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LQ Robust Controllers for Polynomially Uncertain Systems

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Abstract— **This paper is concerned with the LQ optimal robust control of discrete-time LTI systems with uncertainties belonging to a semi-algebraic set. Given a prescribed cost function, the problem of designing a gain-scheduled static controller, whose gain depends polynomially on the uncertain parameters is formulated as a SOS problem. This indeed requires solving two hierarchies of SDP problems, which may in turn introduce high computational burden, and hence complicate the optimal robust controller design. To bypass this barrier, an alternative approach is developed which is much less involved, at the cost of obtaining a near-optimal controller (as opposed to an optimal one). The efficacy of this work is elucidated in two numerical examples.**

I. Introduction

It is axiomatic that every real-world system is subject to uncertainty and perturbations, to some extent. Robustness analysis for different classes of uncertain systems has been extensively studied in the literature [1], [2], [3], [4]. Robust stability verification can be envisaged as one of the most important problems in this area, which is concerned with the conditions under which a controller designed for a nominal model can also stabilize the corresponding uncertain system. This problem has been addressed in the literature for different types of uncertainties (e.g., structured and unstructured [5]) in the past several years. More recently, the important class of parametric uncertainty has drawn much attention in this field.

Presently, the most efficient technique for verifying the robust stability of a system under a nominal controller is to check the existence of a proper Lyapunov function for the closed-loop system [6], [7], [8]. For the sake of computational simplicity, the pioneer works sought a *constant* Lyapunov function. While this method may work satisfactorily for some systems, it is known that the corresponding robust stability results can be quite conservative in general. As an alternative, many of the recent works consider parameter-dependent Lyapunov functions in order to achieve less conservative results. Notice that various types can be considered for the Lyapunov function being sought, e.g. sinusoidal, exponential, etc. Nonetheless, it is shown in [9] that the *polynomial-type* Lyapunov functions are always capable of detecting the robust stability of any robustly stable system.

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As the simplest scenario, assume that the region of uncertainty is polytopic. The works [10] and [11] search for a Lyapunov function in the form of a first-order polynomial to determine the robust stability of the system. These works present relatively simple sufficient LMI conditions which are proved to be very conservative in numerous examples. As a more sophisticated but less conservative approach, it is shown in [6] that robust stability over a polytope is tantamount to the existence of a Lyapunov function in the form of a homogeneous polynomial with a certain bound on its degree. Hence, the seminal work [6] appropriately characterizes all the essential candidates for the desired Lyapunov function in question. A sufficient LMI condition is subsequently derived in [6] to check the robust stability of the system. A method similar to [6] is also proposed in [8], which seeks the same type of Lyapunov function. Nevertheless, the work [8] further simplifies the required LMI conditions at the cost of introducing more conservatism.

More recently, it is asserted in [12] that the robust stability verification of a system over any semi-algebraic set satisfying a mild condition is equivalent to checking the existence of a set of polynomials for which a specific SOS matrix equation holds. The feasibility of this matrix equation can be determined systematically as long as some bounds on the degrees of the relevant polynomials are known *a priori*. The work [12] also presents important results on how to compute these bounds. It is worth mentioning that the conditions obtained in [12] encompass those derived in both [6] and [8] for the particular case of polytopic uncertainty.

In addition to the surveyed papers dealing with the robust stability problem, there have been some other works concerned with the robust control synthesis and robust performance analysis [14], [15], [16], [17], [18], [19]. The latter synthesis problem is known to be more sophisticated than the former one (i.e. the robust stability problem), and the available design techniques are not concrete, in general. For instance, a method is proposed in [14] to design a near-optimal controller for systems with polytopic uncertainties. However, due to the aforementioned points, this work is based on only some sufficient LMI conditions and may lead to poor control performance. There are some other results in the literature dealing with H_2 or H_∞ robust controller design, which normally suffer from the same weak points, e.g. see [16], [19].

This work deals with LTI discrete-time systems which are uncertain over a semi-algebraic set. Consider a LQ performance index defined over the whole region of uncertainty. It is desired to design a static controller minimizing this cost function. This controller can be a gain-scheduled one, whose gain is a polynomial in the uncertain variables (with a prescribed degree). To this end, it is shown that the closed-loop Lyapunov function is a rational function, which can be approximated by a polynomial matrix satisfying an elegant inequality. An important bound is also derived in order to determine the accuracy of this approximation. A SOS method is then proposed to design the optimal controller. Due to the nonconvexity of the controller design and the uncertainty region, this SOS method may need very high computation. In a situation where an initial stabilizing controller is available, another SOS method is developed whose complexity is strikingly less. The latter approach is only able to design a near-optimal controller; however, as demonstrated in numerical examples, it normally results in huge improvement in the performance of the initial controller. It is worth noting that the idea of designing a *parameter-dependent* static controller in the context of this paper has also been utilized in a number of papers; for instance, see [16] for the filtering application. The results of this paper encompasses the outcome of a recent work [14].

This paper is organized as follows. The problem is formulated in Section II, where some definitions and convenient notations are also introduced. The main results are provided in Section III, followed by two illustrative examples in Section IV. Finally, some concluding remarks are drawn in Section V.

II. Problem formulation

Consider an uncertain discrete-time system $\mathcal{S}(\boldsymbol{\alpha})$ with the following state-space representation:

$$\begin{aligned} x[\kappa+1] &= A(\boldsymbol{\alpha})x[\kappa] + B(\boldsymbol{\alpha})u[\kappa] \\ y[\kappa] &= C(\boldsymbol{\alpha})x[\kappa] \end{aligned} \tag{1}$$

where $\kappa \in \mathbb{Z}$, and:

- $x[\kappa] \in \Re^n, u[\kappa] \in \Re^m$ and $y[\kappa] \in \Re^r$ are the state, the input and the output of $\mathcal{S}(\boldsymbol{\alpha})$, respectively.
- The vector $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_\mu \end{bmatrix}$ accounts for the uncertain parameters of the system.
- $A(\boldsymbol{\alpha}), B(\boldsymbol{\alpha})$ and $C(\boldsymbol{\alpha})$ are matrix polynomials in the variable $\boldsymbol{\alpha}$.

