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A Hamiltonian approximation method for the reduction of controlled systems

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Abstract— **This paper considers the problem of model reduction for controlled systems. The paper considers a dual/adjoint formulation of the general optimization problem to minimize a criterion function subject to plant dynamics and system constraints. By carrying out an approximation on the Lagrangian or Hamiltonian system that is inferred from the dual optimization problem, a reduced Hamiltonian system is obtained that approximates the optimally controlled dynamical system. The merits of the method are illustrated on an example of a controlled binary distillation process.**

Index Terms— **Model reduction, Hamiltonian systems, Optimization theory**

I. INTRODUCTION

In control system design one generally faces the paradigm that high quality controllers will have a considerable complexity because they are inferred from high quality models. Indeed, model-based controllers are often as least as complex as the model for which they are designed. For large-scale systems that are represented by high dimensional state space models this often means that the synthesis of controllers easily turns into a computationally infeasible or numerically intractable task. In addition, the maintenance and numerical robustness of controllers is often a serious practical consideration that leads one to prefer simple low order controllers over complex high order ones.

The problem of designing low order controllers for high order or large-scale dynamical systems has received considerable attention in the model reduction community [10]. The most common approach towards solving this problem amounts to first approximating the high order plant by a low order one and subsequently designing a (low order) controller for the low order plant. This 'reduce-then-optimize' strategy of model-based control system design has found widespread applications. The alternative is an 'optimize-then-reduce' approach in which first a model-based controller is synthesized on the basis of a high order plant, which is then reduced in complexity. Due to the complexity of the plant model, this approach is numerically demanding and often infeasible from a practical point of view. Other approaches for the synthesis of low order controllers include the so called 'direct methods'. These methods involve a direct optimization of the parameters that represent the controller as an a priori structured component in a controlled system configuration. See, e.g., [6].

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This paper is motivated by a different approach towards the construction of low order controllers. Using classical approaches in variational analysis, the formulation of an optimal control (or an optimization) problem for a given plant results in a set of Lagrangian or *adjoint equations* whose solution determines the optimally controlled system. The adjoint equations represent the interconnection of plant and optimal controller. However, they are often defined in terms of differential or partial differential equations that are inadequate or cumbersome for the purpose of control system synthesis. Indeed, the classical Hamilton-Jacobi-Bellman [1] differential equations in dynamic programming and variational analysis lead to problems with two-point boundary conditions, non-causal solution structures or non-smooth solutions. The area of dynamic programming has proposed and developed various notions of generalized solutions to cover such situations.

Reduction of the adjoint system corresponds to a more direct approximation of the optimally controlled system and may be a feasible strategy to infer simpler representations of the optimally controlled system. This reduction approach will be the topic of this paper. Earlier work in the realm of unconstrained linear systems has been discussed in [13]. Here, we consider the general optimization problem to minimize a cost function subject to arbitrary equality and inequality constraints and subject to the dynamics of a given system. We adopt the structure of a constrained port-Hamiltonian system to represent the optimal controlled system. A reduction strategy is proposed in which an empirical basis is used to reduce the complexity of the Hamiltonian system.

Notation and terminology

We denote by $\mathbb{R}$ the set and field of real numbers. For $\mathbb{T} \subseteq \mathbb{R}$ let $L_2(\mathbb{T})$ denote the set of Lebesgue measurable real valued functions $v : \mathbb{T} \to \mathbb{R}^n$ that are square integrable in the sense that

$$\|v\|_2 := \sqrt{\int_{\mathbb{T}} \|v(t)\|^2 \ \mathrm{d}t} \ < \ \infty.$$

When equipped with pointwise algebraic operations and inner product

$$\langle v, w \rangle := \int_{\mathbb{T}} v(t)^* w(t) \ \mathrm{d}t$$

this becomes a Hilbert space which we also denote by $L_2(\mathbb{T}, \mathbb{R}^n)$, or by L_2 for short. We denote by $C^p(\mathbb{T})$ the class of real valued functions on $\mathbb{T}$ that are p times continuously differentiable. Partial derivatives and partial gradients of a multivariable differentiable functional v in the arguments

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$x_1, \ldots, x_n$ with $x_i \in \mathbb{R}^{n_i}$ are denoted $\frac{\partial v}{\partial x_i}$ and $\nabla_{x_i} v$, respectively. The partial gradient $\nabla_{x_i} v = [\frac{\partial v}{\partial x_i}]^\top$ consists of the column vector of all partial derivatives of v with respect to the n_i components of x_i. Hence, $\nabla_{x_i} v : \mathbb{R}^N \to \mathbb{R}^{n_i}$ where $N = \sum_{i=1}^n n_i$. Finally, the function $\operatorname{col}(\cdot)$ proves useful to stack vector valued arguments in a column vector as in $\operatorname{col}(a, b) = (a^\top \quad b^\top)^\top$.

