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On the Existence of Zeno Behavior in Hybrid Systems with Non-Isolated Zeno Equilibria

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***Abstract*— This paper presents proof-certificate based sufficient conditions for the existence of Zeno behavior in hybrid systems near non-isolated Zeno equilibria. To establish these conditions, we first prove sufficient conditions for Zeno behavior in a special class of hybrid systems termed *first quadrant interval hybrid systems*. The proof-certificate sufficient conditions are then obtained through a collection of functions that effectively "reduce" a general hybrid system to a first quadrant interval hybrid system. This paper concludes with an application of these ideas to Lagrangian hybrid systems, resulting in easily verifiable sufficient conditions for Zeno behavior.**

I. Introduction

This paper was motivated by the lack of analytic tools for proving the existence of Zeno behavior in nontrivial hybrid systems. In particular, mechanical systems undgergoing impacts, modeled by Lagrangian hybrid systems [1], provide a large class of systems that often appear to display Zeno behavior. While Zeno behavior is often intuitively clear and supported by simulation results [2], formal proofs of Zeno behavior were limited to very simple systems such as the bouncing ball.

The objects of study in this paper are *Zeno equilibria*—subsets of the continuous domains of a hybrid system that are fixed points of the discrete dynamics but not the continuous dynamics—which are defined in analogy to equilibria of dynamical systems. Given the success of studying isolated equilibria in dynamical systems, a natural starting point for studying Zeno behavior is a detailed analysis of *isolated Zeno equilibria*—those Zeno equilibria with no other nearby Zeno equilibria. Recently, however, it was observed that Lagrangian hybrid systems with isolated Zeno equilibria must have one dimensional configuration manifolds [3]. Thus, most interesting Lagrangian hybrid systems believed to show Zeno behavior cannot be studied with attention restricted to isolated Zeno equilibria.

On the other hand, first quadrant hybrid systems [4]—hybrid systems with the first quadrant of $\mathbb{R}^2$ as continuous domains—provide a simple class of hybrid systems that can demonstrate many of the subtleties of Zeno behavior. Recent work [5], provides very simple sufficient conditions for Zeno behavior depending only on the value of the vector fields at the Zeno equilibrium.

This paper builds on a variant of first quadrant hybrid systems to develop sufficient conditions for Zeno behavior near non-isolated Zeno equilibria that are general enough to handle Lagrangian hybrid systems of arbitrary dimension. In particular, we study first quadrant hybrid systems with dynamics governed by simple differential inclusions termed *first quadrant interval hybrid systems*. We find sufficient conditions for Zeno behavior in first quadrant interval hybrid systems extending those in [5].

Our first main result is a technique for "reducing" hybrid systems to first quadrant interval hybrid systems resulting in sufficient conditions for Zeno behavior near non-isolated Zeno equilibria. The reduction consists of functions from the continuous domains into the first quadrant of $\mathbb{R}^2$ mapping executions of the original hybrid system to executions of a first quadrant interval hybrid system. Thus Zeno behavior in the first quadrant interval hybrid system implies Zeno behavior in the original hybrid system, and the conditions for Zeno behavior in first quadrant interval systems yield sufficient conditions for Zeno behavior in hybrid systems.

For our other main result, we obtain sufficient conditions for Zeno behavior in Lagrangian hybrid systems of arbitrary dimension by explicitly constructing the proof-certificates implying Zeno behavior. These conditions for Lagrangian hybrid systems generalize those in [3], but remain remarkably simple. When applied to examples, such as a ball bouncing on a sinusoidal surface or a pendulum on a cart, the conditions for Zeno behavior are easily verifiable and intuitively appealing.

Due to the subtle and complex nature of Zeno behavior, it has been studied in many forms and from many different perspectives. Most of the conditions for Zeno behavior are necessary and tend to be very conservative; see [6], [7], [8] for general hybrid systems, and [9], [10] for linear complementarity systems. Until recently, sufficient conditions for Zeno behavior were more rare [11]. Necessary and sufficent conditions for Zeno behavior in a significantly different class of controlled hybrid systems were found in [12]. Interestingly, their study of bounded rate hybrid systems helped motivate our study of first quadrant interval hybrid systems used in proving our main results.

We also note that this paper studies Zeno behavior in Lagrangian hybrid systems, which were studied in [2], [13], [14] as motivated by [1]. Finally, the characterization of Zeno behavior presented in this paper complements the topological characterization of Zeno behavior presented in [15].

II. Hybrid Systems & Zeno Equilibria

In this section, we introduce the basic notations on which the rest of the paper will build. That is, we define hybrid

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systems, executions, and Zeno equilibria. For a more on hybrid systems see [16] and for more on Zeno behavior see [2], [3], [4], [5], [13].

