions and the level sets $\Omega_1, \cdots, \Omega_n$ thus constructed.

III. SGPS BY LINEAR STATE FEEDBACK

In this section, we turn our attention to the question of when the nonlinear system (1.1), without Assumptions 2.1-2.2, can still be controlled by *linear* state feedback. It has been known that in the absence of Assumptions 2.1-2.2, system (1.1) may exhibit inherent nonlinearities in the sense that the first approximation of (1.1) may contain uncontrollable modes associated with eigenvalues on the right-half plane. Such nonlinearities prevent the nonlinear system (1.1) from being stabilizable by any *linear or smooth* state feedback, even locally [3].

Although a linear controller cannot stabilize the nonlinear system (1.1) that fails to satisfy Assumptions 2.1-2.2, the main result of this section is to point out that if a less aggressive control objective – SGPS instead of GAS and/or SGAS – is pursued, linear controllers still do the job.

Due to the limited space, in what follows we will discuss only an illustrating case, that is, the planar system

$$\begin{aligned} \dot{x}_1 &= x_2^p + f_1(t,x,u) \\ \dot{x}_2 &= u \end{aligned} \tag{3.1}$$

with p being a positive odd integer and $f_1(\cdot)$ satisfying the following assumption:

Assumption 3.1: There exists a smooth function $\bar{f}_1(x_1) \geq 0$, with $\bar{f}_1(0)=0$, such that

$$|f_1(t,x,u)| \leq \frac{1}{2}|x_2|^p + \bar{f}_i(x_1),$$

where $x_2 := u$.

Theorem 3.2: Under Assumption 3.1, SGPS of the nonlinear system (1.1) is achievable by linear state feedback.

Proof. The proof also relies on the Lyapunov argument and the recursive design of a SGPS linear controller. The major difference between the proof of Theorem 3.2 and that of Theorem 2.3 lies on the subtle construction of the level sets of the control Lyapunov functions in order to guarantee semi-global practical stabilizability. Nevertheless, due to the need of overcoming the singularity at the origin, the proof of SGPS result is more involved and subtle than its SGAS counterpart.

For simplicity, it is assumed that the bounds $M>1$ and $0<\varepsilon<1$.

Step 1. Choose the quadratic Lyapunov function $V_1 = \frac{x_1^2}{2}$ as well as the associated level set

$$\Omega_1 \triangleq \left\{(x_1,x_2)\in \mathbb{R}^2 \middle| V_1(x_1) \leq N\right\}, N = \frac{M^2}{2} + \frac{(2M)^2}{2}.$$

From Assumption 3.1, it follows that

$$\begin{aligned} \dot{V}_1 \;\leq\; & x_1 x_2^{*p} + x_1(x_2^p - x_2^{*p}) \\ & + \frac{1}{2}|x_1||x_2^*|^p + \frac{1}{2}|x_1|\Big||x_2|^p - |x_2^*|^p\Big| + x_1\bar{f}_1(x_1), \end{aligned}$$

where x_2^* is a virtual controller to be designed.

Note that $\bar{f}_1(0)=0$ and $\bar{f}_1(x_1)$ is a smooth function. Then, there exists a generic constant $K>0$ such that on the level set Ω_1,

$$|\bar{f}_1(x_1)| \leq K|x_1|.$$

Similar to the proof of Theorem 2.3, design $x_2^* = -\beta_1 x_1$ with $\beta_1 \geq 1$ being a constant to be assigned later and define

$$\xi_2 = \frac{x_2 - x_2^*}{\beta_1} = \frac{x_2}{\beta_1} + x_1 \Leftrightarrow x_2 = \beta_1(\xi_2 - x_1).$$

Then,

$$\dot{V}_1\Big|_{\Omega_1} \leq -\frac{1}{2}\beta_1^p x_1^{p+1} + x_1^{p+1} + K\beta_1^p \xi_2^{p+1} + Kx_1^2. \tag{3.2}$$

To handle the lower-degree terms such as Kx_1^2, we employ Lemma 1.1 to arrive at the following estimate:

$$Kx_1^2 \leq K\tau^{-\frac{p-1}{2}}x_1^{p+1} + \frac{1}{2}\tau, \quad \forall \tau \in (0,1).$$