II. Hamiltonian representations of controlled systems

This section aims to formulate a general optimal control problem for a given dynamical system and to show that, under suitable conditions, its solution admits a representation as a constrained port-controlled Hamiltonian system that is interconnected with a system that consists of a static nonlinearity.

A. Optimization problem

Consider the general optimization problem to minimize the cost function

$$J(x, u) = \int_0^{t_e} s(x(t), u(t)) \, \mathrm{d}t + e(x(t_e)) \qquad (1)$$

over all continuously differentiable state and input trajectories (x, u) that satisfy the constraints

$$\dot{x} = f(x, u), \qquad x(0) = x_0 \qquad (2a)$$

$$g(x, u) = 0 \qquad (2b)$$

$$h(x) \leq 0 \qquad (2c)$$

Here, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ the input, and

$$s : \mathbb{R}^{n+m} \to \mathbb{R}, \qquad e : \mathbb{R}^n \to \mathbb{R}$$
$$f : \mathbb{R}^{n+m} \to \mathbb{R}^n, \quad g : \mathbb{R}^{n+m} \to \mathbb{R}^q, \quad h : \mathbb{R}^n \to \mathbb{R}^p$$

are given functions that are assumed to be continuously differentiable. Following standard terminology, $s(x, u)$ is called the *stage cost* and $e(x)$ is the *end-point weighting*. It is assumed that both s and e are non-negative. We refer to (2a) as the *state evolution* or the *system equation*, to (2b) as the *equality constraint* and to (2c) as the *inequality constraint*. The q equality constraints (2b) and the p inequality constraints (2c) are assumed to hold for all time instances $t \in [0, t_e]$ and the inequality (2c) is interpreted componentwise: that is $h_j(x(t)) \leq 0$ for all $t \in [0, t_e]$ and for all $j = 1, \ldots, p$. If no confusion can arise, we suppress the time dependence of variables.

The constraints (2) define a *feasible set* of candidate state and input trajectories that is given by

$$\mathcal{F} := \{(x, u) \in L_2([0, t_e]) \mid (2) \text{ holds}\}$$

and is assumed to be non-empty throughout the paper. We aim to minimize J subject to the state evolution, the equality and inequality constraints defined in (2). Precisely, we consider the primal optimization problem

$$P_{\text{opt}} := \inf_{(x,u) \in \mathcal{F}} \quad J(x, u) \qquad (3)$$

and wish to find, if possible, $(x^*, u^*) \in \mathcal{F}$ such that $J(x^*, u^*) = P_{\text{opt}}$.

B. Port-controlled Hamiltonian systems

A solution of the optimization problem (3) can be derived using techniques from variational analysis. Define the *Hamiltonian function* $H : \mathbb{R}^{n+n+m+q} \to \mathbb{R}$ by setting

$$H(x, \lambda, u, \mu) := s(x, u) + \lambda^\top f(x, u) + \mu^\top g(x, u). \qquad (4)$$

Further, define the Lagrangian functional

$$L(x, \lambda, u, \mu, \nu) := \langle 1, s(x, u) \rangle + e(x(t_e)) +$$
$$+ \langle \lambda, f(x, u) - \dot{x} \rangle + \langle \mu, g(x, u) \rangle + \langle \nu, h(x) \rangle \qquad (5)$$

where $\langle \cdot, \cdot \rangle$ is the $L_2([0, t_e])$ inner product, together with the *Lagrange dual cost*

$$\ell(\lambda, \mu, \nu) := \inf_{(x,u) \in \mathcal{F}} L(x, u, \lambda, \mu, \nu).$$

The Lagrange dual cost is defined on the domain

$$\mathcal{G} := \{(\lambda, \mu, \nu) \in L_2([0, t_e]) \mid \nu \geq 0\}$$

and is called *bounded* if there exists a triple $(\lambda, \mu, \nu) \in \mathcal{G}$ for which $\ell(\lambda, \mu, \nu) > -\infty$. It is wel known [2] that the Lagrange dual cost is a concave function and satisfies $\ell(\lambda, \mu, \nu) \leq P_{\text{opt}}$ for all $(\lambda, \mu, \nu) \in \mathcal{G}$. If we assume that the Lagrange dual cost is bounded, the dual optimization problem amounts to determining

$$D_{\text{opt}} := \max_{(\lambda, \mu, \nu) \in \mathcal{G}} \quad \ell(\lambda, \mu, \nu)$$

and, if possible, trajectories $(\lambda^*, \mu^*, \nu^*) \in \mathcal{G}$ such that $\ell((\lambda^*, \mu^*, \nu^*) = D_{\text{opt}}$. By construction, we have $D_{\text{opt}} \leq P_{\text{opt}}$.