Definition 1: A *hybrid system on a cycle* is a tuple:

$$\mathscr{H} = (\Gamma, D, G, R, F),$$

where

- $\Gamma = (Q, E)$ is a directed cycle, with
$$\begin{aligned} Q &= \{q_0, \ldots, q_{k-1}\}, \\ E &= \{e_0 = (q_0, q_1), e_1 = (q_1, q_2), \\ &\qquad \ldots, e_{k-1} = (q_{k-1}, q_0)\}. \end{aligned}$$
We denote the source of an edge $e \in E$ by $\mathrm{source}(e)$ and the target of an edge by $\mathrm{target}(e)$.
- $D = \{D_q\}_{q \in Q}$ is a set of *domains*, where D_q is a smooth manifold.
- $G = \{G_e\}_{e \in E}$ is a set of *guards*, where $G_e \subseteq D_{\mathrm{source}(e)}$ is and embedded submanifold of $D_{\mathrm{source}(e)}$.
- $R = \{R_e\}_{e \in E}$ is a set of *reset maps*, where $R_e : G_e \subseteq D_{\mathrm{source}(e)} \to D_{\mathrm{target}(e)}$ is a smooth map.
- $F = \{f_q\}_{q \in Q}$, where $f_q : D_q \to TD_q$ is a Lipschitz vector field on D_q.

Remark 1: Note that if a hybrid system over a finite graph displays Zeno behavior, the graph must contain a cycle (see [6] and [8]). Therefore, beginning with hybrid systems defined on cycles greatly simplifies our analysis, while still capturing characteristic types of Zeno behavior.

Definition 2: An *execution* of a hybrid system $\mathscr{H} = (\Gamma, D, G, R, F)$ is a tuple:

$$\chi = (\Lambda, I, \rho, C)$$

where

- $\Lambda = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$ is a finite or infinite indexing set,
- $I = \{I_i\}_{i \in \Lambda}$ where for each $i \in \Lambda$, I_i is defined as follows: $I_i = [\tau_i, \tau_{i+1}]$ if $i, i+1 \in \Lambda$ and $I_{N-1} = [\tau_{N-1}, \tau_N]$ or $[\tau_{N-1}, \tau_N)$ or $[\tau_{N-1}, \infty)$ if $|\Lambda| = N$, N finite. Here, for all $i, i+1 \in \Lambda$, $\tau_i \leq \tau_{i+1}$ with $\tau_i, \tau_{i+1} \in \mathbb{R}$, and $\tau_{N-1} \leq \tau_N$ with $\tau_{N-1}, \tau_N \in \mathbb{R}$.
- $\rho : \Lambda \to Q$ is a map such that for all $i, i+1 \in \Lambda$, $(\rho(i), \rho(i+1)) \in E$. This is the *discrete component* of the execution.
- $C = \{c_i\}_{i \in \Lambda}$ is a set of *continuous trajectories*, and they must satisfy $\dot{c}_i(t) = f_{\rho(i)}(c_i(t))$ for $t \in I_i$.

We require that when i, $i+1 \in \Lambda$,

$$\begin{array}{ll} \text{(i)} & c_i(t) \in D_{\rho(i)} \ \forall \ t \in I_i \\ \text{(ii)} & c_i(\tau_{i+1}) \in G_{(\rho(i), \rho(i+1))} \\ \text{(iii)} & R_{(\rho(i), \rho(i+1))}(c_i(\tau_{i+1})) = c_{i+1}(\tau_{i+1}). \end{array} \tag{1}$$

When $i = |\Lambda| - 1$, we still require that (i) holds.

The object of study in this paper will be Zeno executions, which are defined in the following manner:

Definition 3: An execution χ is *Zeno* if $\Lambda = \mathbb{N}$ and

$$\lim_{i \to \infty} \tau_i - \tau_0 = \sum_{i=0}^{\infty} \tau_{i+1} - \tau_i = \tau_\infty < \infty.$$

Here τ_∞ is called the *Zeno time*.

A hybrid system $\mathscr{H}$ is *Zeno*[1] if there exists a Zeno execution χ such that $\tau_{i+1} - \tau_i \neq 0$ for some $i \in \mathbb{N}$.

Zeno behavior can be likened to stability, in that both involve convergence. This motivates the study of the type of equilibria associated to Zeno behavior: Zeno equilibria. For more on Zeno equilibria, see [13], [4], [3].

Definition 4: A *Zeno equilibria* of a hybrid system $\mathscr{H} = (\Gamma, D, G, R, F)$ is a set $z = \{z_q\}_{q \in Q}$ satisfying the following conditions for all $q \in Q$:

- For the unique edge $e = (q, q') \in E$
  - $z_q \in G_e$,
  - $R_e(z_q) = z_{q'}$,
- $f_q(z_q) \neq 0$.

Note that, in particular, the conditions given in Definition 4 imply that for all $i \in \{0, \ldots, k-1\}$,

$$R_{e_{i-1}} \circ \cdots \circ R_{e_0} \circ R_{e_{k-1}} \circ \cdots \circ R_{e_i}(z_i) = z_i.$$

That is, the element z_i is a fixed point under the reset maps composed in a cyclic manner.

III. First Quadrant Interval Hybrid Systems

This section gives conditions for the existence of Zeno behavior in a simple class of hybrid systems termed first quadrant interval hybrid systems. These systems are easy to analyze, yet flexible enough to capture important characteristics of nontrivial systems. Note that first quadrant interval hybrid systems are a variant on first quadrant hybrid systems which have been studied in [5], [17].