Given the scalar polynomials $q_1(\boldsymbol{\alpha}), q_2(\boldsymbol{\alpha}), ..., q_\eta(\boldsymbol{\alpha})$, define the semi-algebraic set $\mathcal{D}$ as follows:

$$\mathcal{D} := \left\{ \boldsymbol{\alpha} \;\middle|\; q_1(\boldsymbol{\alpha}) \geq 0, ..., q_\eta(\boldsymbol{\alpha}) \geq 0 \right\} \tag{2}$$

The region $\mathcal{D}$ represents the behavior of uncertainty for the system $\mathcal{S}(\boldsymbol{\alpha})$. It can be easily shown that a wide variety of uncertainty regions can be represented as (2).

Definition 1: Consider the system $\mathcal{S}(\boldsymbol{\alpha})$ under a (dynamic or static) LTI controller $\mathbf{K}$. Define the following performance index for the closed-loop system:

$$J(\boldsymbol{\alpha}, \mathbf{K}) = \mathcal{E}\left\{ \sum_{\kappa=0}^{\infty} \left(x[\kappa]^T Q(\boldsymbol{\alpha}) x[\kappa] + u[\kappa]^T R u[\kappa] \right) \right\} \tag{3}$$

for any $\boldsymbol{\alpha} \in \mathcal{D}$, subject to:

$$\mathcal{E}\{x(0)x(0)^T\} = X_0 \tag{4}$$

where R is a fixed positive definite matrix, and $Q(\boldsymbol{\alpha})$ is a matrix polynomial which is positive definite over the region $\mathcal{D}$. Note that $\mathcal{E}\{\cdot\}$ represents the expectation operator, and $X_0 \in \Re^{n\times n}$.

Notice that the performance index $J(\boldsymbol{\alpha}, \mathbf{K})$ is defined in such a way that the state of the system $\mathcal{S}(\boldsymbol{\alpha})$, unlike the input, has a non-constant weight for different operating points $\boldsymbol{\alpha}$ in the region $\mathcal{D}$.

Definition 2: As opposed to $J(\boldsymbol{\alpha}, \mathbf{K})$ which is defined for a given $\boldsymbol{\alpha} \in \mathcal{D}$, define the following performance index for the whole region of uncertainty (associated with the system $\mathcal{S}(\boldsymbol{\alpha})$ and the controller $\mathbf{K}$):

$$J(\mathbf{K}) = \int_{\mathcal{D}} J(\boldsymbol{\alpha}, \mathbf{K}) f(\boldsymbol{\alpha}) d\boldsymbol{\alpha} \tag{5}$$

where $f(\boldsymbol{\alpha})$ is a given weighting function which specifies the relative importance of the performance of the system at any point $\boldsymbol{\alpha} \in \mathcal{D}$ in the overall cost function $J(\mathbf{K})$.

The objective of this work is to design a controller $\mathbf{K}$ for which $J(\mathbf{K})$ is minimized. This controller is required to be a static one of the form $u[\kappa] = K(\boldsymbol{\alpha})y[\kappa]$, where the gain $K(\boldsymbol{\alpha})$ is a matrix polynomial to be obtained and is of degree n_0 (for a given n_0). It is to be noted that in the particular case when the nonnegative integer n_0 is equal to zero, the controller $\mathbf{K}_0$ is a simple constant gain; otherwise, it is a parameter-dependent static controller.

In what follows, a few definitions and mild assumptions will be made, which will prove convenient in developing the results of this work.

Definition 3: A matrix polynomial $C(\boldsymbol{\alpha})$ is said to be sum-of-squares (SOS) if there exists a matrix polynomial $E(\boldsymbol{\alpha})$ such that:

$$C(\boldsymbol{\alpha}) = E^T(\boldsymbol{\alpha})E(\boldsymbol{\alpha}) \tag{6}$$

Definition 4: For a matrix W, define vec$\{W\}$ as a column vector obtained from W by placing its column vectors below each other successively. For example, vec$\{\text{eye}(2)\} = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$.

Notation 1: Let $\|\cdot\|$ denote an arbitrary norm on $\Re^{w_1\times w_2}$. For a given $w_1 \times w_2$ matrix polynomial $C(\boldsymbol{\alpha})$, define the induced norm $\|\cdot\|_e$ on the polynomial $C(\boldsymbol{\alpha})$ to be the maximum of the norm $\|\cdot\|$ of the coefficients of $C(\boldsymbol{\alpha})$. In addition, let the degree of this polynomial be denoted by $\deg(C)$.

Notation 2: For the sake of simplicity, the upper block entries of a symmetric block matrix will be displayed by the symbol "*" throughout the paper.

Notation 3: Given a SOS matrix polynomial $M(\boldsymbol{\alpha})$, it is known that $M(\boldsymbol{\alpha})$ can be expressed as $\Pi_M(\boldsymbol{\alpha})M\Pi_M(\boldsymbol{\alpha})^T$, where M is a constant positive semi-definite matrix and $\Pi_M(\boldsymbol{\alpha})$ is a block row vector of monomials, which any of its block entries is a scalar monomial times an identity matrix of proper dimension. Definite the pairing function $< \Pi_M, M >$ to be $\Pi_M(\boldsymbol{\alpha})M\Pi_M(\boldsymbol{\alpha})^T$. The terms $M(\boldsymbol{\alpha})$ and $< \Pi_M, M >$ may be interchangeably used henceforth. Note that if $M(\boldsymbol{\alpha})$ is not SOS, the functions $< \cdot, \cdot >$ and Π can still be defined in a similar fashion, but the matrix M will no longer be positive semi-definite.

Assumption 1: The set $\mathcal{D}$ is compact, and there exist SOS scalar polynomials $w_0(\boldsymbol{\alpha}), w_1(\boldsymbol{\alpha}), ..., w_\eta(\boldsymbol{\alpha})$, such that the set of all vectors $\boldsymbol{\alpha}$ satisfying the inequality:

$$w_0(\boldsymbol{\alpha}) + w_1(\boldsymbol{\alpha})q_1(\boldsymbol{\alpha}) + \cdots + w_\eta(\boldsymbol{\alpha})q_\eta(\boldsymbol{\alpha}) \geq 0 \quad (7)$$

is compact.

Assumption 2: The region $\mathcal{D}$ is the closure of some open connected set.

It is noteworthy that Assumption 2 may be violated if any of the polynomials $q_1(\boldsymbol{\alpha}), ..., q_\eta(\boldsymbol{\alpha})$ can be factorized over the field of real numbers. In such cases, the region $\mathcal{D}$ can often be partitioned into a number of subregions, each of which satisfies Assumption 2; therefore, the robust stability problem may be required to be addressed more than once, depending on the number of partitioned subregions.