Under suitable convexity conditions on the cost and constraint functions a sufficient condition for the existence of a global minimizer for the primal optimization problem P_{opt} is given by the generalized Karush-Kuhn-Tucker theorem. See, e.g., [2], [4]. Precisely, the constraints (2b) and (2c) are said to satisfy the *constraint qualification condition* if there exist at least one pair (x, u) such that $g(x, u) = 0$ and $h_j(x) < 0$ for all components h_j, $j = 1, \ldots, r$ of h.

Theorem II.1 *Suppose g and h are affine and J is convex. Assume that the primal optimization problem* (3) *satisfies the constraint qualification. Then $D_{\text{opt}} = P_{\text{opt}}$. Moreover, there exist functions λ^*, μ^* and $\nu^* \geq 0$ defined on $[0, t_e]$ such that $D_{\text{opt}} = \ell(\lambda^*, \mu^*, \nu^*)$, i.e., the dual optimization problem admits an optimal solution. In addition, (x^*, u^*) is an optimal solution of the primal optimization problem and $(\lambda^*, \mu^*, \nu^*)$ is an optimal solution of the dual optimization problem, if and only if for all time instances $t \in [0, t_e]$:*

1) $g(x^*, u^*) = 0$ *and* $h(x^*) \leq 0$,
2) $\nu^* \geq 0$ *and* (x^*, u^*) *minimizes* $L(x, \lambda^*, u, \mu^*, \nu^*)$ *over all* $(x, u) \in L_2([0, t_e])$ *and*
3) $\nu_j^* h_j(x^*) = 0$ *for all* $j = 1, \ldots, p$.

Using partial integration, the conditions on $z^* := \operatorname{col}(x^*, \lambda^*, u^*, \mu^*, \nu^*)$ translate into stationary conditions of the Lagrangian functional. Precisely, under the assumptions given in Theorem II.1 the optimal trajectory z^* allows a representation as the (unique) solution $z = \mathrm{col}(x, \lambda, u, \mu, \nu)$ of the equations

$$0 = \nabla_\lambda L = f(x,u) - \dot{x} = \nabla_\lambda H - \dot{x} \quad (6a)$$
$$0 = \nabla_x L = \nabla_x H + \dot{\lambda} + \frac{\partial h}{\partial x}(x)^\top \nu \quad (6b)$$
$$0 = \nabla_u L = \nabla_u H \quad (6c)$$
$$0 = \nabla_\mu L = g(x,u) \quad (6d)$$
$$0 \geq h(x) \quad (6e)$$
$$0 \leq \nu \quad (6f)$$
$$0 = \nu_j h_j(x) \quad \text{for } j = 1, \ldots, p \quad (6g)$$

where the differential equations (6a) and (6b) are subject to the two-point boundary conditions $x(0) = x_0$ and $\lambda(t_e)^\top x(t_e) = \nabla_x e(x(t_e))$. We refer to (6) as the *adjoint system* corresponding to the optimization. We stress that (6) is an autonomous system in the sense that solutions of (6) only depend on boundary conditions in (6a) and (6b).

Using the Hamiltonian function (4), the adjoint system is equivalently represented by

$$\Sigma_H : \begin{cases} \dot{x} &= \nabla_\lambda H(z) \\ \dot{\lambda} &= -\nabla_x H(z) - \frac{\partial h}{\partial x}(x)^\top \cdot \nu \\ \rho &= -\frac{\partial h}{\partial x}(x) \cdot \nabla_\lambda H(z) \\ 0 &= \nabla_u H(z) \\ 0 &= \nabla_\mu H(z) \end{cases} \quad (7)$$

together with the static nonlinear constraints

$$\Sigma_{\text{NL}} : \begin{cases} 0 &\geq h(x) \\ 0 &\leq \nu \\ 0 &= \nu_j h_j(x) \quad \text{for } j = 1, \ldots, p. \end{cases} \quad (8)$$