Definition 5: We define a *first quadrant interval* (**FQI**) *hybrid system* to be a tuple

$$\mathscr{H}_{FQI} = (\Gamma, D, G, R, F)$$

where

- $\Gamma = (Q, E)$ is a directed cycle as in Definition 1.
- $D = \{D_q\}_{q \in Q}$ where for all $q \in Q$,
$$D_q = \mathbb{R}^2_{\geq 0} = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 \geq 0, \ x_2 \geq 0\}.$$
- $G = \{G_e\}_{q \in Q}$ where for all $e \in E$,
$$G_e = \{(x_1, x_2)^T \in \mathbb{R}^2_{\geq 0} : x_1 = 0, \ x_2 \geq 0\}.$$
- $R = \{R_e\}_{e \in E}$ where for all $e \in E$, R_e is a set valued function defined by
$$\begin{aligned} R_e(0, x_2) &= \{(y_1, y_2)^T \in D_{q'} : y_1 = 0, \\ &\qquad y_2 \in [\gamma_e^l x_2, \gamma_e^u x_2]\}, \end{aligned}$$

[1]The motivation for this definition is that we want to exclude the possibility that a hybrid system is "trivally" Zeno, i.e., the only Zeno executions are executions that begin at a Zeno Equilibria

for $\gamma_e^u \geq \gamma_e^l > 0$ and for all $(0, x_2)^T \in G_e$.

- $F = \{f_q\}_{q \in Q}$ where for all $q \in Q$, f_q is the (constant) differential inclusion defined for all $x \in D_q$ by

$$f_q(x) = \{(y_1, y_2)^T \in \mathbb{R}^2 : y_1 \in [\alpha_q^l, \alpha_q^u], y_2 \in [\beta_q^l, \beta_q^u]\}.$$

Definition 6: An *execution* of a first quadrant interval system, $\mathscr{H}_{FQI}$ is a tuple $\chi_{FQI} = (\Lambda, I, \rho, C)$ where

- Λ, I and ρ are defined as in Definition 2.
- $C = \{c_i\}_{i \in \Lambda}$ is a set of *continuous trajectories*, and they must satisfy $\dot{c}_i(t) \in f_{\rho(i)}(c_i(t))$ for $t \in I_i$.

We require that when i, $i+1 \in \Lambda$,

$$\begin{array}{ll} \text{(i)} & c_i(t) \in D_{\rho(i)} \ \forall \ t \in I_i \\ \text{(ii)} & c_i(\tau_{i+1}) \in G_{(\rho(i), \rho(i+1))} \\ \text{(iii)} & c_{i+1}(\tau_{i+1}) \in R_{(\rho(i), \rho(i+1))}(c_i(\tau_{i+1})). \end{array} \tag{2}$$

When $i = |\Lambda| - 1$, we still require that (i) holds.

Theorem 1: *Let $\mathscr{H}_{FQI} = (\Gamma, D, G, R, F)$ be a first quadrant interval hybrid system with $|Q| = k$. If $\alpha_q^u < 0 < \beta_q^l$ for all $q \in Q$, $\gamma_e^l > 0$ for all $e \in E$ and*

$$\prod_{i=0}^{k-1} \left| \gamma^u_{(q_i, q_{i+1})} \frac{\beta^u_{q_i}}{\alpha^u_{q_i}} \right| < 1$$

then every execution χ_{FQI} extends to an infinite execution, and every infinite execution is Zeno.

Further, if χ_{FQI} is a Zeno execution, then there exists $T_{\text{Zeno}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $\tau_\infty \leq T_{\text{Zeno}}(\|c_0(\tau_0)\|)$ and $T_{\text{Zeno}}(\|c_0(\tau_0)\|) \to 0$ as $\|c_0(\tau_0)\| \to 0$.

Proof: [Sketch] Consider the hybrid system defined by $\mathscr{H}_C = (\Gamma, D, R_C, F_C)$ with $R_C = \{\hat{R}_e\}$ defined by $\hat{R}_e(0, x_2) = (\gamma_e^u x_2, 0)^T$ and $F_C = \{\hat{f}_q\}$ defined by $\hat{f}_q(x) = (\alpha_q^u, \beta_q^u)^T$. By integration, every execution of $\mathscr{H}_C$ extends to a unique Zeno execution. If $\chi = (\mathbb{N}, I, \rho, C)$ is an execution of $\mathscr{H}_{FQI}$ and $\hat{\chi} = (\mathbb{N}, \hat{I}, \hat{\rho}, \hat{C})$ is an execution of $\mathscr{H}_C$ with $c_0(\tau_0) = \hat{c}_0(\hat{\tau}_0)$, then for all $i \in \mathbb{N}$, $\tau_i - \tau_0 \leq \hat{\tau}_i - \hat{\tau}_0$. Therefore χ is Zeno with time bound inherited from $\hat{\chi}$. ■

IV. Sufficient Conditions for Zeno Behavior through Reduction to FQI Hybrid Systems

The main result of this paper is presented in this section, i.e., we give sufficient conditions for the existence of Zeno behavior in hybrid systems "reducing" them to **FQI** hybrid systems. In particular, we prove that if a hybrid system satisfies certain conditions then, given a Zeno equilibria, every execution starting near this Zeno equilibria is Zeno.

