Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

The Method of Monotone Structural Evolution for Dynamic Optimization of Switched Systems

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Abstract— **The paper presents application of the Monotone Structural Evolution (MSE) to dynamic optimization of switched systems. The MSE is a direct computational method of optimal control which automatically identifies the optimal structure. Its distinctive feature is that the decision space undergoes gradual evolution, driven by the discrepancy from the Maximum Principle conditions. The approach is illustrated with two examples: the benchmark fishing problem, and a three-mode problem with right-hand sides linear in the state.**

I. INTRODUCTION

Optimization problems for switched systems are widely studied in the literature (see, e.g., [2], [10], [19]). This paper is confined to autonomous systems with continuous state trajectories and switchings independent of state. The choice of modes, their number and sequence as well as switching times are decision variables. Such problems can be approached directly, by nonlinear programming (NLP) of mixed type. They can also be formulated as optimal control problems with constraints of special type, to which the Maximum Principle is applied [1]. We follow the latter approach which gives a possibility to study the limits of optimizing sequences by means of singular control theory applied to relaxed problems [9], and to employ computational optimal control methods for approximation of solutions.

Computational methods of optimal control are divided into *direct* and *indirect* [3]. In the indirect approach, the boundary value problem of the Maximum Principle is solved, usually by multiple shooting. The rate of convergence is high, but the area of convergence is relatively small. Practically, the optimal control structure has to be determined beforehand. In the direct methods, approximating finite-dimensional optimization problems are solved by NLP algorithms. The direct *simultaneous* approach discretizes both controls and state trajectories, and frequently leads to large-scale computations. In the *sequential* approach only the controls are discretized and the states are computed by numerical integration. The direct methods feature large areas of convergence but in general, they are rather slow. A special class of sequential methods uses switching times and, possibly, the end points of singular arcs as the NLP decision variables. The derivatives of cost for gradient optimization are obtained from adjoint solutions [8], [11] – [14] or by variational and difference techniques [7], [17]. Such parameterization usually results in a low-dimensional decision space and good convergence, but the optimal control structure has to be known to start the NLP computations.

Our main goal is to show that the method of *monotone structural evolution* (MSE) [6], [8], [12] – [16] is an effective tool for optimization of switched systems. The MSE is a direct sequential method for optimal control computations which detects the optimal control structure in an automatic way, within the NLP algorithm. Its distinctive feature is that the decision space undergoes gradual evolution in the course of optimization, with changing the control parameterization and the number of decision variables. Such structural changes, which locally increase efficacy of gradient optimization, are followed by periods of gradient optimization in a constant decision space. The changes are driven by current discrepancy from the Maximum Principle conditions. The control is preserved by every structural change so that monotone decrease of the cost is achieved in the whole algorithm. A basic idea behind the application of the MSE to switched systems is that of spike generations, with the *rule of minimum positive efficiency* which prevents the algorithm from converging to chattering modes if the optimal solution has a singular arc. A concept similar to that of spike generation (*mode insertions*) can be found in [4].

We begin with the statement of the optimization problem. Next, we explain the basics of the MSE and present the general algorithm, together with the spike generations. Two examples are then solved. The first is the well-known fishing problem [9], and the second is a generalization of the problem discussed in [4] to arbitrary three-mode systems with the right-hand sides linear in the state.

II. PROBLEM STATEMENT AND OPTIMALITY CONDITIONS

Let $q : \mathbb{R}^n \to \mathbb{R}$ and $f_i : \mathbb{R}^n \to \mathbb{R}^n,\ i = 1, ..., M$ be given functions of class C^1, and T, a positive real. The problem is to find a positive integer N, a strictly increasing sequence of reals $\tau_i \in [0, T],\ i = 0, ..., N,\ \tau_0 = 0,\ \tau_N = T$ and a function $\{1, ..., N\} \ni i \mapsto k_i \in \{1, ..., M\},\ k_i \neq k_{i-1},\ i = 2, ..., N$ which minimize a cost functional

$$Q(N, \tau, k) = q(x(T))$$

on the trajectories of a switched system

$$\dot{x}(t) = f_{k_i}(x(t)),\ \tau_{i-1} \leq t < \tau_i,\ i = 1, ..., N\,,$$

$x(0) = x_0$. The initial state x_0 is fixed. This formulation also covers integral costs, considered in further sections, which are transformed by introducing an additional state variable.

This work was not supported by any organization

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978-1-4244-3124-3/08/$25.00 ©2008 IEEE

It is standard to restate this hybrid problem as a control problem: minimize the cost

$$S(u) = q(x(T)) \tag{1}$$

subject to

$$\dot{x}(t) = f(x(t))\,u(t), \quad t \in [0,T], \quad x(0) = x_0 \tag{2}$$

where $f = [f_1\ ...\ f_M]$, the control u is piecewise continuous, $u \in PC(0,T;D_o)$,

$$D_o = \left\{ v \in \mathbb{R}^M : v_i \in \{0,1\}, i = 1,...,M, \sum\nolimits_{i=1}^{M} v_i = 1 \right\}.$$

This control problem will be called original (OCP). We also consider a relaxed problem (RCP) in which the control takes values in the set

$$D_r = \left\{ v \in \mathbb{R}^M : v_i \geq 0,\ i = 1,...,M,\ \sum\nolimits_{i=1}^{M} v_i = 1 \right\}. \tag{3}$$

There are obvious relationships between optimal solutions of the OCP and RCP. First, $\inf\{S(u) : u \in PC(0,T;D_o)\} = \inf\{S(u) : u \in PC(0,T;D_r)\}$. Secondly, suppose that u is an optimal control of the RCP. If u has only boundary arcs, then it is also optimal in the OCP. If u has interior arcs, then the OCP has an optimal sliding regime $\hat{u}$ whose chattering arcs coincide with the interior arcs of u, and besides, $\hat{u}$ is identical with u. Moreover, u and $\hat{u}$ generate identical state trajectories.