III. LQ OPTIMAL ROBUST CONTROLLER

The problem of minimizing the performance index introduced in (5) with the initial state satisfying (4) will be addressed in this section. It is essential to note that if $J(\mathbf{K})$ is finite for a given controller $\mathbf{K}$, then the system $\mathcal{S}(\boldsymbol{\alpha})$ under the controller $\mathbf{K}$ is robustly stable over the region $\mathcal{D}$ (by virtue of the positive definiteness of $Q(\boldsymbol{\alpha})$). Hence, minimization of $J(\mathbf{K})$ leads to the robust stability of the closed-loop system as well.

Lemma 1: Assume that a controller $\mathbf{K}$ with the control law $u[\kappa] = K(\boldsymbol{\alpha})y[\kappa]$ stabilizes the uncertain system $\mathcal{S}(\boldsymbol{\alpha})$ over the region $\mathcal{D}$. Then, the performance index $J(\mathbf{K})$ satisfies the following equation:

$$J(\mathbf{K}) = \text{trace}\left(X_0 \int_{\mathcal{D}} G(\boldsymbol{\alpha})f(\boldsymbol{\alpha})d\boldsymbol{\alpha}\right) \quad (8)$$

where $G(\boldsymbol{\alpha})$ is the solution of the discrete Lyapunov equation given below:

$$\begin{aligned} &\big(A(\boldsymbol{\alpha}) + B(\boldsymbol{\alpha})K(\boldsymbol{\alpha})C(\boldsymbol{\alpha})\big)^T G(\boldsymbol{\alpha}) \\ &\times \big(A(\boldsymbol{\alpha}) + B(\boldsymbol{\alpha})K(\boldsymbol{\alpha})C(\boldsymbol{\alpha})\big) \\ &- G(\boldsymbol{\alpha}) + Q(\boldsymbol{\alpha}) + C(\boldsymbol{\alpha})^T K(\boldsymbol{\alpha})^T R K(\boldsymbol{\alpha})C(\boldsymbol{\alpha}) = 0 \end{aligned} \quad (9)$$

Proof: It is well-known that $J(\boldsymbol{\alpha}, \mathbf{K})$ can be expressed as trace$(X_0 G(\boldsymbol{\alpha}))$, where $G(\boldsymbol{\alpha})$ satisfies the discrete Lyapunov equation (9) (see [20] for more details). Thus, one can write:

$$\begin{aligned} J(\mathbf{K}) &= \int_{\mathcal{D}} J(\boldsymbol{\alpha}, \mathbf{K})f(\boldsymbol{\alpha})d\boldsymbol{\alpha} \\ &= \text{trace}\left(X_0 \int_{\mathcal{D}} G(\boldsymbol{\alpha})f(\boldsymbol{\alpha})d\boldsymbol{\alpha}\right) \end{aligned}$$

This completes the proof. ■

Lemma 2: Assume that the system $\mathcal{S}(\boldsymbol{\alpha})$ is robustly stable under a static controller $\mathbf{K}$ with the control law $u[\kappa] = K(\boldsymbol{\alpha})y[\kappa]$. There exist a matrix polynomial $H(\boldsymbol{\alpha})$ and a scalar polynomial $h(\boldsymbol{\alpha})$ such that the matrix $G(\boldsymbol{\alpha})$ satisfying the equation (9) can be written as $G(\boldsymbol{\alpha}) = \frac{H(\boldsymbol{\alpha})}{h(\boldsymbol{\alpha})}$, where both of the polynomials $H(\boldsymbol{\alpha})$ and $h(\boldsymbol{\alpha})$ are positive definite over the region $\mathcal{D}$.

Proof: The proof follows from [12], and is omitted here due to space restrictions. ■

Definition 5: Given a positive integer i and a parameter-dependent static controller $\mathbf{K}$ with the polynomial gain $K(\boldsymbol{\alpha})$, consider an optimization problem whose objective is to minimize the function:

$$\text{trace}\left(X_0 \int_{\mathcal{D}} P(\boldsymbol{\alpha})f(\boldsymbol{\alpha})d\boldsymbol{\alpha}\right) \quad (10)$$

for a symmetric matrix polynomial $P(\boldsymbol{\alpha}) \in \Re^{n \times n}$ of degree i, subject to the following constraint:

$$\Gamma(\boldsymbol{\alpha}) := \begin{bmatrix} \Gamma_{11}(\boldsymbol{\alpha}) & * & * \\ \Gamma_{21}(\boldsymbol{\alpha}) & \Gamma_{22}(\boldsymbol{\alpha}) & * \\ \Gamma_{31}(\boldsymbol{\alpha}) & \Gamma_{32}(\boldsymbol{\alpha}) & -I \end{bmatrix} \leq 0, \quad \forall \boldsymbol{\alpha} \in \mathcal{D} \quad (11)$$

where:

$$\begin{aligned} &\Gamma_{11}(\boldsymbol{\alpha}) = -P(\boldsymbol{\alpha}) + Q(\boldsymbol{\alpha}),\ \Gamma_{21}(\boldsymbol{\alpha}) = P(\boldsymbol{\alpha})A(\boldsymbol{\alpha}), \\ &\Gamma_{22}(\boldsymbol{\alpha}) = -P(\boldsymbol{\alpha}) - P(\boldsymbol{\alpha})B(\boldsymbol{\alpha})R^{-1}B(\boldsymbol{\alpha})^T P(\boldsymbol{\alpha}), \\ &\Gamma_{31}(\boldsymbol{\alpha}) = R^{\frac{1}{2}}K(\boldsymbol{\alpha})C(\boldsymbol{\alpha}),\ \Gamma_{32}(\boldsymbol{\alpha}) = R^{-\frac{1}{2}}B(\boldsymbol{\alpha})^T P(\boldsymbol{\alpha}) \end{aligned} \quad (12)$$

Denote the infimum of this minimization problem with $J_i(\mathbf{K})$.

Theorem 1: Consider a static controller $\mathbf{K}$ with the parameter-dependent gain $K(\boldsymbol{\alpha})$. Then, the following statements are true:

i) The infinite sequence $J_1(\mathbf{K}), J_2(\mathbf{K}), J_3(\mathbf{K}), ...$ is non-increasing.
ii) $J_i(\mathbf{K})$ is always greater than or equal to $J(\mathbf{K})$, for any natural number i.