Here, we introduced the auxiliary output ρ and set $z = \mathrm{col}(x, \lambda, u, \mu, \nu)$. By doing so, the system Σ_H becomes a *constrained port-controlled Hamiltonian system* that satisfies the conservation law

$$\begin{aligned} \frac{dH}{dt} &= \nabla_x H(z)^\top \dot{x} + \nabla_\lambda H(z)^\top \dot{\lambda} + \nabla_u H(z)^\top \dot{u} + \\ &+ \nabla_\mu H(z)^\top \dot{\mu} = -\nu^\top \frac{\partial h}{\partial x}(x) \dot{x} = \\ &= -\nu^\top \frac{\partial h}{\partial x}(x) \cdot \nabla_\lambda H(z) = \nu^\top \rho \end{aligned}$$

for all time instances $t \in [0, t_e]$. See, e.g., [3], [8], [9]. In particular, we infer that

$$H(z(t_1)) = H(z(t_0)) + \int_{t_0}^{t_1} \nu^\top(t) \rho(t) \ \mathrm{d}t \quad (9)$$

for all $0 \leq t_0 \leq t_1 \leq t_e$ and all ν. This shows that the system Σ_H is *conservative* with respect to the supply rate $\nu^\top \rho$. Note that the complementarity condition in (8) implies that, in fact, $\nu^\top(t)\rho(t) = 0$ for all $t \in [0, t_e]$, so that in the combined equations (7) and (8) the quantity H is conserved along optimal trajectories.

III. Hamiltonian systems and their reduction

The adjoint system (6) represents the optimal controlled system as it incorporates information about the plant, the optimization criterion and the optimization constraints. In this section we wish to reduce the complexity of the adjoint system (6) by finding a lower order state space representation for the port-Hamiltonian system (7). In the canonical coordinates $s := \mathrm{col}(x, \lambda)$ the state space of (7) has dimension $2n$. Projection based model reduction strategies aim to reduce this dimension by projecting the state on a suitably defined lower dimensional manifold.

A. Canonical state transformations

A straightforward implementation of this strategy first involves a canonical transformation of the state variable $s = \mathrm{col}(x, \lambda)$ according to a bijective and continuously differentiable mapping $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ where

$$s = \Phi(v) = \begin{pmatrix} \Phi_x(v) \\ \Phi_\lambda(v) \end{pmatrix} = \begin{pmatrix} x \\ \lambda \end{pmatrix}. \quad (10)$$

Define the state transformed Hamiltonian

$$\bar{H}(v, u, \mu) := H(\Phi_x(v), \Phi_\lambda(v), u, \mu).$$

Then one easily verifies that in the transformed coordinates the port-controlled Hamiltonian system Σ_H is represented by

$$\Sigma_H : \begin{cases} \dot{v} &= J(v) \nabla_v \bar{H}(v, u, \mu) + b(v)\nu \\ \rho &= b(v)^\top \nabla_v \bar{H}(v, u, \mu) \\ 0 &= \nabla_u \bar{H}(v, u, \mu) \\ 0 &= \nabla_\mu \bar{H}(v, u, \mu) \end{cases} \quad (11)$$

where

$$J(v) = \left(\frac{\partial \Phi}{\partial v}(v)\right)^{-1} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \left(\frac{\partial \Phi}{\partial v}(v)\right)^{-\top}$$

is a skew-symmetric matrix function (i.e., $J(v) + J(v)^\top = 0$) and where

$$b(v) = \left(\frac{\partial \Phi}{\partial v}(v)\right)^{-1} \begin{pmatrix} 0 \\ -\frac{\partial h}{\partial x}(\Phi_x(v))^\top \end{pmatrix}.$$

The transformed system is again a port-Hamiltonian system, as this property is not dependent on the choice of canonical basis in the state space [5].

B. Reduced order systems

It is our purpose to reduce the complexity of the port-controlled Hamiltonian system (7) while retaining the port-Hamiltonian structure. In particular, we aim to replace the Hamiltonian function H that satisfies the conservation law (9) by a function $\hat{H}$ that satisfies the same conservation law (9) for the reduced order system.