Define the hamiltonian for both control problems

$$H(\psi(t), x(t), u(t)) = \psi(t)^\top f(x(t))u(t).$$

The adjoint function ψ is a solution of the terminal value problem

$$\begin{aligned} \dot{\psi}(t) &= -\nabla_x H(\psi(t), x(t), u(t)), \quad t \in [0,T] \\ \psi(T) &= -\nabla q(x(T)). \end{aligned} \tag{4}$$

The Maximum Principle says that if u is optimal, then for every $t \in [0,T]$

$$H(\psi(t), x(t), u(t)) \geq H(\psi(t), x(t), v) \quad \forall v \in D$$

where $D = D_o$ for the OCP and $D = D_r$ for the RCP. Let $\phi(x,\psi) = f(x)^\top \psi$ be the switching function and

$$K_t = \{i : \phi_i|_t = \max\{\phi_j|_t,\ j = 1,...,M\}\}.$$

Then, for every optimal control u

$$\sum\nolimits_{i \in K_t} u_i(t) = 1\,.$$

In particular, if $\phi_i|_t > \phi_j|_t$ for every $j \in \{1,...,M\}\backslash\{i\}$, then $u_i(t) = 1$.

The ith optimal control component is *singular* on a time interval σ, if $\#K_t > 1$ and $i \in K_t$ for every $t \in \sigma$. To characterize singular controls one can use a set of equations valid for every $t \in \sigma$

$$\phi_i(x(t),\psi(t)) = \phi_j(x(t),\psi(t)), \quad i,j \in K_t, \quad i \neq j \tag{5}$$

together with equations produced from (5) by subsequent differentiations w.r.t. t, with substitutions of the right-hand sides of (2) and (4) for $\dot{x}(t)$ and $\dot{\psi}(t)$. In typical cases the resulting set can be solved w.r.t. the singular controls and so they can be expressed in terms of $x(t)$ and $\psi(t)$. Sometimes it is possible to eliminate $\psi(t)$ and obtain the singular controls in a state feedback form, which is particularly desirable for analytical and computational purposes.

The control problems can be stated in a reduced form. For every admissible control we have $u_M = 1 - \sum_{i=1}^{M-1} u_i$ and the state equation can be written as $\dot{x} = f(x)\,u + f_M(x)$ where $u = \text{col}(u_1,...,u_{M-1})$, $f = [f_1 - f_M\ ...\ f_{M-1} - f_M]$. The hamiltonian reads $H(\psi,x,u) = \psi^\top f(x)u + \psi^\top f_M(x)$. In the reduced relaxed problem (RRCP), the control takes values in

$$D_r = \left\{ v \in \mathbb{R}^{M-1} : v_i \geq 0,\ \ i = 1,...,M-1,\ \sum\nolimits_{i=1}^{M-1} v_i \leq 1 \right\}.$$

III. MONOTONE STRUCTURAL EVOLUTION

Let us recall a few basic ideas of the MSE which will be used in this work [16]. The *control structure* is a sequence of procedures P_i, $i = 1,...,N$ which determine the control u in successive time intervals $[\tau_{i-1},\tau_i[$, $u(t) = P_i(x(t),t,p_i)$ where x is the state trajectory generated by u and p_i is a vector of parameters. The points $\tau_0,\tau_1,...,\tau_N$ are called *structural nodes*, $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_N = T$. The restrictions of controls to intervals $[\tau_{i-1},\tau_i[$ are called *arcs*. The procedures (taken from a fixed, finite set $\mathbf{P}$), their number, order and parameters, as well as the nodes $\tau_1,...,\tau_{N-1}$ and, possibly, τ_N are decision variables. The contents of the set $\mathbf{P}$ may be suggested by the optimality conditions of the Maximum Principle, and by general approximation techniques.

In the sequel we only use procedures where control components in every interval $[\tau_{i-1},\tau_i[$ are either constant on a boundary or *candidate singular*, determined from the singularity conditions in a state feedback form. For a given control structure, we can thus write

$$u(t) = P_i(x(t)),\ t \in [\tau_{i-1},\tau_i[,\ i = 1,...,N\,.$$

The internal structural nodes are the only decision variables, $d = \text{col}(\tau_1,...,\tau_{N-1})$. Let

$$g_i(x) = f(x)\,P_i(x),\ i = 1,...,N$$
$$g(x,t) = g_i(x),\ t \in [\tau_{i-1},\tau_i[,\ i = 1,...,N\,.$$

The system equation (2) takes the form

$$\dot{x}(t) = g(x(t),t),\ t \in [0,T].$$

The respective hamiltonian is defined as $\hat{H}(\hat{\psi}(t),x(t),t) = \hat{\psi}(t)^\top g(x(t),t)$ where $\hat{\psi}$ is the solution of the adjoint boundary problem

$$\begin{aligned} \dot{\hat{\psi}}(t) &= -\nabla_x \hat{H}(\hat{\psi}(t),x(t),t), \quad t \in [0,T] \\ \hat{\psi}(T) &= -\nabla q(x(T)). \end{aligned}$$

The adjoint $\hat{\psi}$ is identical with ψ determined by (4) if the control u is purely bang-bang, or its every candidate singular arc is singular.

An optimal control approximation in the direct approach is a value of an *approximation mapping* $A : \mathbf{D}_a \to \mathcal{U}$, from the admissible set $\mathbf{D}_a$ in a finite-dimensional space of decision variables $\mathbf{D}$, $\mathbf{D}_a \subset \mathbf{D}$ into a functional control space $\mathcal{U}$. Once $\mathbf{D}$, $\mathbf{D}_a \subset \mathbf{D}$ and A are chosen, the cost functional may be redefined as a function of the decision vector $\Sigma(d) = S(A(d))$, $d \in \mathbf{D}_a$. In the MSE, the decision space is gradually adapted to the accumulated knowledge on optimal solution in a series of *structural changes*, separated by periods of gradient optimization in a constant space. Assume that Δ is a given family of decision spaces $\mathbf{D}$. Each structural change is determined by a function $(\mathbf{D}, d) \mapsto (\bar{\mathbf{D}}, \bar{d})$ where $d \in \mathbf{D}_a \subset \mathbf{D}$, $\bar{d} \in \bar{\mathbf{D}}_a \subset \bar{\mathbf{D}}$, $\mathbf{D}, \bar{\mathbf{D}} \in \Delta$. Let the approximation mappings A and $\bar{A}$ be assigned to $\mathbf{D}$ and $\bar{\mathbf{D}}$, respectively. It is required that the *condition of control preservation* holds:

$$\bar{A}(\bar{d}) = A(d), \quad d \in \mathbf{D}_a, \quad \bar{d} \in \bar{\mathbf{D}}_a.$$

Thus, the control as an element of $\mathcal{U}$ is not immediately affected, and in consequence, the cost functional monotonously decreases.