Proof: It is straightforward to substantiate that if the system $\mathcal{S}(\boldsymbol{\alpha})$ is not robustly stable under the controller $\mathbf{K}$, then the elements of the sequence $J_1(\mathbf{K}), J_2(\mathbf{K}), ...$ as well as $J(\mathbf{K})$ are all equal to infinity. Thus, assume that the closed-loop system is robustly stable. Part (i) follows from the fact that the class of matrix polynomials of degree i is a subset of the class of polynomials of degree $i+1$. To prove Part (ii), choose a polynomial $P(\boldsymbol{\alpha})$ of degree i which satisfies the inequality (11). It can be easily concluded by applying the Schur complement formula to the relation (11) that:

$$\begin{aligned} &\begin{bmatrix} \Gamma_{11}(\boldsymbol{\alpha}) & * \\ \Gamma_{21}(\boldsymbol{\alpha}) & \Gamma_{22}(\boldsymbol{\alpha}) \end{bmatrix} - \begin{bmatrix} \Gamma_{31}(\boldsymbol{\alpha})^T \\ \Gamma_{32}(\boldsymbol{\alpha})^T \end{bmatrix} (-I) \\ &\times \begin{bmatrix} \Gamma_{31}(\boldsymbol{\alpha}) & \Gamma_{32}(\boldsymbol{\alpha}) \end{bmatrix} \leq 0, \quad \forall \boldsymbol{\alpha} \in \mathcal{D} \end{aligned}$$

which gives rise to the negative definiteness of the following:

$$\begin{bmatrix} -P(\boldsymbol{\alpha}) + Q(\boldsymbol{\alpha}) + C(\boldsymbol{\alpha})^T K(\boldsymbol{\alpha})^T R K(\boldsymbol{\alpha})C(\boldsymbol{\alpha}) & * \\ P(\boldsymbol{\alpha})\big(A(\boldsymbol{\alpha}) + B(\boldsymbol{\alpha})K(\boldsymbol{\alpha})C(\boldsymbol{\alpha})\big) & -P(\boldsymbol{\alpha}) \end{bmatrix}$$

Now, employing the Schur complement formula one more time leads to:

$$\begin{aligned} &\big(A(\boldsymbol{\alpha}) + B(\boldsymbol{\alpha})K(\boldsymbol{\alpha})C(\boldsymbol{\alpha})\big)^T P(\boldsymbol{\alpha}) \\ &\times \big(A(\boldsymbol{\alpha}) + B(\boldsymbol{\alpha})K(\boldsymbol{\alpha})C(\boldsymbol{\alpha})\big) - P(\boldsymbol{\alpha}) + Q(\boldsymbol{\alpha}) \\ &+ C(\boldsymbol{\alpha})^T K(\boldsymbol{\alpha})^T R K(\boldsymbol{\alpha})C(\boldsymbol{\alpha}) < 0, \quad \forall \boldsymbol{\alpha} \in \mathcal{D} \end{aligned} \quad (13)$$

It follows from the relations (9) and (13) that:

$$\begin{aligned} &\big(A(\boldsymbol{\alpha}) + B(\boldsymbol{\alpha})K(\boldsymbol{\alpha})C(\boldsymbol{\alpha})\big)^T (P(\boldsymbol{\alpha}) - G(\boldsymbol{\alpha})) \\ &\times \big(A(\boldsymbol{\alpha}) + B(\boldsymbol{\alpha})K(\boldsymbol{\alpha})C(\boldsymbol{\alpha})\big) \\ &- (P(\boldsymbol{\alpha}) - G(\boldsymbol{\alpha})) < 0, \quad \forall \boldsymbol{\alpha} \in \mathcal{D} \end{aligned} \quad (14)$$

(note that $G(\boldsymbol{\alpha})$ is defined right after the equation (8)). Since the system $\mathcal{S}(\boldsymbol{\alpha})$ is assumed to be robustly stable under the controller $\mathbf{K}$, it can be inferred from the above inequality that $P(\boldsymbol{\alpha}) - G(\boldsymbol{\alpha})$ is positive definite over the region $\mathcal{D}$. The proof is an immediate consequence of this result and the following facts:

- $J(\mathbf{K})$ is equal to trace $\left(X_0 \int_{\mathcal{D}} G(\boldsymbol{\alpha}) f(\boldsymbol{\alpha}) d\boldsymbol{\alpha}\right)$.
- $J_i(\mathbf{K})$ is equal to the infimum of trace $\left(X_0 \int_{\mathcal{D}} P(\boldsymbol{\alpha}) f(\boldsymbol{\alpha}) d\boldsymbol{\alpha}\right)$ over every $P(\boldsymbol{\alpha})$ satisfying the inequality (11). ■

In the case when the system $\mathcal{S}(\boldsymbol{\alpha})$ is not uncertain and all the system matrix polynomials are of degree 0, it can be inferred from [22] that the relation $J_i(\mathbf{K}) = J(\mathbf{K})$ holds for all positive integers i. In the general case, however, $J(\mathbf{K})$ is obtained from a *rational* function (i.e. $G(\boldsymbol{\alpha})$), whereas $J_i(\mathbf{K}_k)$ depends on a *polynomial* Lyapunov function (i.e. $P(\boldsymbol{\alpha})$). Therefore, it is axiomatic that one should not expect to reach the equality $J_i(\mathbf{K}) = J(\mathbf{K})$, for some integer i. However, an elegant result will be presented next, which aims to address this issue.

Consider $h(\boldsymbol{\alpha})$ and $H(\boldsymbol{\alpha})$ introduced in Lemma 2. It results from the compactness of $\mathcal{D}$ that there exist two positive numbers μ_1 and μ_2 with the property:

$$0 < \mu_1 \leq h(\boldsymbol{\alpha}) \leq \mu_2, \quad \forall \boldsymbol{\alpha} \in \mathcal{D} \tag{15}$$

Define also $\rho(i) := (2i-1)\text{deg}(h) + \text{deg}(H)$, for any natural number i.