With the (linear or non-linear) coordinate change Φ, the reduced system is defined by decomposing the transformed state variable v according to $v = \mathrm{col}(v', v'')$ and by projecting v on its first $r = \dim(v')$ canonical coordinates. Here, $r < 2n$ is the dimension of the reduced order system. With this decomposition, the system (11) is equivalently described by

$$\Sigma_H : \begin{cases} \begin{pmatrix} \dot{v}' \\ \dot{v}'' \end{pmatrix} &= \begin{pmatrix} J_{11}(v) & J_{12}(v) \\ J_{21}(v) & J_{22}(v) \end{pmatrix} \begin{pmatrix} \nabla_{v'} \bar{H}(v,u,\mu) \\ \nabla_{v''} \bar{H}(v,u,\mu) \end{pmatrix} + \\ & \quad + \begin{pmatrix} b'(v) \\ b''(v) \end{pmatrix} \nu \\ \rho &= \begin{pmatrix} b'(v)^\top & b''(v)^\top \end{pmatrix} \begin{pmatrix} \nabla_{v'} \bar{H}(v,u,\mu) \\ \nabla_{v''} \bar{H}(v,u,\mu) \end{pmatrix} \\ 0 &= \nabla_u \bar{H}(v,u,\mu) \\ 0 &= \nabla_\mu \bar{H}(v,u,\mu) \end{cases} \qquad (12)$$

The reduced system is then defined by the equations

$$\Sigma_H^r : \begin{cases} \dot{v}' = \begin{pmatrix} J_{11}(v',0) & J_{12}(v',0) \end{pmatrix} \begin{pmatrix} \nabla_{v'} \bar{H}(v',0,u,\mu) \\ \nabla_{v''} \bar{H}(v',0,u,\mu) \end{pmatrix} + \\ \quad + b'(v',0)\nu \\ \rho = \begin{pmatrix} b'(v',0)^\top & b''(v',0)^\top \end{pmatrix} \begin{pmatrix} \nabla_{v'} \bar{H}(v',0,u,\mu) \\ \nabla_{v''} \bar{H}(v',0,u,\mu) \end{pmatrix} \\ 0 = \nabla_u \bar{H}(v',0,u,\mu) \\ 0 = \nabla_\mu \bar{H}(v',0,\mu) \end{cases} . \qquad (13)$$

In general, the reduced system (13) is no longer a port controlled Hamiltonian system. The following theorem gives a sufficient condition that guarantees a Hamiltonian structure of the reduced order system.

Theorem III.1 *In the decomposition of* (12), *the reduced system* (13) *is a constrained port-Hamiltonian system of state dimension r whenever*

$$\nabla_{v''} \bar{H}(v',0,u,\mu) = 0 \qquad (14)$$

for all v' and for all u and μ that satisfy the last two equations in (13).

Proof: Under the hypothesis (14), the state evolution of (13) reads

$$\dot{v}' = J_{11}(v',0)\nabla_{v'} \bar{H}(v',0,u,\mu) + b'(v',0)\nu$$
$$\rho = b'(v',0)^\top \nabla_{v'} \bar{H}(v',0,u,\mu).$$

It thus follows that

$$\begin{aligned} \frac{d\bar{H}}{dt} &= \nabla_{v'} \bar{H}^\top \dot{v}' + \nabla_u \bar{H}^\top \dot{u} + \nabla_\mu \bar{H}^\top \dot{\nu} \\ &= \nabla_{v'} \bar{H}^\top [J_{11} \nabla_{v'} \bar{H} + b'(v',0)\nu] + 0 + 0 \\ &= \rho^\top \nu \end{aligned}$$

where we used that the skew symmetry of $J(v)$ implies the skew symmetry of $J_{11}(v',v'')$ for all pairs (v',v'') and therefore also for all pairs $(v',0)$. Then integrate to obtain the result. ■

If we assume that the primal and dual optimization problems P_{opt} and D_{opt} admit optimal solutions, then the control input u and the Lagrange multiplier μ can be explicitly solved from the equations (11). This means that there exist functions $c : \mathbb{R}^{2n} \times \mathbb{R}^p \to \mathbb{R}^m$ and $d : \mathbb{R}^{2n} \times \mathbb{R}^p \to \mathbb{R}^q$ with the property that $u = c(v,\nu)$ and $\mu = d(v,\nu)$ if and only if (v,u,μ,ν) satisfy (11). The Hamiltonian $\bar{H}$ can then be written as $\bar{H}(v,u,\mu) = \bar{H}(v,c(v,\nu),d(v,\nu))$ which becomes a function $\widehat{H}(v,\nu)$ of state-input pairs (v,ν). With this substitution, the condition (14) becomes $\frac{\partial \widehat{H}}{\partial v''}(v',0,\nu) = 0$ for all pairs (v',ν).

Obviously, the choice of a suitable, possibly nonlinear, coordinate transformation Φ together with the value of r define the quality of the reduced order model (13). Candidate nonlinear coordinate transformations Φ include the nonlinear balanced realization that has been proposed in [11]. An empirical basis transformation similar to the one proposed in [7], leads to a linear coordinate transformation and will be discussed below.