To define the *efficiency E* of a structural change, denote $\bar{\Sigma} = S(\bar{A}(\cdot))$. If the antigradient $\gamma = -\nabla\Sigma(d)$ points to int$\mathbf{D}_a$ and $\bar{\gamma} = -\nabla\bar{\Sigma}(\bar{d})$ to int$\bar{\mathbf{D}}_a$, the efficiency is given by

$$E = \|\bar{\gamma}\|^2 - \|\gamma\|^2. \tag{6}$$

In the general case the antigradients are replaced by their orthogonal projections onto the local conical approximations of the admissible sets.

Two kinds of structural changes are typical for the MSE: the set of decision variables increases in *generations*, and is diminished in *reductions*. A *driving generation* takes place when its efficiency E exceeds a given threshold, $E > \varepsilon(\|\gamma\|)$ where $\varepsilon : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function vanishing at 0. Additional rules limit the number of new decision variables and, possibly, impose selected regularity properties on controls, such as continuity or smoothness. The MSE also admits *saturation generations*, enforced by the requirement that at the moment of gradient computation each control component arc has to be either purely boundary or purely interior.

Typical reductions consist of eliminating arcs of zero length when they are not promising, or unification of two adjacent arcs described by identical procedures. To facilitate the detection of optimal control structure, we sometimes admit reductions which slightly violate the condition of control preservation. Very short control arcs can be reduced even if it brings a temporary increase of cost. If a cost improvement does not follow in a prescribed, small number of iterations, the reduction is withdrawn.

IV. BASIC MSE ALGORITHM AND SPIKE GENERATIONS

The basic algorithm of the MSE consists of the following steps.

1^0 Selection of initial decision space $\mathbf{D}$ and starting point $d \in \mathbf{D}_a \subset \mathbf{D}$.
2^0 Termination, if optimality conditions in $\mathcal{U}$ are satisfied.
3^0 Generation, if it is sufficiently promising or needed.
4^0 Iteration of gradient optimization in current decision space $\mathbf{D}$.
5^0 Reduction, if necessary.
6^0 Return to 2^0.

In step 2^0 it is verified if the condition of hamiltonian maximization of the Maximum Principle is fulfilled with sufficient accuracy. This can be also formulated as a condition of existence of appropriately efficient generations. Step 3^0 is distinctive for the MSE algorithms and crucial for their convergence. The algorithm is equipped with procedures for gradient computation and evaluation of efficiency of generations, based on the solutions of the adjoint equations. While the gradients $\nabla\Sigma(d)$ can be computed by other techniques, such as variational equations or discrete approximation, the adjoint trajectories are indispensable for the estimation of accuracy of fulfillment of the Maximum Principle conditions, as well as for choosing generations with satisfactory efficiency.

In this application of the MSE to switched problems we only use *spike generations*, in which the new control arcs are of zero length, and saturation generations. To explain the technique of spike generations, recall that the control structure is defined by a sequence of procedures $(P_1, ..., P_N)$, $P_i \in \mathbf{P}$ for $i = 1, ..., N$. The gradient of the cost is given by [11]

$$\begin{aligned}
&\nabla_{\tau_i}\Sigma(d) = \\
&H(\hat{\psi}(\tau_i), x(\tau_i), P_{i+1}(x(\tau_i))) - H(\hat{\psi}(\tau_i), x(\tau_i), P_i(x(\tau_i))) = \\
&\qquad \phi(x(\tau_i), \hat{\psi}(\tau_i))^\top (P_{i+1}(x(\tau_i)) - P_i(x(\tau_i))) = \\
&\qquad\qquad \hat{\psi}(\tau_i)^\top G_i(x(\tau_i)), \quad i = 1, ..., N-1
\end{aligned} \tag{7}$$

where $G_i = g_{i+1} - g_i$. If $\tau_{i-1} = \tau_i < \tau_{i+1}$, formula (7) determines the right partial derivative, and if $\tau_{i-1} < \tau_i = \tau_{i+1}$, it determines the left partial derivative. The case $\tau_{i-1} = \tau_i = \tau_{i+1}$ is excluded from consideration. Let $H_P(\tau) = H(\hat{\psi}(\tau), x(\tau), P|_\tau)$ for every $P \in \mathbf{P}$ and $\tau \in [0, T]$. Define $\pi(\tau) \in \mathbf{P}$ by the equality

$$H_{\pi(\tau)}(\tau) = \min\{H_P(\tau) : P \in \mathbf{P},\ H_P(\tau) > \hat{H}|_\tau\}. \tag{8}$$

This recipe for selecting control procedures is called the *rule of minimum positive efficiency*. If the set $\{P \in \mathbf{P} : H_P(\tau) > \hat{H}|_\tau\}$ is empty, we put $\pi(\tau) = P_i$ with i such that $\tau \in [\tau_{i-1}, \tau_i[$. Assume $0 = \tau_0 < \tau_1 < ... < \tau_N = T$ and define $e(\tau) = (H_{\pi(\tau)}(\tau) - \hat{H}|_\tau)^2$.

Let us first consider a generation in which only one spike is inserted. We wish it located at a point τ^* where the function e attains a possibly large value, and not at the internal structural nodes τ_i, $0 < i < N$. Additionally, it is required that $e(\tau^*) > \varepsilon_0 ||\gamma||^4$ where γ is the current gradient of Σ and $\varepsilon_0 > 0$, a case dependent constant. A spike generation at τ^* changes the control structure to $(P_1, ..., P_i, \pi(\tau^*), P_i, ..., P_N)$ if $\tau^* \in]\tau_{i-1}, \tau_i[$,

$(\pi(\tau^*), P_1, ..., P_N)$ if $\tau^* = \tau_0$, and $(P_1, ..., P_N, \pi(\tau^*))$ if $\tau^* = \tau_N$. Respectively, the sequence of structural nodes becomes $(\tau_0, ..., \tau_{i-1}, \tau^*, \tau^*, \tau_i, ..., \tau_N)$, $(\tau_0, \tau^*, \tau_1, ..., \tau_N)$, or $(\tau_0, \tau_1, ..., \tau_N, \tau^*)$. Note that $e(\tau^*)$ is the efficiency $E(\tau^*)$ of a spike generation for $\tau^* = 0$ and $\tau^* = T$, and is equal to $\frac{1}{2}E(\tau^*)$ for $\tau^* \neq \tau_i$, $i = 0, ..., N$ (see (6) and (7)). The factor $\frac{1}{2}$ is introduced to give some preference to inserting spikes at 0 and T since the number of decision variables is then increased only by one. It should be observed that for a spike thus generated at any point of positive efficiency, the cost is a strictly decreasing (increasing) function of the right (left) structural node of this spike.