Theorem 2: Assume that the system $\mathcal{S}(\boldsymbol{\alpha})$ is robustly stable under a static controller $\mathbf{K}$ with the control law $u[\kappa] = K(\boldsymbol{\alpha}) y[\kappa]$. The subsequence $\{J_{\rho(i)}(\mathbf{K})\}_1^{\infty}$ of the sequence $\{J_i(\mathbf{K})\}_1^{\infty}$ converges exponentially to $J(\mathbf{K})$ from above. More precisely:

$$J(\mathbf{K}) \leq J_{\rho(i)}(\mathbf{K}) \leq \frac{1}{1 - \left(1 - \frac{\mu_1}{\mu_2}\right)^{2i}} J(\mathbf{K}) \tag{16}$$

Proof: Define the following functions:

$$P_i(\boldsymbol{\alpha}) := \frac{H(\boldsymbol{\alpha})}{h(\boldsymbol{\alpha})} \times \frac{1 - \left(1 - \frac{h(\boldsymbol{\alpha})}{\mu_2}\right)^{2i}}{1 - \left(1 - \frac{\mu_1}{\mu_2}\right)^{2i}}, \quad i = 1, 2, ... \tag{17}$$

It is straightforward to show that $\text{deg}(P_i) = \rho(i)$, and that $P_i(\boldsymbol{\alpha})$ is a polynomial (as opposed to a non-polynomial rational function). Moreover, it can be concluded from (15) that:

$$\frac{1 - \left(1 - \frac{h(\boldsymbol{\alpha})}{\mu_2}\right)^{2i}}{1 - \left(1 - \frac{\mu_1}{\mu_2}\right)^{2i}} \geq 1, \quad \forall \boldsymbol{\alpha} \in \mathcal{D} \tag{18}$$

Thus, since $G(\boldsymbol{\alpha}) = \frac{H(\boldsymbol{\alpha})}{h(\boldsymbol{\alpha})}$ satisfies the equation (9), one can write:

$$\begin{aligned}
&\big(A(\boldsymbol{\alpha}) + B(\boldsymbol{\alpha})K(\boldsymbol{\alpha})C(\boldsymbol{\alpha})\big)^T P_i(\boldsymbol{\alpha}) \\
&\times \big(A(\boldsymbol{\alpha}) + B(\boldsymbol{\alpha})K(\boldsymbol{\alpha})C(\boldsymbol{\alpha})\big) - P_i(\boldsymbol{\alpha}) + Q(\boldsymbol{\alpha}) \\
&+ C(\boldsymbol{\alpha})^T K(\boldsymbol{\alpha})^T R K(\boldsymbol{\alpha}) C(\boldsymbol{\alpha}) \\
&= \big(Q(\boldsymbol{\alpha}) + C(\boldsymbol{\alpha})^T K(\boldsymbol{\alpha})^T R K(\boldsymbol{\alpha}) C(\boldsymbol{\alpha})\big) \\
&\times \left(1 - \frac{1 - \left(1 - \frac{h(\boldsymbol{\alpha})}{\mu_2}\right)^{2i}}{1 - \left(1 - \frac{\mu_1}{\mu_2}\right)^{2i}}\right) \leq 0
\end{aligned} \tag{19}$$

for any $\boldsymbol{\alpha} \in \mathcal{D}$. This means that the inequality (13) is satisfied by choosing $P(\boldsymbol{\alpha})$ equal to $P_i(\boldsymbol{\alpha})$, and so is the inequality (11). Therefore, it can be deduced from Definition 5 that:

$$J_{\rho(i)}(\mathbf{K}) \leq \text{trace}\left(X_0 \int_{\mathcal{D}} P_i(\boldsymbol{\alpha}) f(\boldsymbol{\alpha}) d\boldsymbol{\alpha}\right) \tag{20}$$

On the other hand:

$$1 - \left(1 - \frac{h(\boldsymbol{\alpha})}{\mu_2}\right)^{2i} \leq 1 - \left(1 - \frac{\mu_2}{\mu_2}\right)^{2i} = 1, \quad \forall \boldsymbol{\alpha} \in \mathcal{D} \tag{21}$$

The inequality (16) follows from the relations (8), (17), (20) and (21). Moreover, applying the squeezing theorem to the relation (16) yields that $\lim_{i \to +\infty} J_{\rho(i)}(\mathbf{K}) = J(\mathbf{K})$. ■

Remark 1: In order to investigate the tightness of the bound given in (16), consider the case when the Lyapunov function $\frac{H(\boldsymbol{\alpha})}{h(\boldsymbol{\alpha})}$ turns out to be a polynomial. In this case, μ_1 and μ_2 are equal (after cancelling the common factors of $H(\boldsymbol{\alpha})$ and $h(\boldsymbol{\alpha})$, if any). Hence, the relation (16) concludes that $J(\mathbf{K}) = J_{\rho(i)}(\mathbf{K})$, $\forall i > 0$, which together with the monotone property of the sequence $\{J_i(\mathbf{K})\}$ yields that $J_i(\mathbf{K})$ is always equal to $J(\mathbf{K})$. This implies that the bound obtained is tight in the sense that equality can be reached in the special case of polynomial Lyapunov functions.

Remark 2: The result of Theorem 2 can be interpreted as follows: consider an optimization problem aiming at minimizing the cost function (8) subject to the constraint (9) for a rational variable $G(\boldsymbol{\alpha})$ and a matrix polynomial $K(\boldsymbol{\alpha})$. The rational variable $G(\boldsymbol{\alpha})$ can be replaced by a polynomial variable $P(\boldsymbol{\alpha})$, resulting in the replacement of the constraint (9) with (11) such that: (i) the solution of the latter optimization problem converges *from above* to that of the former one as the degree of $P(\boldsymbol{\alpha})$ increases; (ii) this convergence is more or less exponential, roughly speaking. Notice that property (i) could have been deduced from [9], if the constraint (9) were not an equality, or equivalently, if the constraint (11) were a strict inequality. However, this is not true here (one can observe that the results of [9] do not succeed to prove the convergence from above as well as the stability of the closed-loop system). This shows the importance of Theorem 2.

Since the objective function $J(\mathbf{K})$ to be minimized in this work involves an unknown rational function $G(\boldsymbol{\alpha})$ (as a result of Lemma 1), SOS techniques cannot directly be employed to solve the problem. Hence, the problem of minimizing the performance index $J_i(\mathbf{K})$ will be treated at this point for any fixed value of i, as an alternative strategy for addressing the optimal controller design (i.e., minimizing $J(\mathbf{K})$). It is worth mentioning that no matter how small or large the number i is, $J_i(\mathbf{K})$ is an upper bound on $J(\mathbf{K})$. To minimize $J_i(\mathbf{K})$, one should solve a minimization problem in two variables $K(\boldsymbol{\alpha})$ and $P(\boldsymbol{\alpha})$ with the objective function (10) subject to the constraint (11). The main difficulty of this problem is that one of the block entries of the constraint matrix $\Gamma(\boldsymbol{\alpha})$ is a nonlinear function with respect to the coefficients of the matrix polynomial $P(\boldsymbol{\alpha})$. To get rid of this difficulty, two approaches will be presented in the sequel.