Clearly, one can choose $\beta_1 = -C\tau^{-\frac{p-1}{2p}} \geq 1$ with a sufficiently large constant $C>0$, such that

$$\dot{V}_1\Big|_{\Omega_1} \leq -2\tau^{-\frac{p-1}{2}}x_1^{p+1} + K\beta_1^p\xi_2^{p+1} + \frac{1}{2}\tau.$$

Step 2. Choose the quadratic Lyapunov function

$$V_2 = V_1 + \frac{\xi_2^2}{2}$$

and define the associated level set

$$\Omega_2 = \left\{(x_1,x_2)\middle| V_2(x_1,x_2) \leq N\right\}.$$

Similar to the proof of Theorem 2.3, it can be shown that

$$B_M \subset \Omega_2 \subset \Omega_1, \qquad \forall \tau.$$

A straightforward calculation gives

$$\begin{aligned} \dot{V}_2\Big|_{\Omega_2} \;\leq\; & -2\tau^{-\frac{p-1}{2}}x_1^{p+1} + K\beta_1^p\xi_2^{p+1} + \frac{1}{2}\tau \\ & + \xi_2\Big[\frac{u}{\beta_1} + x_2^p + f_1(\cdot)\Big] \\ \leq\; & -2\tau^{-\frac{p-1}{2}}x_1^{p+1} + K\beta_1^p\xi_2^{p+1} + \frac{1}{2}\tau \\ & + \frac{\xi_2 u}{\beta_1} + |\xi_2|(\frac{3}{2}|x_2|^p + \bar{f}_1(x_1)). \end{aligned}$$

Using Lemma 1.1 and the fact that $\bar{f}_1(0)=0$, it is deduced that on Ω,

$$\begin{aligned} |\xi_2\bar{f}_1(x_1)|\Big|_{\Omega_2} &\leq K|\xi_2 x_1| \leq \tau^{-\frac{p-1}{2}}x_1^{p+1} + \frac{1}{2}\tau + K\xi_2^2, \\ \xi_2^{p+1}\Big|_{\Omega_2} &\leq K\xi_2^2, \end{aligned}$$

where $K>0$ is a fixed constant independent of τ. Therefore,

$$\begin{aligned} \dot{V}_2\Big|_{\Omega_2} \;\leq\; & -\tau^{-\frac{p-1}{2}}x_1^{p+1} + K\beta_1^p\xi_2^{p+1} + \tau + \frac{\xi_2 u}{\beta_1} \\ \leq\; & -\tau^{-\frac{p-1}{2}}x_1^{p+1} + K\tau^{-\frac{p-1}{2}}\xi_2^2 + \tau + \frac{\xi_2 u}{\beta_1}. \end{aligned}$$

Obviously, the *linear* controller

$$u = -\beta_2\beta_1\xi_2 = -C\tau^{-\frac{p-1}{2}-\frac{p-1}{2p}}\xi_2 = -\beta_2 x_2 - \beta_2\beta_1 x_1$$

with $\beta_2 = C\tau^{-\frac{p-1}{2}}$, renders

$$\dot{V}_2\Big|_{\Omega_2} \le -\tau^{-\frac{p-1}{2}}x_1^{p+1} - K\xi_2^2 + \tau.$$

On the other hand, it follows from Lemma 1.1 that

$$-\tau^{-\frac{p-1}{2}}x_1^{p+1} \le -Kx_1^2 + \tau.$$

Consequently,

$$\dot{V}_2\Big|_{\Omega_2} \le -Kx_1^2 - K\xi_2^2 + 2\tau = -2KV_2 + 2\tau,$$

which implies that if $V_2(x_1, x_2) > \frac{\tau}{2K} \triangleq K_0\tau$ and $(x_1, x_2) \subset \Omega_2$, $\dot{V}_2$ is negative definite.