An *empirical basis* for the $2n$ dimensional Hamiltonian system (7) is obtained by generating trajectories $s_j(t) = \text{col}(x_j(t), \lambda_j(t))$ of the port-controlled Hamiltonian system (7), for $j = 1,\ldots,M$ and $t \in [0,t_e]$. The trajectories s_j are generated by varying the Lagrange multiplier ν and/or by varying the boundary conditions in (7). Given such a collection of state trajectories the data correlation matrix $X : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is defined by setting

$$X = \sum_{j=1}^{M} \int_0^{t_e} s_j(t) s_j^\top(t) \; \mathrm{d}t. \qquad (15)$$

Since X is a self-adjoint and positive semi-definite operator its singular value decomposition assumes the form $X = \Phi^\top \Sigma \Phi$ in which Φ is a unitary matrix and $\Sigma = \text{diag}(\sigma_1,\ldots,\sigma_{2n})$ is a diagonal matrix with singular values ordered according to

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2n} \geq 0.$$

A linear state space transformation (10) is then defined by setting

$$\Phi(v) := \Phi v$$

and transforms (7) to (11).

Remark III.2 It is important to remark that this reduction strategy leads to a reduced order system that retains its Hamiltonian structure under the conditions of Theorem III.1, but that does not, in general, respect the constraints (2). Reduction methods in which the equality and inequality constraints (2b) and (2c) are respected are subject of investigation.

C. The linear case

For linear systems with quadratic cost function and affine constraints the previous analysis specializes to

$$\begin{aligned} s(x,u) &= \frac{1}{2} \begin{pmatrix} x \\ u \end{pmatrix}^\top \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \quad e(x) = \frac{1}{2} x^\top E x \\ f(x,u) &= Ax + Bu \\ g(x,u) &= A_g x + B_g u + C_g \\ h(x) &= A_h x + C_h \end{aligned}$$

where (A,B) is assumed to be stabilizable, $E \geq 0$, $Q \geq 0$ and $R > 0$. The Hamiltonian system (7) is then represented by

$$\Sigma_H : \begin{cases} \dot{x} &= Ax + Bu \\ \dot{\lambda} &= -Qx - Su - A^\top \lambda - A_g^\top \mu - A_h^\top \nu \\ \rho &= -A_h Ax - A_h Bu \\ 0 &= Ru + S^\top x + B^\top \lambda + B_g^\top \mu \\ 0 &= A_g x + B_g u + C_g \end{cases}$$

This can be rewritten in the form of an affine state space system

$$\Sigma_H : \begin{cases} \dot{s} = \mathcal{A}s + \mathcal{B}\nu + b \\ y = \mathcal{C}s + d \end{cases} \tag{16}$$

where $s = \mathrm{col}(x, \lambda)$ is the state, ν the input, $y = \mathrm{col}(\rho, u)$ the output and

$$\mathcal{A} = \begin{pmatrix} \Gamma_0 - BR^{-1}B_g^\top R_g^{-1}\Gamma_g & BR^{-1}(B_g^\top R_g^{-1}B_g - R)R^{-1}B^\top \\ SR^{-1}S^\top - Q - \Gamma_g^\top R_g^{-1}\Gamma_g & \Gamma_g^\top B_g R^{-1}B^\top - \Gamma_0^\top \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} 0 \\ -A_h^\top \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} A_h(BR^{-1}B_g^\top R_g^{-1}\Gamma_g - \Gamma_0) & A_h(BR^{-1}(R - B_g^\top R_g^{-1}B_g)R^{-1}B^\top) \\ -R^{-1}(S^\top + B_g^\top R_g^{-1}\Gamma_g) & R^{-1}(B_g^\top R_g^{-1}B_g - R)R^{-1}B^\top \end{pmatrix}$$

$$b = \begin{pmatrix} -BR^{-1}B_g^\top R_g^{-1}C_g \\ SR^{-1}B_g^\top R_g^{-1}C_g \end{pmatrix}$$

$$d = \begin{pmatrix} A_h BR^{-1}B_g^\top R_g^{-1}C_g \\ -R^{-1}B_g^\top R_g^{-1}B_g C_g \end{pmatrix}.$$

Here, we set $\Gamma_0 = A - BR^{-1}S^\top$, $\Gamma_g = A_g - B_g R^{-1}S^\top$ and it assumed that $R_g = B_g^\top R^{-1}B_g$ is invertible. (If R_g is not invertible, a similar affine representation can be inferred that involves a factorization of R_g).