A. Exact formulation toward the optimal control design

It follows from Definition 5 and Theorem 2 that the underlying optimal control problem amounts to minimizing the cost function (10) subject to the constraint (11). This constraint can be recast as two relations as follows:

$$\Gamma_d(\boldsymbol{\alpha}) \leq 0, \quad \forall \boldsymbol{\alpha} \in \mathcal{D}, \tag{22a}$$
$$P_d(\boldsymbol{\alpha}) = P(\boldsymbol{\alpha})B(\boldsymbol{\alpha})R^{-1}B(\boldsymbol{\alpha})^T P(\boldsymbol{\alpha}) \tag{22b}$$

where:

$$\Gamma_d(\boldsymbol{\alpha}) := \begin{bmatrix} -P(\boldsymbol{\alpha})+Q(\boldsymbol{\alpha}) & * & * \\ P(\boldsymbol{\alpha})A(\boldsymbol{\alpha}) & -P(\boldsymbol{\alpha})-P_d(\boldsymbol{\alpha}) & * \\ R^{\frac{1}{2}}K(\boldsymbol{\alpha})C(\boldsymbol{\alpha}) & R^{-\frac{1}{2}}B(\boldsymbol{\alpha})^T P(\boldsymbol{\alpha}) & -I \end{bmatrix}$$

Using the matrix version of Putinar's theorem (see Theorem 2 of [24]), one can conclude that the above constraints are tantamount to the existence of some SOS matrix polynomials $M_0(\boldsymbol{\alpha}), M_1(\boldsymbol{\alpha}), ..., M_\eta(\boldsymbol{\alpha})$ subject to:

$$-\Gamma_d(\boldsymbol{\alpha}) = M_0(\boldsymbol{\alpha}) + q_1(\boldsymbol{\alpha})M_1(\boldsymbol{\alpha}) + \cdots + q_\eta(\boldsymbol{\alpha})M_\eta(\boldsymbol{\alpha}),$$
$$P_d(\boldsymbol{\alpha}) = P(\boldsymbol{\alpha})B(\boldsymbol{\alpha})R^{-1}B(\boldsymbol{\alpha})^T P(\boldsymbol{\alpha})$$

The above relations can be expressed in terms of the monomial and the coefficient matrices as follows:

$$-<\Pi_{\Gamma_d}, \Gamma_d> = <\Pi_{M_0}, M_0> + \sum_{i=1}^{\eta} <\Pi_{M_i}, M_i> \cdot <\Pi_{q_i}, q_i> \tag{23a}$$
$$<\Pi_{P_d}, P_d> = <\Pi_P, P> \cdot <\Pi_B, B> R^{-1} \times <\Pi_B, B>^T \cdot <\Pi_P, P> \tag{23b}$$

where the pairing function $< \cdot, \cdot >$ and Π are both introduced in Notation 3, and $M_0, M_1, ..., M_\eta$ are positive semi-definite matrices of proper dimensions. Note that having fixed the monomial matrices $\Pi_P, \Pi_{M_0}, ..., \Pi_{M_\eta}, \Pi_K$, the coefficient matrices $M_0, ..., M_\eta, K, P$ are to be obtained, which leads to a hierarchy of optimization problems (see the discussion following Remark 1 in [13]). It can be observed that the relations given in (23) can be simplified so that only the coefficient matrices are remained, because the equalities hold for every value of $\boldsymbol{\alpha}$. Therefore, assume that the equations in (23) have been rearranged as $\boldsymbol{f}_1(K, P, P_d, M_0, ..., M_\eta) = 0$ and $\boldsymbol{f}_2(P_d) + \boldsymbol{f}_3(P) = 0$, where $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are linear, and $\boldsymbol{f}_3$ is quadratic.

The optimal control problem can now be regarded as the minimization of (10) subject to the following constraints:

$$\begin{aligned} &\boldsymbol{f}_1(K, P, P_d, M_0, ..., M_\eta) = 0, \\ &\boldsymbol{f}_2(P_d) + \boldsymbol{f}_3(P) = 0, \\ &M_i \geq 0, \quad \forall i \in \{0, 1, ..., \mu\} \end{aligned} \tag{24}$$

Observe that the objective function as well as all constraints except for a quadratic term are convex.. Due to the linearity of the underlying optimization in most of the variables (indeed in all variables $M_0, M_1, ..., M_\mu$), this problem is expected to be among the *simplest* NP-hard problems. It is shown in [25] how this category of problems can be recast as constrained optimization of scalar polynomials, for which there exist many SOS approaches to obtain the global solution. Therefore, a hierarchy of SDP problems should be solved to obtain the global solution being sought. It is worth noting that the idea of converting the nonlinear matrix constraints into scalar polynomial constraints has been proved in [25] to work reasonably well.

Nevertheless, this technique clearly requires two sets of hierarchies: one for the utilization of Putinar's theorem in (23) (because of fixing the monomial matrices Π) and one for obtaining the global solution of the constrained polynomial optimization problem. Hence, this method requires high computational burden in general, with the advantage of giving the global solution without having to search a local solution. Now, let an alternative approach be developed which is much faster at the cost of obtaining a near-optimal solution.

B. Descent algorithm for performance improvement

It is desired now to investigate how the proposed technique can be simplified. Assume that $\boldsymbol{\alpha}_0$ is the nominal parameter of the uncertain system $\mathcal{S}(\boldsymbol{\alpha})$. Consider a static controller $\mathbf{K}_0$ stabilizing the nominal system $\mathcal{S}(\boldsymbol{\alpha}_0)$, and denote the corresponding control law with $u[\kappa] = K_0(\boldsymbol{\alpha})y[\kappa]$, where $K_0(\boldsymbol{\alpha})$ is a matrix polynomial of degree n_0. Assume that the controller $\mathbf{K}_0$ is known to robustly stabilize the uncertain system $\mathcal{S}(\boldsymbol{\alpha})$ over the region $\mathcal{D}$ (this hypothesis can, for instance, be verified by using the result of [12]).