Assumption. In this section, we assume that each D_q is a subset of $\mathbb{R}^{n_q}$ with $n_q = \dim(D_q)$ and $z_q = 0$. No generality is lost because we can work locally in coordinate charts.

Reduction conditions. Let $z = \{z_q\}_{q \in Q}$ be a Zeno equilibrium (not necessarily isolated) of a hybrid system $\mathscr{H} = (\Gamma, D, G, R, F)$, $\{W_q\}_{q \in Q}$ be a collection of sets with $z_q \in W_q \subseteq D_q$ and $\{\psi_q\}_{q \in Q}$ be a collection of C^1 maps; these are "proof-certificates", with

$$\psi_q : W_q \subseteq D_q \to \mathbb{R}^2_{\geq 0}.$$

Consider the following conditions:

R1: $\psi_q(z_q) = 0$ for all $q \in Q$.

R2: If $(q, q') \in E$, then $\psi_q(x)_1 = 0$ if and only if $x \in G_{(q,q')} \cap W_q$.

R3: $d\psi_q(z_q)_1 f_q(z_q) < 0 < d\psi_q(z_q)_2 f_q(z_q)$ for all $q \in Q$.

R4: $\psi_{q'}(R_{(q,q')}(x))_2 = 0$ and there exist constants $0 < \gamma_e^l \leq \gamma_e^u$ such that

$$\psi_{q'}(R_{(q,q')}(x))_1 \in \left[\gamma^l_{(q,q')} \psi_q(x)_2, \gamma^u_{(q,q')} \psi_q(x)_2 \right]$$

for all $x \in G_{(q,q')} \cap W_q$ and all $(q, q') \in E$.

R5:

$$\prod_{i=0}^{|Q|-1} \left| \gamma^u_{(q_i, q_{i+1})} \frac{d\psi_{q_i}(z_{q_i})_2 f_{q_i}(z_{q_i})}{d\psi_{q_i}(z_{q_i})_1 f_{q_i}(z_{q_i})} \right| < 1.$$

R6: There exists $K \geq 0$ such that

$$\|R_{(q,q')}(x) - z_{q'}\| \leq \|x - z_q\| + K \psi_q(x)_2$$

for all $x \in G_{(q,q')} \cap W_q$ and all $(q, q') \in E$.

Theorem 2: *Let $\mathscr{H}$ be a hybrid system with a Zeno equilibria $z = \{z_q\}_{q \in Q}$. If there exists a collection of sets $\{W_q\}_{q \in Q}$ with $z_q \in W_q \subseteq D_q$ and maps $\{\psi_q\}_{q \in Q}$ satisfying conditions **R**1-**R**6, then there exists $\eta > 0$ such that for all $q \in Q$ and $x_0 \in D_q$ such that $\|x_0 - z_q\| < \eta$ there exist an execution χ of $\mathscr{H}$ with $c_0(\tau_0) = x_0$, $\rho(0) = q$ and $\Lambda = \mathbb{N}$, and every such execution is Zeno. Therefore, $\mathscr{H}$ is Zeno.*

We prove Theorem 2 in the following manner:

1) Construct a Zeno first quadrant interval system $\mathscr{H}_{\text{FQI}}$ from the hybrid system $\mathscr{H}$ and project executions of the hybrid system to executions of the **FQI** hybrid system (Lemma 1).
2) Prove that executions of $\mathscr{H}$ stay "close" to the Zeno equilibria for a bounded period of time (Lemma 2).
3) Use (2) and (1) to show that $\mathscr{H}$ is Zeno exactly because $\mathscr{H}_{\text{FQI}}$ is Zeno due to conditions **R**1-**R**6.

Constructing a FQI hybrid system. We define a first quadrant interval system $\mathscr{H}_{\text{FQI}}$ from a hybrid system $\mathscr{H}$ based on the reduction conditions. Assume that $\mathscr{H}$ is a hybrid system satisfying **R**1-**R**5. Pick α_q^l, α_q^u, β_q^l and β_q^u such that

$$\alpha_q^l < d\psi_q(0)_1 f_q(0) < \alpha_q^u < 0 < \beta_q^l < d\psi_q(0)_2 f_q(0) < \beta_q^u$$

for all $q \in Q$ and

$$\prod_{i=0}^{k-1} \left| \gamma^u_{(q_i, q_{i+1})} \frac{\beta^u_{q_i}}{\alpha^u_{q_i}} \right| < 1,$$

where $\gamma^u_{(q_i, q_{i+1})}$ is given by **R**4. The constants α_q^l, α_q^u, β_q^l, β_q^u, $\gamma^l_{(q,q')}$ and $\gamma^u_{(q,q')}$ (with $\gamma^l_{(q,q')}$ also given by **R**4) thus define a first quadrant interval system $\mathscr{H}_{FQI}$, on the same graph Γ as $\mathscr{H}$, satisfying the conditions of Theorem 1 due to conditions **R**3-**R**5. Thus all executions of $\mathscr{H}_{FQI}$ extend to Zeno executions.