Computational practice shows that the convergence of the algorithm is faster, if several spikes are allowed in one generation. In the numerical implementation of the MSE we use the following rules. Let $\Theta_1 = [\,0, \tau_1[\,$, $\Theta_i =]\,\tau_{i-1},\ \tau_i[\,$, $1 < i < N$ and $\Theta_N =]\,\tau_{N-1}, T]$. Define I as the set of all integers i in $\{1, ..., N\}$ such that $e(\tau_i^*) > \varepsilon_0 ||\gamma||^4$ where τ_i^* is a maximizer of e in Θ_i. Let θ and θ_0 be strictly increasing sequences built of all elements of $\{\tau_i^* : i \in I\}$ and $\{\tau_i^* : i \in I\} \backslash \{0, T\}$, respectively. To obtain the sequence of the new structural nodes $(\bar\tau_0, ..., \bar\tau_{\bar N})$, sort the concatenation of the sequences $(\tau_0, ..., \tau_N)$, θ and θ_0 in a nondecreasing order. The new control structure $(\bar P_1, ..., \bar P_{\bar N})$ which includes all the procedures P_i, $i \in \{1, ..., N\}$ and $\pi(\tau_i^*)$, $i \in I$ is characterized as follows. Let $j \in \{1, ..., \bar N\}$. If $\bar\tau_{j-1} = \bar\tau_j = \tau_i^*$ for some $i \in I$, then $\bar P_j = \pi(\tau_i^*)$. If $\bar\tau_j \neq \tau_k^* \quad \forall k \in I$, there is exactly one i in $\{1, ..., N\}$ such that $\bar\tau_j = \tau_i$. Then $\bar P_j = P_i$, and $\bar P_{j+1} = P_{i+1}$ for $j < \bar N$.

The rule of minimum positive efficiency (8), introduced in [15], [16] is an essential element of the MSE approach. This rule may be suppressed in early stages of optimization, but becomes vital in the final stage to avoid convergence to chattering modes, when singular arcs are expected in the optimal solution.

V. HESSIAN OF COST, AND NEWTON METHOD

We determine the hessian of cost for the non-reduced relaxed problem RCP (1) – (3), assuming that all necessary functions are of class C^2. The hessian is not indispensable for the MSE, but allows the Newton method for optimization which considerably speeds up convergence. It can be also used for verification of sufficient conditions of optimality. We compute the second derivatives of the cost analytically, using formulas adapted from [14]. In a different framework, similar results were independently obtained in [17]. Define for $i = 1, ..., N-1$

$$\pi_i = \hat\psi(\tau_i)^\top \big(\nabla g_{i+1}(x(\tau_i))^\top g_i(x(\tau_i)) - \nabla g_i(x(\tau_i))^\top g_{i+1}(x(\tau_i))\big)$$

$$\rho_i = V_1(\tau_i)^\top \nabla G_i(x(\tau_i))\, \hat\psi(\tau_i) + V_2(\tau_i)^\top G_i(x(\tau_i))$$

$$\sigma_i = V_1(\tau_i)^{-1} G_i(x(\tau_i))$$

where V_1 and V_2 are $n \times n$ matrix functions determined from the terminal value problem

$$\dot V_1(t) = A(t)\, V_1(t), \qquad V_1(T) = I$$
$$\dot V_2(t) = B(t)\, V_1(t) - A(t)^\top\, V_2(t) \quad V_2(T) = -\nabla^2 q(x(T))$$

$$A(t) = \nabla_x^\top g(x(t), t)$$

$$B(t) = -\nabla_x^2 (\hat\psi(t)^\top g(x(t), t)) = -\nabla_x ((\nabla_x g(x(t), t))\, \hat\psi(t)).$$

The hessian is a symmetric $(N-1) \times (N-1)$ matrix $h = \nabla^2 \Sigma(d)$. Its diagonal elements satisfy

$$h_{ii} = \pi_i - \rho_i^\top \sigma_i, \quad i = 1, ..., N-1.$$

The elements above the main diagonal are calculated from

$$h_{ij} = -\rho_j^\top \sigma_i, \quad j = 2, ..., N-1, \quad i = 1, ..., j-1.$$

For gradient optimization, we use the Newton method with analytical hessians, and compare its performance with the BFGS method. In the case when the hessian has some negative eigenvalues, the Newton method performs a curvilinear search along a path $y(\sigma) = \tau + \arg\min\{\, \frac{1}{2} z^\top h\, z + \gamma^\top z : z \in K(\sigma)\}$, $\sigma > 0$ where $\tau = y(0)$ is the current point, γ is the gradient of the cost and $K(\sigma)$ denotes the ball with radius σ, centered at the origin [5].

VI. FISHING PROBLEM

We first illustrate the MSE method with the fishing problem [9] where the state equations take two forms. The corresponding reduced optimal control problem can be written as follows

$$\dot x_1(t) = x_1(t) - x_1(t)x_2(t) - c_1 x_1(t) u(t), \qquad x_1(0) = 0.5$$
$$\dot x_2(t) = -x_2(t) + x_1(t)x_2(t) - c_2 x_2(t) u(t), \quad x_2(0) = 0.7$$

$$S(u) = \tfrac{1}{2} \int_0^T ((x_1(t) - 1)^2 + (x_2(t) - 1)^2)\, \mathrm{d}t$$

$$c_1 = 0.4, \quad c_2 = 0.2, \quad T = 12.$$

The control is subject to a constraint $u(t) \in \{0,\ 1\}$. In the relaxed problem RRCP, $0 \le u(t) \le 1$. It can be shown [9] that the original non-relaxed problem has no optimal solution. The infimum of cost is attained on a 'sliding regime', that is, a control with chattering on some time intervals. To examine this phenomenon, consider now the relaxed version. The hamiltonian is given by

$$H = \psi_1 x_1 (1 - x_2 - c_1 u) + \psi_2 x_2 (-1 + x_1 - c_2 u) - \tfrac{1}{2}(x_1 - 1)^2 - \tfrac{1}{2}(x_2 - 1)^2$$

where the adjoint variables ψ_1 and ψ_2 satisfy

$$\dot\psi_1 = -\psi_1 (1 - x_2 - c_1 u) - \psi_2 x_2 + x_1 - 1, \quad \psi_1(T) = 0$$
$$\dot\psi_2 = \psi_1 x_1 - \psi_2 (-1 + x_1 - c_2 u) + x_2 - 1, \quad \psi_2(T) = 0.$$