The purpose of this part is to tune the coefficients of the polynomial $K_0(\boldsymbol{\alpha})$ in order to arrive at a near-optimal static parameter-dependent controller $\mathbf{K}_k$ with a small performance index $J(\mathbf{K}_k)$.

The celebrated technique of introducing a slack variable [22], [23] will be exploited in the subsequent theorem in order to handle the nonlinear term encountered in the formulation of the preceding section.

Theorem 3: There exist a matrix $K(\boldsymbol{\alpha})$ and a symmetric matrix polynomial $P(\boldsymbol{\alpha})$ satisfying (11) and (12) if and only if there exist a matrix $K(\boldsymbol{\alpha})$ and symmetric matrix polynomials $P(\boldsymbol{\alpha})$ and $\bar{P}(\boldsymbol{\alpha})$ of the same degree with the following property:

$$\bar{\Gamma}(\boldsymbol{\alpha}) := \begin{bmatrix} \Gamma_{11}(\boldsymbol{\alpha}) & * & * \\ \Gamma_{21}(\boldsymbol{\alpha}) & \bar{\Gamma}_{22}(\boldsymbol{\alpha}) & * \\ \Gamma_{31}(\boldsymbol{\alpha}) & \Gamma_{32}(\boldsymbol{\alpha}) & -I \end{bmatrix} \leq 0, \quad \boldsymbol{\alpha} \in \mathcal{D} \tag{25}$$

where:

$$\begin{aligned}\bar{\Gamma}_{22}(\boldsymbol{\alpha}) = & -P(\boldsymbol{\alpha}) - \bar{P}(\boldsymbol{\alpha})B(\boldsymbol{\alpha})R^{-1}B(\boldsymbol{\alpha})^T P(\boldsymbol{\alpha}) \\ & - P(\boldsymbol{\alpha})B(\boldsymbol{\alpha})R^{-1}B(\boldsymbol{\alpha})^T \bar{P}(\boldsymbol{\alpha}) \\ & + \bar{P}(\boldsymbol{\alpha})B(\boldsymbol{\alpha})R^{-1}B(\boldsymbol{\alpha})^T \bar{P}(\boldsymbol{\alpha})\end{aligned} \quad (26)$$

Proof: For any arbitrary matrix $\bar{P}(\boldsymbol{\alpha})$, one can write:

$$\left(P(\boldsymbol{\alpha}) - \bar{P}(\boldsymbol{\alpha})\right) B(\boldsymbol{\alpha})R^{-1}B(\boldsymbol{\alpha})^T \left(P(\boldsymbol{\alpha}) - \bar{P}(\boldsymbol{\alpha})\right) \geq 0 \quad (27)$$

The proof is primarily contingent upon this inequality (by expanding its left side). Since the proof can be carried out in a way similar to the proof of Theorem 3 in [23] for systems with known parameters, the details are omitted here. ■

The advantage of $\bar{\Gamma}(\boldsymbol{\alpha})$ over $\Gamma(\boldsymbol{\alpha})$ is that all its entries are linear, provided the slack variable $\bar{P}(\boldsymbol{\alpha})$ is fixed. Furthermore, Putinar's theorem can be applied to the constraint (25) in order to convert its feasibility to a SOS problem [24]. This concept leads to the following lemma.

Lemma 3: Given a matrix polynomial $\bar{P}(\boldsymbol{\alpha})$, there exist matrix polynomials $K(\boldsymbol{\alpha})$ and $P(\boldsymbol{\alpha})$ for which the inequality (25) is satisfied if and only if there exist SOS matrix Polynomials $Q_0(\boldsymbol{\alpha}), ..., Q_\eta(\boldsymbol{\alpha})$, in addition to $P(\boldsymbol{\alpha})$, such that:

$$\bar{\Gamma}(\boldsymbol{\alpha}) = -Q_0(\boldsymbol{\alpha}) - \sum_{i=1}^{\eta} q_i(\boldsymbol{\alpha})Q_i(\boldsymbol{\alpha}), \quad \forall \boldsymbol{\alpha} \in \Re^\mu \quad (28)$$

Proof: The proof follows immediately from Theorem 2 of [24], and on noting that the only requirement of this theorem is that Assumption 1 holds. ■

The subsequent algorithm takes advantage of all the theorems and lemmas presented so far, in order to delineate a systematic procedure for addressing the objective of this subsection.

Algorithm 1:

Step 1) Find a Lyapunov polynomial $P_0(\boldsymbol{\alpha})$ for which there exist a positive scalar ε and SOS matrix polynomials $Q_0(\boldsymbol{\alpha}), ..., Q_\eta(\boldsymbol{\alpha})$ so that the equation (29) (given in the next page) holds for all $\boldsymbol{\alpha} \in \Re^\mu$:
Step 2) Set $\bar{P}(\boldsymbol{\alpha}) = P_0(\boldsymbol{\alpha})$.
Step 3) Choose a natural number i greater than or equal to deg(P_0).
Step 4) Solve a LMI optimization problem with the variables:

- a matrix polynomial $K(\boldsymbol{\alpha})$ of degree n_0;
- SOS matrix Polynomials $Q_0(\boldsymbol{\alpha}), ..., Q_\eta(\boldsymbol{\alpha})$;
- a matrix polynomial $P(\boldsymbol{\alpha})$ of degree i;

to minimize the linear objective function:

$$\text{trace}\left(X_0 \int_{\mathcal{D}} P(\boldsymbol{\alpha})f(\boldsymbol{\alpha})d\boldsymbol{\alpha}\right) \quad (30)$$

under the SOS constraint:

$$\bar{\Gamma}(\boldsymbol{\alpha}) = -Q_0(\boldsymbol{\alpha}) - \sum_{i=1}^{\eta} q_i(\boldsymbol{\alpha})Q_i(\boldsymbol{\alpha}) \quad (31)$$

where the block entries of the matrix $\bar{\Gamma}(\boldsymbol{\alpha})$ are defined in (12) and (26), as noted in (25).