Now we show how an execution of $\mathscr{H}$ remaining near the Zeno equilibria gives rise to an execution of $\mathscr{H}_{FQI}$.

Lemma 1: *Suppose $\mathscr{H}$ is a hybrid system satisfying the conditions of Theorem 2. Then there exists $\mu > 0$ such that if $\chi = (\Lambda, \rho, I, C)$ is an execution of $\mathscr{H}$ with $\|c_i(t)\| < \mu$ for all $t \in I_i$ and all $i \in \Lambda$, then $\chi_{FQI} = (\Lambda, \rho, I, \Psi \circ C)$, where $\Psi \circ C = \{\psi_{\rho(i)} \circ c_i\}_{i\in\Lambda}$, is an execution of of $\mathscr{H}_{FQI}$*

Proof: [Sketch] By continuity, it follows that there exists $\mu > 0$ such that for all $q \in Q$ and for all $x \in W_q$ with $\|x\| < \mu$,

$$\alpha_q^l < d\psi_q(x)_1 f_q(x) < \alpha_q^u < 0 < \beta_q^l < d\psi_q(x)_2 f_q(x) < \beta_q^u.$$

wherein it follows that χ_{FQI} satisfies the conditions of $\mathscr{H}_{FQI}$ by construction. ■

Lemma 2: *Let $\mathscr{H}$ satisfy the conditions of Theorem 2. Then for all $0 < \mu$ sufficiently small and all $\eta > 0$ sufficiently smaller than μ, there exists $T_{\text{escape}}(\eta, \mu) > 0$ such that any execution χ of $\mathscr{H}$ with $\|c_0(\tau_0)\| < \eta$ satisfies $\|c_i(t)\| < \mu$ for all $t \in I_i$ with $t - \tau_0 < T_{\text{escape}}(\eta, \mu)$.*

Furthermore, if $\hat{\eta} \le \eta$, then $T_{\text{escape}}(\hat{\eta}, \mu) \ge T_{\text{escape}}(\eta, \mu)$.

Proof: [Sketch] Let χ be such that $\|c_0(\tau_0)\| < \eta$ and $\|c_{i^*}(\tau_{i^*})\| \ge \mu$. By continuity of f_q and **R**6, it can be shown that for some $M > 0$,

$$M(\tau - \tau_0) \ge \|c_{i^*}(\tau)\| - \|c_0(\tau_0)\| - K \sum_{i=0}^{i^*-1} \psi_{\rho(i)}(c_i(\tau_{i+1}))_2.$$

The proof is completed by showing the summation on the right hand side approaches zero as $\eta \to 0$. ■

Proof: [*of Theorem 2*, Sketch] Because we assume that the f_q, $q \in Q$, are Lipschitz, the continuous dynamics are always well-defined on each domain. Furthermore, since Γ is a directed cycle the dynamics of $\mathscr{H}$ are completely deterministic. Thus given $x_0 \in D_q$, there exists a unique execution χ of $\mathscr{H}$ with $c_0(\tau_0) = x_0$ such that either χ is defined for all $t \ge \tau_0$ or χ is Zeno.

Pick small constants η and μ such that $T_{\text{Zeno}}(g(\eta)) < T_{\text{escape}}(\eta, \mu)$, where $g(\eta) = \max_{\|x\| \le \eta, q \in Q} \|\psi_q(x)\|$. Assume for the sake of contradiction that there is an execution χ with $\|c_0(\tau_0)\| < \eta$ that is not Zeno. Let $\hat{\chi}$ be χ restriced to $t - \tau_0 < T_{\text{escape}}(\eta, \mu)$. Then Lemmas 1 and 2 combined with Theorem 1 imply that $\hat{\chi}$ gives rise to an execution χ_{FQI} of $\mathscr{H}_{FQI}$ that is defined past its Zeno time, a contradiction. ■

V. Application to Simple Hybrid Mechanical Systems

Mechanical systems undergoing impacts are naturally modeled as hybrid systems. In this section, we will consider hybrid systems of this form and demonstrate how one obtains such systems from hybrid Lagrangians, which are the hybrid analogue of Lagrangians. For more on hybrid Lagrangians and Lagrangian hybrid systems, see [2], [13], [14].

Lagrangians. Consider a configuration space[2] Θ and a Lagrangian $L : T\Theta \to \mathbb{R}$ given in coordinates by:

$$L(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^T M(\theta)\dot{\theta} - U(\theta) \tag{3}$$

where $M(\theta)$ is positive definite and symmetric and $U(\theta)$ is the potential energy. For the sake of simplicity, we assume $\Theta \subset \mathbb{R}^n$ since all our results are local, i.e., we can work within a coordinate chart. The equations of motion are then given in coordinates by the Euler-Lagrange equations, $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$. In the case of Lagrangians of the form given in (3), the Lagrangian vector field, f_L, associated to L takes the familiar form

$$\dot{x} = f_L(x) = \begin{pmatrix} \dot{\theta} \\ M(\theta)^{-1}(-C(\theta,\dot{\theta})\dot{\theta} - N(\theta)) \end{pmatrix}. \tag{4}$$

where $x = (\theta^T, \dot{\theta}^T)^T$, $C(\theta, \dot{\theta})$ is the *Coriolis matrix* and $N(\theta) = \frac{\partial U}{\partial \theta}(\theta)$.