Define the switching function $\phi = -c_1 \psi_1 x_1 - c_2 \psi_2 x_2$. The optimal control u maximizes the hamiltonian H, whence $u(t) = \frac{1}{2}(1 + \mathrm{sgn}\phi|_t)$ for $\phi|_t \neq 0$. To characterize the optimal control on a singular interval we use the fact that ϕ and its time derivatives are equal to zero there. The resulting system of equations can be solved yielding an expression for singular control in a state feedback form (see [9])

$$u_s(x) = 5 \frac{8x_1^3 - x_2^3 - x_1 x_2 (6x_1^2 + 3x_1 x_2 - 3x_2^2 - 8x_1 + 13x_2 - 4)}{16x_1^3 + 6x_1^2 x_2 + 2x_1 x_2 + x_2^3}.$$

In candidate singular intervals, we substitute $u_s(x)$ for u in the state equations. In consequence, the hamiltonian takes the form

$$\hat{H} = \hat{\psi}_1 x_1(1-x_2) + \hat{\psi}_2 x_2(-1+x_1) - \tfrac{1}{2}(x_1-1)^2 - \tfrac{1}{2}(x_2-1)^2 + \phi(x,\hat{\psi})u_s(x)$$

and the adjoint equations

$$\dot{\hat{\psi}}_1 = -\hat{\psi}_1(1-x_2-c_1u_s) - \hat{\psi}_2 x_2 + x_1 - 1 - \phi\,\nabla_{x_1}u_s$$
$$\dot{\hat{\psi}}_2 = \hat{\psi}_1 x_1 - \hat{\psi}_2(-1+x_1-c_2u_s) + x_2 - 1 - \phi\,\nabla_{x_2}u_s.$$

The variables $\hat{\psi}_1,\ \hat{\psi}_2$ satisfy unchanged adjoint equations outside these intervals and $\hat{\psi}_1(T) = \hat{\psi}_2(T) = 0$. The differential equations are solved by the Runge-Kutta 3/8 method with moving mesh, consistent with rhs discontinuities. All computations are performed in MATLAB.

In the first numerical experiment only bang-bang controls are admissible. The maximum allowed number of switchings is equal to 20. The history of optimization is illustrated by Fig. 1. The initial approximation is $u \equiv 0$, and after 17 iterations of the Newton method (18 gradient and hessian computations) the solution presented at the bottom is obtained, with the cost value 0.672135. The situations at the moments of the first three spike generations are shown in the upper part of the figure. Note that the switching function ϕ in all figures is normalized (divided by its maximum). A blowup in Fig. 2 shows details of the control and switching function on the final solution which prove that the infimum of cost (without an upper limit on the number of switchings) has not been achieved. The hessian w.r.t. switching times, computed on the final solution is positive definite, with the minimum eigenvalue equal to $1.09 \cdot 10^{-5}$. The evolution of control structure is depicted in Fig. 3.

The BFGS method finds the optimal solution in 84 iterations (85 gradient evaluations), i.e., it needs 4.9 times more iterations than the Newton method. It also requires 3.6 times more cost evaluations and 2.9 times more CPU time.

In the second experiment, bang-bang and candidate singular arcs are admitted. After 46 iterations of the Newton method, we arrive at the optimal solution depicted in Fig. 5, with the cost value 0.672041. Fig. 4 shows the history of optimization, starting with iteration 10 when the first candidate singular spikes appear, and ending with the optimal control. The eigenvalues of the hessian on the optimal solution are equal to $5.80 \cdot 10^5$ and 0.102. The evolution of control structure is presented in Fig. 6.

The BFGS method finds the optimal solution in 114 iterations. It needs 2.5 times more iterations, 2.2 times more cost evaluations and 2.0 times more CPU time.

VII. LINEAR SYSTEMS

Consider a linear switched system with $M = 3$ right-hand sides. The state equation in the RCP reads

$$\dot{x} = (u_1A_1 + u_2A_2 + u_3A_3)\,x$$

where A_i, $i = 1, 2, 3$ are constant matrices. A cost functional S is to be minimized,

$$S(u) = \tfrac{1}{2}\int_0^T x^\top x \mathrm{d}t.$$

The controls satisfy $u_1 + u_2 + u_3 = 1$, $u_i \geq 0$, $i = 1, 2, 3$. We write the hamiltonian

$$H = \psi^\top(u_1A_1 + u_2A_2 + u_3A_3)x - \tfrac{1}{2}x^\top x,$$

and the adjoint equation (4)

$$\dot{\psi} = x - (u_1A_1 + u_2A_2 + u_3A_3)^\top\psi, \quad \psi(T) = 0.$$

Define the switching functions $\phi_i = \psi^\top A_i x$, $i = 1, 2, 3$. The controls that maximize the hamiltonian can be characterized as follows. Let $i, j, k \in \{1,\ 2,\ 3\}$ be three pairwise different integers. (i) If $\phi_i(t) > \max\{\phi_j(t), \phi_k(t)\}$, then $u_i(t) = 1$, $u_j(t) = u_k(t) = 0$. (ii) If $\phi_i(t) = \phi_j(t) > \phi_k(t)$, then $u_i(t) + u_j(t) = 1$, $u_i(t), u_j(t) \geq 0$, $u_k(t) = 0$. (iii) If $\phi_i(t) = \phi_j(t) = \phi_k(t)$, then every admissible control vector is a hamiltonian maximizer. Cases (ii) and (iii) are singular and will be further analyzed for second order systems and systems with commuting matrices.