For the convenience, define the level set

$$\Omega_0 = \Big\{(x_1, x_2)\Big|V_2(x_1, x_2) \le K_0\tau\Big\},$$

which shrinks to the origin as $\tau \searrow 0$, because

$$\begin{aligned} V_2 \le K_0\tau \quad &\Rightarrow \quad |x_1| \le K\tau^{\frac{1}{2}} \searrow 0, \ |\xi_2| \le K\tau^{\frac{1}{2}} \searrow 0 \\ &\Rightarrow \quad |x_2| \le |\beta_1|(|\xi_2| + |\xi_1|) \le K\tau^{\frac{1}{2p}} \searrow 0, \end{aligned}$$

as $\tau \searrow 0$. Hence, there exists a sufficiently small $\tau \in (0, 1)$ such that $\Omega_0 \subset B_\varepsilon$ and

$$\dot{V}_2\Big|_{\Omega_2 - \Omega_0} < 0, \quad B_M \subset \Omega_2.$$

This implies the SGPS property of the closed-loop system (see Fig. 2 for detailed illustrations). ∎

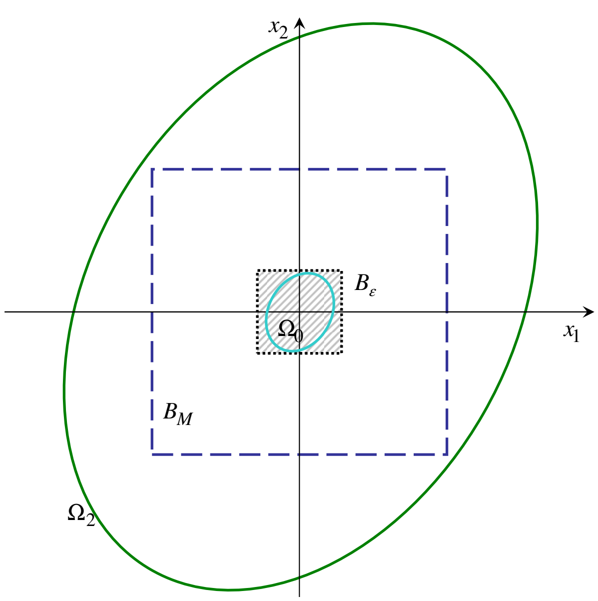

Fig. 2 The level set on x-space: SGPS case

The proof in the higher-dimensional case can be carried out, conceptually, in a similar manner, but technically, is much more complicated and hence is omitted here.

IV. Discussions and Examples

For the purpose of illustration, we present in this section several examples to demonstrate how the linear state feedback control schemes developed so far can be employed to achieve SGAS or SGPS for highly nonlinear systems with uncontrollable unstable linearization.

Example 4.1: Consider the planar non-triangular system

$$\begin{aligned} \dot{x}_1 &= x_2^p + x_2^{p-1}\phi_{p-1}(x_1) + \cdots + x_2\phi_1(x_1) + \phi_0(x_1) \\ \dot{x}_2 &= u, \end{aligned} \qquad (4.1)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $u \in \mathbb{R}$ are the system state and input, respectively, $p \ge 1$ is an odd integer and the mappings $\phi_i : \mathbb{R} \to \mathbb{R}$, $i = 0, \cdots, p-1$ are C^1 with $\phi_i(0) = 0$.

Notably, (4.1) is known to be a normal form of the affine system $\dot{\xi} = f(\xi) + g(\xi)u$ in the plane [14]. It is feedback equivalent to the planar affine system when $\text{rank}[g(0), \text{ad}_f^p g(0)] = 2$. A more general characterization in [4], [19] shows that system (4.1) is indeed a special case of the so-called Hessenberg normal form [4], [19].

A fascinating feature of the planar system (4.1) is that it is usually not smoothly stabilizable when $p > 1$, because its linearization may have uncontrollable modes associated with eigenvalues on the right-half plane; for instance, see the simple example

$$\begin{aligned} \dot{x}_1 &= x_2^3 + x_2 x_1 + x_1 \\ \dot{x}_2 &= u. \end{aligned} \qquad (4.2)$$

As a consequence, there exist no *linear* controllers that stabilize system (4.2), even locally. However, a straightforward calculation shows that the planar system (4.1) satisfies Assumption 3.1. By Theorem 3.2, a linear controller can be designed to achieve semi-global practical stability. Simulations of system (4.2) with $u = -25x_2 - 100x_1$ and $x(0) = (3, -2)^T$ have been conducted, illustrating the effectiveness of the linear controller. The simulation figures are omitted due to the limit of spaces.