This process of associating a dynamical system to a Lagrangian will be mirrored in the setting of hybrid systems. First, we introduce the notion of a hybrid Lagrangian.

Definition 7: A *hybrid Lagrangian* is a tuple, $\mathbf{L} = (\Theta, L, h)$, where

- $\Theta \subset \mathbb{R}^n$ is the configuration space,
- $L : T\Theta \to \mathbb{R}$ is a Lagrangian of the form give in (3),
- $h : \Theta \to \mathbb{R}$ is a *unilateral constraint function*, where we assume that 0 is a regular value of h.

Examples. We now present two examples that will be considered throughout the rest of the paper to illustrate the concepts involved. Note that these examples were studied in the context of hybrid reduction in [14], although these examples have never been formally shown to be Zeno.

Example 1 (Ball): Our first running example is a ball bouncing on a sinusoidal surface (cf. Fig. 1). In this case $\mathbf{B} = (\Theta_{\mathbf{B}}, L_{\mathbf{B}}, h_{\mathbf{B}})$, where $\Theta_{\mathbf{B}} = \mathbb{R}^3$, and for $x = (x_1, x_2, x_3)$,

$$\begin{aligned} L_{\mathbf{B}}(x, \dot{x}) &= \tfrac{1}{2}m\|\dot{x}\|^2 - mgx_3, \\ h_{\mathbf{B}}(x_1, x_2, x_3) &= x_3 - \sin(x_2). \end{aligned}$$

So, for this example, there are trivial dynamics and a nontrivial unilateral constraint function.

Example 2 (Cart): Our second running example is a constrained pendulum on a cart (cf. Fig. 1); this is a variation on the classical pendulum on a cart, where the pendulum is not allowed to "pass through" the cart, i.e., the cart gives physical constraints on the configuration space. In this case $\mathbf{C} = (\Theta_{\mathbf{C}}, L_{\mathbf{C}}, h_{\mathbf{C}})$, where $\Theta_{\mathbf{C}} = \mathbb{S}^1 \times \mathbb{R}$, $q = (\theta, x)$, and

$$L_{\mathbf{C}}(\theta, \dot{\theta}, x, \dot{x}) = -mgR\cos(\theta) + \frac{1}{2}\begin{pmatrix} \dot{\theta} & \dot{x} \end{pmatrix}\begin{pmatrix} mR^2 & mR\cos(\theta) \\ mR\cos(\theta) & M + m \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{x} \end{pmatrix}.$$

[2]Note that we denote the configuration space by Θ rather than Q, due to the fact that Q denotes the vertices of the graph of a hybrid system.

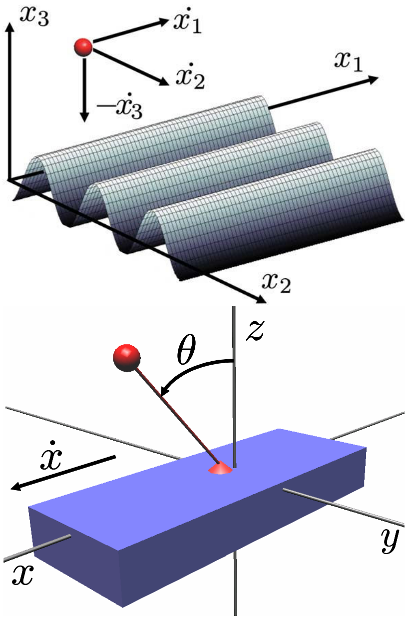

Fig. 1. Ball bouncing on a sinusoidal surface (left). Pendulum on a cart (right).

where m is the mass of the pendulum, M is the mass of the cart and R is the length of the pendulum. Finally, the constraint function $h_{\mathbf{C}}(\theta, x) = \cos(\theta)$ implies that the pendulum is not allowed to pass through the cart.

Domains from constraints. Given a smooth (unilateral constraint) function $h : \Theta \to \mathbb{R}$ on a configuration space Θ such that 0 is a regular value of h (so $h^{-1}(0)$ is a smooth manifold), we can construct a domain and a guard explicitly. Define the domain, D_h, as the manifold (with boundary):

$$D_h = \{(\theta, \dot{\theta}) \in T\Theta : h(\theta) \geq 0\}.$$

Similarly, we have an associated guard, G_h, defined as the following submanifold of D_h:

$$G_h = \{(\theta, \dot{\theta}) \in T\Theta : h(\theta) = 0 \text{ and } dh(\theta)\dot{\theta} \leq 0\},$$

where $dh(\theta) = \begin{pmatrix} \frac{\partial h}{\partial \theta_1}(\theta) & \cdots & \frac{\partial h}{\partial \theta_n}(\theta) \end{pmatrix}$. Note that the requirement that 0 is a regular value of h is equivalent to requiring that $dh(\theta) \neq 0$ when $h(\theta) = 0$.