Assume that $n = 2$. Consider first the case (ii), that is, $\phi_i(t) = \phi_j(t) > \phi_k(t)$ on some time interval. By differentiating twice the identity $\phi_i(t) = \phi_j(t)$ we arrive at a system of algebraic equations

$$\psi^\top A_{ij}x = 0$$
$$x^\top A_{ij}x + \psi^\top\langle A_i, A_j\rangle x = 0$$
$$x^\top\left((A_{ij}^\top + A_{ij})(u_iA_{ij} + A_j) + \langle A_i, A_j\rangle\right)x +$$
$$\psi^\top\langle\langle A_i, A_j\rangle, u_iA_{ij} + A_j\rangle\,x = 0$$

where $A_{ij} = A_i - A_j$ and $\langle A_i, A_j\rangle = A_iA_j - A_jA_i$. Assuming that the vectors $A_{ij}x$ and $\langle A_i, A_j\rangle x$ are linearly independent (which is generically true), we can compute

$$\psi^\top = -[0 \quad x^\top A_{ij}x]\ [A_{ij}x \quad \langle A_i, A_j\rangle x]^{-1} \tag{9}$$

$$u_{si}^{(j)}(x) = -\frac{x^\top\left((A_{ij} + A_{ij}^\top)A_j + \langle A_i, A_j\rangle\right)x + \psi^\top\langle\langle A_i, A_j\rangle, A_j\rangle x}{x^\top(A_{ij} + A_{ij}^\top)A_{ij}x + \psi^\top\langle\langle A_i, A_j\rangle, A_{ij}\rangle x}. \tag{10}$$

We use the convention that the subscript i denotes the number of the (candidate) singular control component $u_{si}^{(j)}$ and the superscripts are the indices of the remaining (candidate) singular control components so that, e.g., $u_{si}^{(j)} + u_{sj}^{(i)} = 1$ in the case of (10).

Suppose now that $\phi_i(t) = \phi_j(t) = \phi_k(t)$ on a time interval, that is, case (iii) occurs. Then

$$\psi^\top A_{ij}x = \psi^\top A_{ik}x = \psi^\top A_{jk}x = 0. \tag{11}$$

After differentiating the equality $\psi^\top A_{ij}x = 0$ and using $u_i + u_j + u_k = 1$ we obtain

$$x^\top A_{ij}x + \psi^\top\langle A_i, A_j\rangle x + u_k\psi^\top\left(\langle A_{ij}, A_k\rangle - \langle A_i, A_j\rangle\right)x = 0 \tag{12}$$

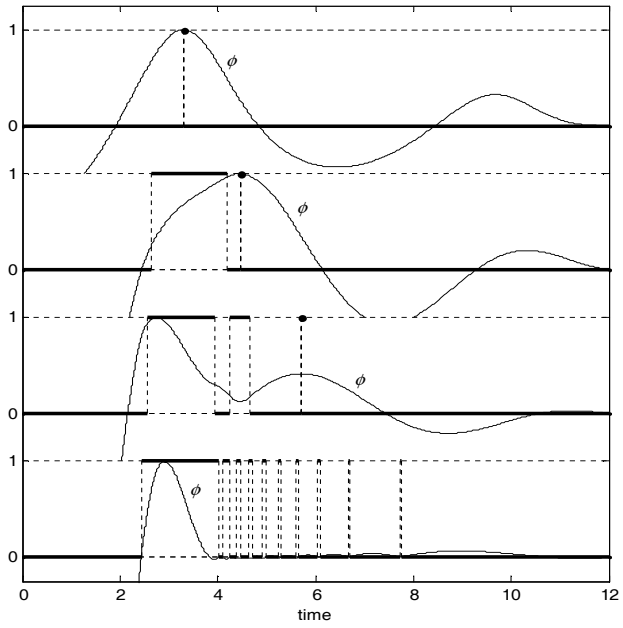

Fig. 1. History of optimization in experiment 1 (u in bold line)

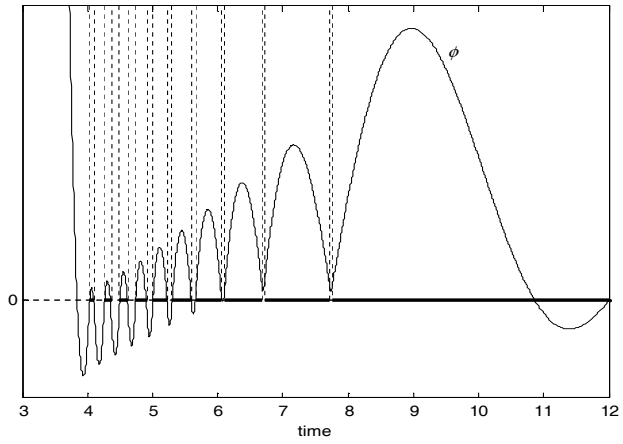

Fig. 2. Details of the last subplot of Fig. 1

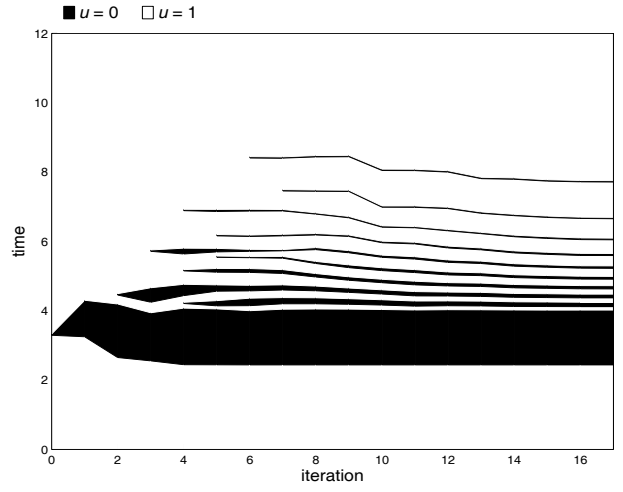

Fig. 3. Evolution of control structure in experiment 1

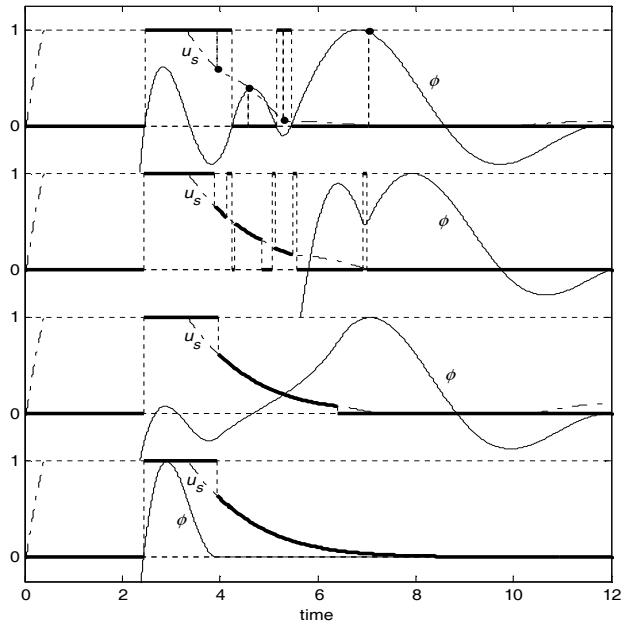

Fig. 4. History of optimization in experiment 2 (u in bold line)