The optimal trajectories of the primal and dual optimization problem are defined by combining (16) with the static nonlinear constraints

$$\Sigma_{\mathrm{NL}} : \begin{cases} 0 &\geq A_h x + C_h \\ 0 &\leq \nu \\ 0 &= \nu_j e_j^\top (A_h x + C_h) \quad \text{for } j = 1, \ldots, p. \end{cases}$$

To define the reduced order system, the affine system (16) is simulated over the time interval $[0, t_e]$ for M different combinations of input trajectories ν and initial conditions x_0. This leads to the time trajectories $s_j(t)$, $t \in [0, t_e]$, $j = 1, \ldots, M$ and the data correlation matrix $X = X^\top \geq 0$ defined in (15). We let $X = \Phi^\top \Sigma \Phi$ be a singular value decomposition of X and transform the state s in (16) to

$$v := \Phi^\top s = \Phi^{-1}s.$$

This gives an affine system that is equivalent to (16). Finally, the reduced order system of order $r < 2n$ is defined by discarding the least $(2n - r)$ dominant directions in the correlation matrix X as in (13).

IV. Example in binary distillation

We illustrate the reduction methodology on a model of a binary distillation process. In binary distillation the separation of a mixture of two components is achieved by controlling the transfer of components between various *trays* in the column, so as to produce output products of a desired purity. In a typical distillation process, two recycle streams are returned to the column. One at the top (the liquid recycle) to feed the downward liquid stream and one at the bottom (the vapor recycle) to feed the upward vapor stream in the column.

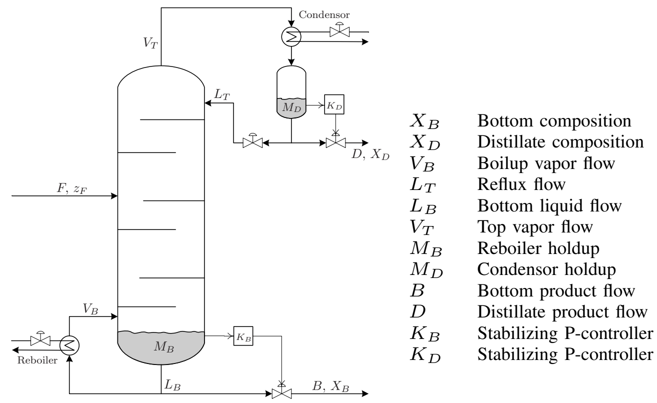

Fig. 1. Distillation column

Here, we use a linearized time-invariant model of a stabilized binary distillation column with 41 stages. A detailed description of this model can be found in [12]. A schematic representation of the distillation column with nomenclature is depicted in Figure 1. Flow units are in kmol/min, holdups in kmol, and tray compositions in mole fraction. The model contains two proportional controllers in order to stabilize the levels in the reboiler and the condensor.

Inputs of the model are

$$u = \mathrm{col}(V_B, L_T)$$

which consists of the flow rate V_B of the reboiler and the reflux flow rate L_T of the condenser. Outputs of the model are taken to be the bottom and top distillate product compositions

$$y = \mathrm{col}(X_B, X_D).$$

The model is inferred from a linearization of a rigorous nonlinear model in which the total material balance at each of the 41 trays in the column is described. A state space representation of the model has a state variable of dimension $n = 82$ that consists of liquid compositions and the component hold-up at each tray.

In this study, we consider only V_B and L_T to exert control over the product compositions X_B and X_D. The resulting plant model is therefore a stable LTI model with 2 inputs, 2 outputs and $n = 82$ states. The stage cost is defined by

$$s(x, u) = \frac{1}{2} \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

where the Q is chosen such that $x^\top Qx = y^\top y$ and $R = 0.001 \cdot I$. We only applied inequality constraints on the holdup conditions in that

$$-0.05 \leq x_j(t) \leq 0.05, \quad j = 42, \ldots, 82$$

That is, the constrained function

$$h(x) = A_h x + C_h \leq 0$$

with A_h of rank 41. No equality constraints where considered in this example.

A reduction has been made of the affine controlled port-Hamiltonian system on the basis of an empirical (linear) state transformation that has been derived in the previous section. The reduction order was set to $r = 4$. The optimal input u^* and the input $\hat{u}$ inferred from the reduced order system where both fed to the original plant to produce outputs y^* and $\hat{y}$, respectively.