Example 4.2: Consider the uncertain planar system with uncontrollable linearization

$$\begin{aligned} \dot{x}_1 &= x_2^3 + x_1^3 d(t) \\ \dot{x}_2 &= u + x_2, \end{aligned} \qquad (4.3)$$

where $d(t)$ is a C^0 time-varying parameter with $|d(t)| \le 1$.

Clearly, Example 4.2 satisfies Assumptions 2.1 and 2.2. If the control objective is global asymptotic stabilization, a smooth *nonlinear controller* must be used according to the work [17]. By comparison, Theorem 2.3 indicates that under the same conditions, a linear controller exists, achieving SGAS for the uncertain system (4.3). In fact, Simulations of system (4.3) with $u = -50x_2 - 250x_1$, $x(0) = (-5, 5)^T$ and $d(t) = \sin t$ demonstrate the performance of the linear controller. The details are omitted for the reason of space.

Example 4.3: Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^{\frac{1}{3}} \\ \dot{x}_2 &= u, \end{aligned} \tag{4.4}$$

which has been shown to be not stabilizable, even *locally*, by any *continuous* state feedback [7], due to the presence of the nonlinearity $x_1^{\frac{1}{3}}$. In other words, all the existing continuous control design methods including the one proposed [18] are invalid and inapplicable to system (4.4). However, if one does not insist to pursue GAS or SGAS but instead, to make a trade-off by seeking a less aggressive control aim such as SGPS, the *linear* state feedback scheme proposed in section 3 works and can be used to control the *non-continuously stabilizable* system (4.4). In fact, SGPS can be achieved by linear state feedback. The proof of this conclusion is very close to that of Theorem 3.2 and omitted for the sake of spaces. The efficiency of a linear controller can be seen from the simulations in Fig. 3.

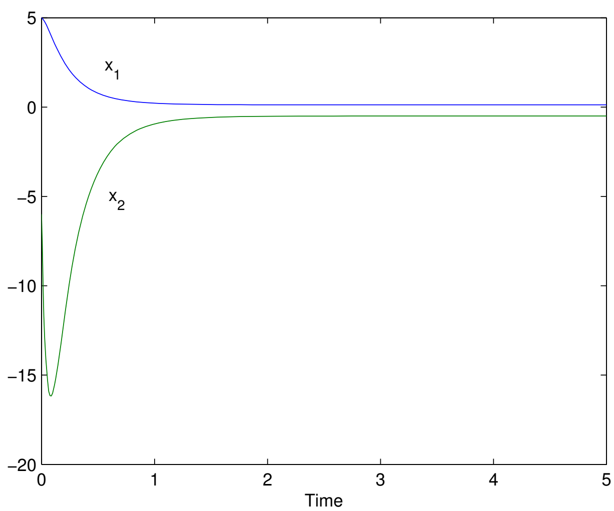

Fig. 3 Simulations of system (4.4) with $u = -25x_2 - 100x_1$ and $x(0) = (5, -6)^T$.

V. Conclusions

In this paper, we have provided some answers to the important questions such as, to what extent, a linear controller would be good enough for the control of nonlinear systems. This was done by presenting two *linear* robust state feedback control schemes, which achieve *SGAS* and *SGPS* for a family of inherently nonlinear systems (1.1), respectively. The main results of the paper are Theorems 2.3 and 3.2, whose proofs are constructive and carried out by designing recursively, a set of linear state feedback stabilizers, control Lyapunov functions and the associated level sets. Because linear controllers are much easier to be implemented than the existing nonlinear controllers [17], [18], the linear feedback design methods proposed in this work have offered an valuable alternative for the control of uncertain systems with higher-order nonlinearity. The effectiveness of our robust linear controllers was illustrated by several examples — some of them such as Example 4.3 are very difficult to be controlled and cannot be dealt with, even locally, by any *continuous* feedback. As a trade-off, the gains of our linear controller should be large enough to overcome the nonlinearity of system.

Finally, it is worth pointing out that under suitable conditions, the main results of this paper can be extended, without many efforts, to a family of uncertain cascade systems involving zero-dynamics. Such generalizations will be presented in a full version of the paper.