Lagrangian Hybrid Systems. Given a hybrid Lagrangian $\mathbf{L} = (\Theta, L, h)$, the *Lagrangian hybrid system associated to* $\mathbf{L}$ is the hybrid system

$$\mathscr{H}_{\mathbf{L}} = (\Gamma = (\{q\}, \{(q,q)\}), D_{\mathbf{L}}, G_{\mathbf{L}}, R_{\mathbf{L}}, F_{\mathbf{L}}),$$

where $D_{\mathbf{L}} = \{D_h\}$, $F_{\mathbf{L}} = \{f_L\}$, $G_{\mathbf{L}} = \{G_h\}$ and $R_{\mathbf{L}} = \{R_h\}$ with $R_h(\theta, \dot{\theta}) = (\theta, P(\theta, \dot{\theta}))$, where

$$P(\theta, \dot{\theta}) = \dot{\theta} - (1+e)\frac{dh(\theta)\dot{\theta}}{dh(\theta)M(\theta)^{-1}dh(\theta)^T} M(\theta)^{-1} dh(\theta)^T. \quad (5)$$

Example 3: From the hybrid Lagrangian $\mathbf{B} = (\Theta_{\mathbf{B}}, L_{\mathbf{B}}, h_{\mathbf{B}})$ we obtain

$$\mathscr{H}_{\mathbf{B}} = (\Gamma = (\{q\}, \{(q,q)\}), D_{\mathbf{B}}, G_{\mathbf{B}}, R_{\mathbf{B}}, F_{\mathbf{B}}),$$

where

$$\begin{aligned} D_{h_{\mathbf{B}}} &= \{(x, \dot{x}) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_3 - \sin(x_2) \geq 0\}, \\ G_{h_{\mathbf{B}}} &= \{(x, \dot{x}) \in \mathbb{R}^3 \times \mathbb{R}^3 : x_3 = \sin(x_2) \\ &\qquad \text{and } \dot{x}_3 - \cos(x_2)\dot{x}_2 \leq 0\}, \end{aligned}$$

and $R_{h_{\mathbf{B}}}(x, \dot{x}) = (x, P_{h_{\mathbf{B}}}(x, \dot{x}))$, where $P_{h_{\mathbf{B}}}$ is computed from (5) with $0 \leq e \leq 1$ the coefficient of restitution. Finally, $f_{L_{\mathbf{B}}}(x, \dot{x}) = \left(\dot{x}, (0, 0, -g)^T\right)$.

One can similarly construct a Lagrangian hybrid system $\mathscr{H}_{\mathbf{C}}$ from the hybrid Lagrangian $\mathbf{C}$.

VI. Sufficient Conditions for Zeno Behavior in Lagrangian Hybrid Systems

In this section, we present sufficient conditions for the existence of Zeno behavior in Lagrangian hybrid systems. Before presenting these conditions, we characterize Zeno equilibria in systems of this form.

Zeno equilibria in Lagrangian hybrid systems. If $\mathscr{H}_{\mathbf{L}}$ is a Lagrangian hybrid system, then due to the special form of these systems we find that $z = \{(\theta^*, \dot{\theta}^*)\}$ is a Zeno equilibria iff $\dot{\theta}^* = P(\theta^*, \dot{\theta}^*)$, with P given in (5). In particular, the special form of P implies that this holds iff $dh(\theta^*)\dot{\theta}^* = 0$. Therefore the set of all Zeno equilibria for a Lagrangian hybrid system is given by the hypersurfaces in G_h:

$$Z = \{(\theta, \dot{\theta}) \in G_h : dh(\theta)\dot{\theta} = 0\}.$$

Note that if $\dim(\Theta) > 1$, the Zeno equilibria in Lagrangian hybrid systems are always non-isolated (see [3])—this motivates the study of such equilibria.

Theorem 3: *Let $\mathscr{H}_{\mathbf{L}}$ be a Lagrangian hybrid system and Let $z = \{(\theta^*, \dot{\theta}^*)\}$ be a Zeno equilibria of $\mathscr{H}_{\mathbf{L}}$. If $0 < e < 1$ and $\ddot{h}(\theta^*, \dot{\theta}^*) < 0$, with*

$$\begin{aligned} \ddot{h}(\theta^*, \dot{\theta}^*) &= (\dot{\theta}^*)^T H(h(\theta^*))\dot{\theta}^* + \\ &\quad dh(\theta^*)M(\theta^*)^{-1}(-C(\theta^*, \dot{\theta}^*)\dot{\theta}^* - N(\theta^*)), \end{aligned}$$

where $H(h(\theta^*))$ is the Hessian of h at θ^*, then there is a neighborhood $W \subset D_h$ of $(\theta^*, \dot{\theta}^*)$ such that for every $(\theta, \dot{\theta}) \in W$, there is a unique Zeno execution χ of $\mathscr{H}_{\mathbf{L}}$ with $c_0(\tau_0) = (\theta, \dot{\theta})$.