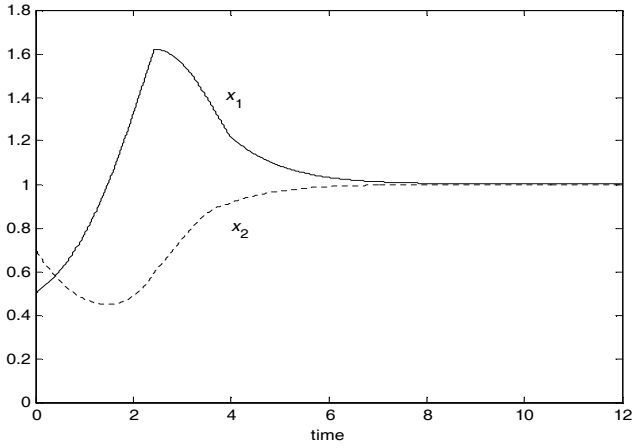

Fig. 5. Optimal state trajectories

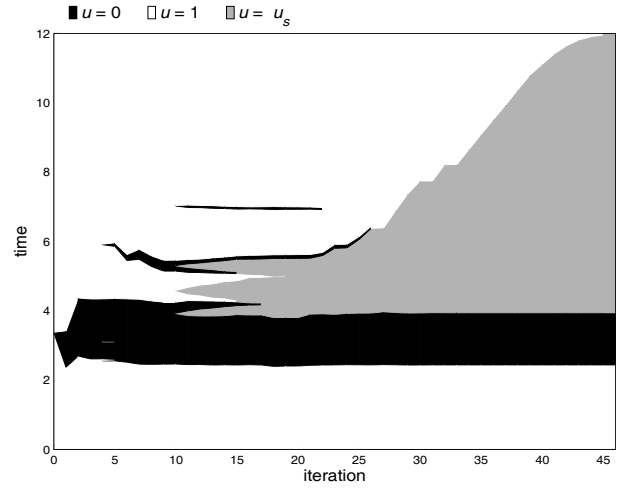

Fig. 6. Evolution of control structure in experiment 2

which generically yields an expression for the singular control $u_{sk}^{(i,j)}$. The singular controls $u_{si}^{(j,k)}$ and $u_{sj}^{(i,k)}$ are easily found by symmetry. As (11) may include at most one linearly independent equation for $\psi^\top$, the adjoints can be fully eliminated from (12) only in special cases.

Let us now consider systems of an arbitrary order n, with commuting matrices $A_1,\ A_2,\ A_3$ which means that $\langle A_1, A_2\rangle = \langle A_1, A_3\rangle = \langle A_2, A_3\rangle = 0$. The analysis in the case (ii) does not change, giving the singular control

$$u_{si}^{(j)}(x) = -\frac{x^\top (A_{ij} + A_{ij}^\top) A_j\, x}{x^\top (A_{ij} + A_{ij}^\top) A_{ij} x}.$$

In the case (iii), the differentiation of (11) gives $x^\top A_{ij} x = x^\top A_{ik} x = x^\top A_{jk} x = 0$. After another differentiation and substitution of $u_k = 1 - u_i - u_j$ we arrive at

$$\begin{bmatrix} x^\top A_{ik}^2 x & x^\top A_{ik} A_{jk} x \\ x^\top A_{jk} A_{ik} x & x^\top A_{jk}^2 x \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} = -\begin{bmatrix} x^\top A_{ik} A_k x \\ x^\top A_{jk} A_k x \end{bmatrix}$$

which yields $u_{si}^{(j,k)}$ and $u_{sj}^{(i,k)}$.

Consider now a numerical example (based on [4]) with

$$A_1 = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix},\ A_3 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$x_1(0) = 0.5, \quad x_2(0) = 0.5, \quad T = 1.$$

In experiment 3 only bang-bang controls are allowed, with the number of switchings not greater than 40. The optimization is started from $u_3 = 1$, $u_1 = u_2 = 0$, and after 16 Newton iterations we obtain the optimal control, presented in Fig. 7 together with the normalized functions

$$\delta_i(t) = \phi_i(x(t), \hat{\psi}(t)) - \max_{j \neq i} \phi_j(x(t), \hat{\psi}(t)).$$

The optimal cost equals 0.494565. The hessian is positive definite with the smallest eigenvalue 0.00011. The BFGS method finds the optimal solution in 127 iterations. It needs 7.9 times more iterations, 8.2 times more cost evaluations and 5.1 times more CPU time.

In experiment 4 we admit bang-bang and candidate singular controls. As case (iii) does not appear in the optimal solution, we do not use candidate singular controls (12).