References

[1] A. Bacciotti, *Local stabilizability of nonlinear control systems*, World Scientific, 1992.

[2] A. Bacciotti, Linear feedback: the local and potentially global stabilization of cascade systems, *Proc. of the 2nd IFAC Symp. on Nonlinear Control Systems*, Bordeaux, pp. 21-25 (1992).

[3] R.W. Brockett, Asymptotic stability and feedback stabilization, in *Differential Geometric Control Theory*, R.W. Brockett, R.S. Millman and H.J. Sussmann, Eds., Birkäuser, Basel-Boston, 181-191, 1983.

[4] D. Cheng and W. Lin, On p-normal forms of nonlinear systems, *IEEE Trans. Automat. Contr.*, Vol. 48, 1242-1248, 2003.

[5] S. Celikovsky and E. Aranda-Bricaire, Constructive nonsmooth stabilization of triangular systems, *Syst. Contr. Lett.*, Vol. 36, 21-37, 1999.

[6] J. M. Coron and L. Praly, Adding an integrator for the stabilization problem, *Syst. Contr. Lett.*, Vol. 17, 89-104, 1991.

[7] J. M. Coron and L. Rosier, A relation between continuous time-varying and discontinuous feedback stabilization, *J. of Math. Systems, Estimation, and Control*, Vol. 4, 67-84, 1994.

[8] P. E. Crouch and M. Irving, Sufficient conditions for local asymptotic stability of nonlinear systems whose linearization is uncontrollable, *Control Theory Centre Report*, Vol. 114, Univ. Warwick, 1983.

[9] W. P. Dayawansa, Recent advances in the stabilization problem for low dimensional systems, *Proc. of the 2nd IFAC Symp. on Nonlinear Control Systems*, Bordeaux, pp. 1-8 (1992).

[10] W. P. Dayawansa, C. F. Martin and G. Knowles, Asymptotic stabilization of a class of smooth two dimensional systems, *SIAM. J. Optimiz. Contr.*, Vol. 28, 1321-1349, 1990.

[11] H. Hermes, Homogeneous coordinates and continuous asymptotically stabilizing feedback controls, in: S. Elaydi (Ed.), *Differential Equations Stability and Control*, Vol. 109, Marcel Dekker, 249-260, 1991.

[12] H. Hermes, Nilpotent and high-order approximations of vector field systems, *SIAM. Review,* Vol. 33, 238-264, 1991.

[13] A. Isidori, *Nonlinear Control Systems*, Springer-Verlag, 1995.

[14] B. Jakubczyk and W. Respondek, Feedback equivalence of planar systems and stability, Robust Control of Linear Systems and Nonlinear Control, Kaashoek M.A. et al. eds., van Schuppen J.H., Ran A.C.M., Birkhäuser, 447-456, 1990.

[15] M. Kawski, Stabilization of nonlinear systems in the plane, *Syst. Contr. Lett.*, Vol. 12, 169-175, 1989.

[16] M. Kawski, Homogeneous stabilizing feedback laws, *Contr. Theory & Advanced Tech.,* Vol. 6, 497-516, 1990.

[17] W. Lin and C. Qian, Adding one power integrator: a tool for global stabilization of high-order lower-triangular systems, *Syst. Contr. Lett.*, Vol. 39, 339-351, 2000.

[18] C. Qian and W. Lin, A continuous feedback approach to global stabilization of nonlinear systems, *IEEE Trans. Automat. Contr.*, Vol. 46, 1061-1079, 2001.

[19] W. Respondek, Transforming a single-input system to a p-normal form via feedback, *Proc. 42nd IEEE CDC*, Maui, 1574-1579, 2003.

[20] L. Rosier, Homogeneous Lyapunov functions for homogeneous continuous vector field, *Syst. Contr. Lett.*, Vol. 19, 467-473, 1992.

[21] A. Teel and L. Praly, Tools for semiglobal stabilization by partial state and output feedback, *SIAM. J. Contr. Optimiz.*, Vol. 33, 1443-1488, 1995.

[22] M. Tzamtzi and J. Tsinias, Explicit formulas of feedback stabilizers for a class of triangular systems with uncontrollable linearization, *Syst. Contr. Lett.*, Vol. 38, 115-126, 1999.