Figure 2 shows the results on a comparison of the optimal and suboptimal output and input trajectories, respectively. Due to the approximation, the inequality constraints $h(x) \leq 0$ are not satisfied in the reduced order system. In particular, this means that the active sets in the reduced order system are different from the active sets in the optimally controlled system. Nevertheless, the responses $(\hat{u}, \hat{y})$ in the 4th order approximation are reasonable approximations of the optimal trajectories (u^*, y^*).

V. CONCLUSIONS

In this paper we considered the problem of model reduction for controlled systems. The controlled system is defined by minimizing a criterion function subject to plant dynamics and system constraints. Using variational analysis, the solution of this optimization problem is expressed in terms of an adjoint system of equations that is associated with the dual optimization problem. We view the adjoint system as the interconnection of a constrained port-controlled Hamiltonian system and a static nonlinear system. By carrying out an approximation on the dynamic port-controlled Hamiltonian system, a reduced order system is obtained that has been proven to be a port-controlled Hamiltonian system under suitable conditions. The merits of the method are illustrated on an example of a controlled binary distillation process that has been reduced by projecting the system on the dominant basis functions inferred from an empiric basis.

Qualitative or quantitative properties of the approximate system are still subject of investigation. We believe that is of particular interest to find reduction methods for the adjoint system in which equality and inequality constraints are left invariant. However, currently, this is a largely open question.

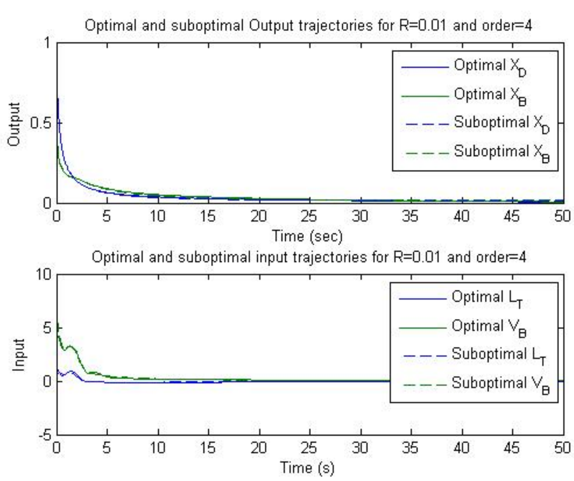

Fig. 2. Approximation results for controlled distillation column

REFERENCES

[1] R.E. Bellman. *Dynamic Programming*. Princeton, NJ, 1957.
[2] D.P. Bertsekas. *Constrained Optimization and Lagrange Multipliers*. Athena Scientific, Belmont (org.: Academic Press), 1996.
[3] Van der Schaft. *L_2-Gain and passivity techniques in nonlinear control*. Springer, London, 2000.
[4] R. Fletcher. *Practical Methods of Optimization*. Chichester: Wiley, 2nd edition, 1987.
[5] K. Fugimoto and T. Sugie. Canonical transformation and stabilization of generalized hamiltonian systems. *System and Control Letters*, 42(3):217–227, 2001.
[6] D.C. Hyland and S. Richter. On direct versus indirect methods for reduced order controller design. *IEEE Trans. Automatic Control*, 35(3):377–379, 1990.
[7] S. Lall, J.E. Marsden, and S. Glovaski. Empirical model reduction of controlled nonlinear systems. *Proc. IFAC World Congress*, pages 473–478, 1999.
[8] B.M. Maschke and A.J. Van der Schaft. Port-controlled hamiltonian systems: modeling origins and system theoretic properties. *Proc. 2nd IFAC NOLCOS, Bordeaux*, pages 282–288, 1992.
[9] B.M. Maschke and A.J. Van der Schaft. *System-theoretic properties of port-controlled Hamiltonian systems*, volume II of *Systems and Networks: Mathematical Theory and Applications*, pages 349–352. Academie-Verlag, Berlin, 1994.
[10] G. Obinata and B.D.O. Anderson. *Model reduction for control system design*. Berlin : Springer, 2001.
[11] J.M.A. Scherpen. Balancing for nonlinear systems. *System and Control Letters*, 21:143–153, 1993.
[12] S. Skogestad. Dynamics and control of distillation columns: A tutorial introduction. *Trans. IChemE (UK)*, 75:539–562, Sept 1997.
[13] S. Weiland, J. Wildenberg, L. Ozkan, and J. Ludlage. A lagrangian method for model reduction of controlled systems. In *Proceedings of the 17th IFAC World Congress, Seoul*, 2008.