Proof: [Sketch] Let W_q be a small neighborhood of $(\theta^*, \dot{\theta}^*)$ and assume (by passing to a coordinate chart) that $W_q \subset \mathbb{R}^{2n}$ with Euclidean norm. Let K satisfy

$$K > \frac{1+e}{2} \frac{\|M(\theta^*)^{-1}dh(\theta^*)T\|}{dh(\theta^*)M(\theta^*)^{-1}dh(\theta^*)^T}.$$

Routine calculation verifies that the constants $\gamma_h^u = \gamma_g^l = e$, K and the function

$$\psi_h(\theta, \dot{\theta}) = \begin{pmatrix} \dot{h}(\theta, \dot{\theta}) + \sqrt{\dot{h}(\theta, \dot{\theta})^2 + 2h(\theta)} \\ -\dot{h}(\theta, \dot{\theta}) + \sqrt{\dot{h}(\theta, \dot{\theta})^2 + 2h(\theta)} \end{pmatrix}$$

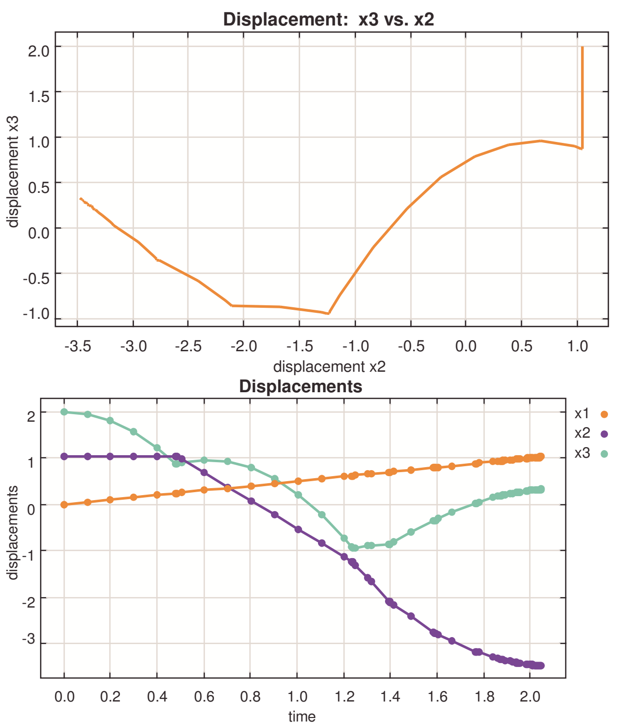

Fig. 2. A Zeno execution of the bouncing ball (left) x_3 vs x_2 and (right) displacements x_1, x_2 and x_3 vs time.

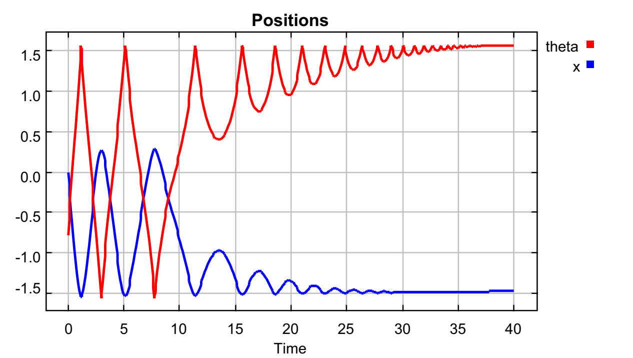

Fig. 3. A Zeno execution of the pendulum on a cart—displacements θ and x vs time.

satisfy conditions **R**1-**R**6 when W_q is small enough. ■

Example 4 (Ball): We first demonstrate that the hybrid system $\mathscr{H}_{\mathbf{B}}$ modeling a ball bouncing on a sinusoidal surface is Zeno. First, the Zeno equilibria of this system are given by the set

$$Z = \{(x, \dot{x}) \in G_{h_{\mathbf{B}}} : \dot{x}_3 - \dot{x}_2 \cos(x_2) = 0\}.$$

Now, one can easily verify that for $(x^*, \dot{x}^*) \in Z$

$$\ddot{h}_{\mathbf{B}}(x^*, \dot{x}^*) = \sin(x_2)\dot{x}_2^2 - g.$$

Therefore, there are clearly Zeno equilibria satisfying the conditions of Theorem 3, namely when $\dot{x}_2$ is small, and thus $\mathscr{H}_{\mathbf{B}}$ is Zeno. A simulation of a Zeno trajectory of the system can be seen in Fig. 2.

Example 5 (Cart): We now demonstrate that the hybrid system modeling a pendulum on a cart, $\mathscr{H}_{\mathbf{C}}$, is Zeno. First, note that the Zeno equilibria are given by the set:

$$Z = \{(\theta, x, \dot{\theta}, \dot{x}) \in G_{h_{\mathbf{C}}} : \sin(\theta)\dot{\theta} = 0\},$$

and for $(\theta^*, x^*, \dot{\theta}^*, \dot{x}^*) \in Z$,

$$\ddot{h}_{\mathbf{C}}(\theta^*, x^*, \dot{\theta}^*, \dot{x}^*) = -\frac{g}{R} < 0.$$

Therefore, for every Zeno equilibria of the pendulum on a cart there a neighborhood of the Zeno equilibria such that every execution with an initial condition in that neighborhood is Zeno. Such a Zeno execution can be seen in Fig. 3.

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