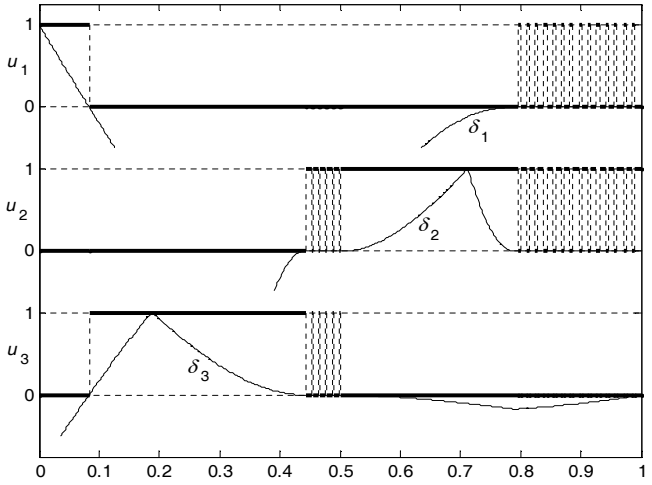

Fig. 7. Optimal control with 40 switchings in experiment 3

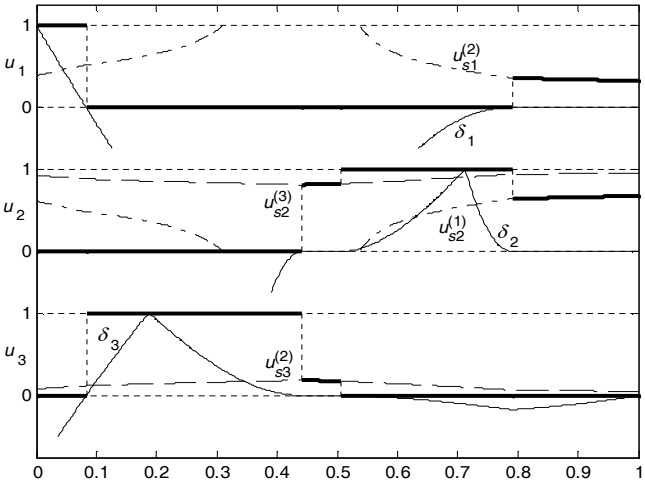

Fig. 9. Optimal control in experiment 4

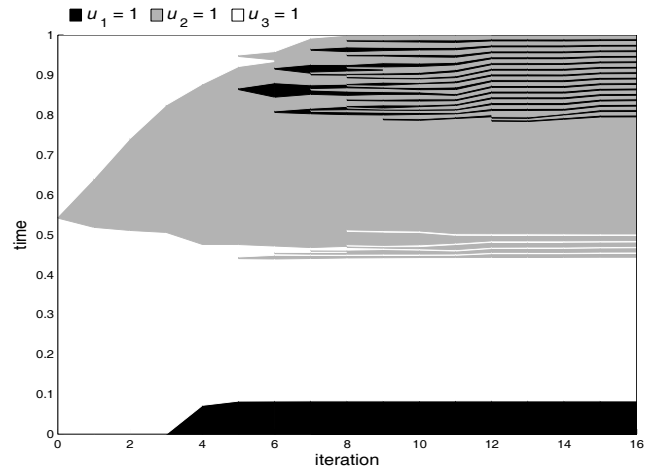

Fig. 8. Evolution of control structure in experiment 3

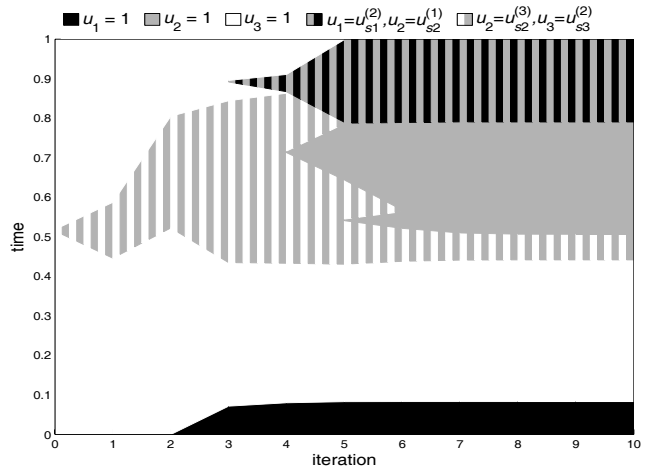

Fig. 10. Evolution of control structure in experiment 4

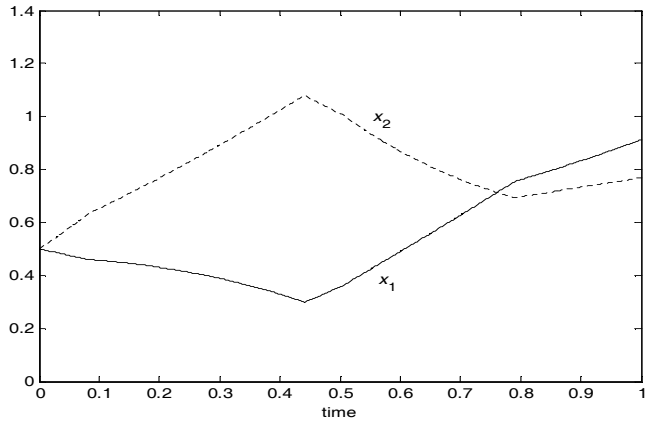

Fig. 11. Optimal state trajectories in experiment 4

The formulas (9) and (10) yield

$$u_{s1}^{(2)} = \frac{8x_1^4 + 116x_1^3x_2 - 178x_1^2x_2^2 - 123x_1x_2^3 + 28x_2^4}{48x_1^4 + 64x_1^3x_2 - 440x_1^2x_2^2 - 136x_1x_2^3 + 83x_2^4}$$

$$u_{s1}^{(3)} = \frac{5x_1^4 - 22x_1^3x_2 + 12x_1^2x_2^2 - 14x_1x_2^3 + x_2^4}{3x_1^4 - 5x_1^3x_2 + 23x_1^2x_2^2 - 13x_1x_2^3 + x_2^4}$$

$$u_{s2}^{(3)} = \frac{2x_1 + 10x_2}{13x_2}.$$

The Newton optimization takes 10 iterations. The pairs of singular control components $(u_{s1}^{(2)}, u_{s2}^{(1)})$ and $(u_{s2}^{(3)}, u_{s3}^{(2)})$ appear in the optimal solution shown in Figs. 9 and 11. The optimal cost value is 0.494562. The hessian is positive definite with the smallest eigenvalue equal to 0.0168.

The BFGS method finds the optimal solution in 25 iterations. It needs 2.5 times more iterations, 1.7 times more cost evaluations and 1.5 times more CPU time.

VIII. CONCLUSIONS

The MSE method has proved relatively efficient for the considered class of switched optimization problems, both for the OCP and RCP. This is due to the fact that it uses, whenever possible, a low-dimensional parameterization of controls and incorporates the necessary optimality conditions of the Maximum Principle. A particularly rapid convergence is observed if the Newton method with analytical hessians is used for gradient optimization. On the other hand, the MSE requires analytical work for the derivation of adjoint equations, especially for the candidate singular arcs.

The MSE, equipped with an appropriate set of procedures $\mathbf{P}$, has quickly found the optimal control structure in all numerical experiments, which confirms earlier experience with this method. The process of structure detection is automatic, in contrast to other approaches known from the literature where more or less heuristic procedures based on homotopy or discrete approximations are used. Since in many practical problems (e.g., in process engineering) the knowledge of optimal structure gives way to economical and exact representation of high quality control, the MSE may be helpful in real time optimizing control.

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