Step 5) If $\|P(\boldsymbol{\alpha}) - \bar{P}(\boldsymbol{\alpha})\|_c \leq \delta$, where δ is a prescribed error margin (which is chosen in line with the design specifications), go to Step 7.
Step 6) Set $\bar{P}(\boldsymbol{\alpha})$ to $P(\boldsymbol{\alpha})$, where $P(\boldsymbol{\alpha})$ is obtained in Step 4. Go to Step 4.
Step 7) The cost function $J_i(\mathbf{K}_k)$ is sufficiently close to a (local) solution. The value obtained for $K(\boldsymbol{\alpha})$ corresponds to the parameters of a static (locally) optimal robust controller.

Remark 3: The reason why such a function $P_0(\boldsymbol{\alpha})$ satisfying the condition in Step 1 of this algorithm exists is thoroughly explained in [12].

Remark 4: The problem tackled here is investigated in [14] for polytopic systems. The method developed therein relies on approximating the rational Lyapunov function by a first-order polynomial, which in turn may be extremely conservative. In any case, it can be observed that by making several relaxations in this particular case, the SOS formula given here encompasses the one obtained in [14].

IV. Numerical Examples

Example 1: Consider a second-order system $\mathcal{S}(\alpha)$ with the following state-space matrices:

$$A(\alpha) = \begin{bmatrix} 0.6 & 0 \\ -0.1 & 0.4 \end{bmatrix}, \; B(\alpha) = \begin{bmatrix} -0.16 & 0.2 \\ 0 & -0.04 \end{bmatrix}, \\ C(\alpha) = \begin{bmatrix} 0.25 & 1.25 \\ 0 & -1 \end{bmatrix} (\alpha^2 - \alpha + 1) \quad (32)$$

In this system, the matrices $A(\alpha)$ and $B(\alpha)$ are constant, while the matrix $C(\alpha)$ is a function of the uncertain scalar variable α. Define the normalized uncertainty region $\mathcal{D}$ as:

$$\mathcal{D} = \left\{\alpha : 1 - \alpha^2 \geq 0\right\} \quad (33)$$

Using the method given in [12], it is straightforward to show that an initial static controller $\mathbf{K}_0$ with the simple control law $u[\kappa] = y[\kappa]$ robustly stabilizes this uncertain system. However, this controller does not necessarily result in a good performance for the control system. Thus, the main objective here is to tune the parameters of this static controller in order to improve the performance of the system over the uncertainty region. To this end, consider the performance index (5), and let:

$$X_0 = I_2, \quad Q(\alpha) = R = I_2, \quad f(\alpha) = 1 \quad \forall \alpha \in \mathcal{D} \quad (34)$$

Algorithm 1 will be exploited now to design an optimal robust controller $\mathbf{K}_k$. Step 1 of this algorithm arrives at the initial Lyapunov function $P_0(\alpha)$ as follows:

$$P_0(\alpha) = \begin{bmatrix} P_{11}(\alpha) & * \\ P_{21}(\alpha) & P_{22}(\alpha) \end{bmatrix} \quad (35)$$

where:

$$\begin{aligned} P_{11}(\alpha) &= 1.725 - 0.4352\alpha + 0.5751\alpha^2, \\ P_{21}(\alpha) &= -0.4318 + 0.4358\alpha - 0.5331\alpha^2, \\ P_{22}(\alpha) &= 6.258 - 14.97\alpha + 19.55\alpha^2 \end{aligned} \quad (36)$$

$$\begin{bmatrix} P_0(\boldsymbol{\alpha}) & (A(\boldsymbol{\alpha})+B(\boldsymbol{\alpha})K_0(\boldsymbol{\alpha})C(\boldsymbol{\alpha}))^T P_0(\boldsymbol{\alpha}) \\ P_0(\boldsymbol{\alpha})(A(\boldsymbol{\alpha})+B(\boldsymbol{\alpha})K_0(\boldsymbol{\alpha})C(\boldsymbol{\alpha})) & P_0(\boldsymbol{\alpha}) \end{bmatrix} = Q_0(\boldsymbol{\alpha}) + \sum_{i=1}^{\eta} q_i(\boldsymbol{\alpha})Q_i(\boldsymbol{\alpha}) + \varepsilon I \quad (29)$$

Since the order of $P_0(\alpha)$ is equal to 2, the integer i in Step 2 should be chosen greater than or equal to 2. As the smallest acceptable value, let i be equal to 2. Moreover, set the required precision for the optimal solution to $\delta = 10^{-4}$ (which is used in Step 5). To solve the optimization problem in Step 4 of Algorithm 1, let the degrees of the polynomials $Q_0(\alpha)$ and $Q_1(\alpha)$ be equal to 4 and 2, respectively. One can simply verify that the initial performance index $J_i(\mathbf{K}_0)$ is equal to 29.3820. Denote with j the number of times that Step 4 of the algorithm is carried out recursively. The performance indices $J_i(\mathbf{K})$ obtained for $j = 0, 1, \ldots, 5$ are tabulated in the second row of Table I. As can be observed from these results, the algorithm converges to the optimal value 5.4550 after only three iterations. Moreover, it takes less than 1 second of CPU time in each iteration to solve the SOS optimization problem in Step 4. The resultant optimal controller is:

$$u[\kappa] = \begin{bmatrix} 0.2725 & 0.3423 \\ -0.3524 & -0.4520 \end{bmatrix} y[\kappa] \quad (37)$$

which corresponds to the Lyapunov function:

$$P(\alpha) = \begin{bmatrix} P_{11}(\alpha) & * \\ P_{21}(\alpha) & P_{22}(\alpha) \end{bmatrix} \quad (38)$$

with:

$$\begin{aligned} P_{11}(\alpha) &= 1.535 + 0.0111\alpha + 0.0084\alpha^2, \\ P_{21}(\alpha) &= -0.0584 - 0.0007\alpha - 0.0008\alpha^2, \\ P_{22}(\alpha) &= 1.1900 + 0.0001\alpha + 0.0008\alpha^2 \end{aligned} \quad (39)$$

This Means that after the third iteration (i.e., solving the SOS problem in Step 4 of Algorithm 1 three successive times), the objective function $J_i(\mathbf{K})$ has been improved remarkably from 29.3820 to 5.4550. To illustrate the improvement in the performance index $J(\mathbf{K})$ (rather than $J_i(\mathbf{K})$), let the interval $[-1, 1]$ be divided into 400 equidistant points. Now, one can approximate the integral in the performance index (5) with a sum including 400 terms. In this case, it can be concluded that $J(\mathbf{K}_0)$ and $J(\mathbf{K}_k)$ are approximately equal to 23.6758 and 5.4346, respectively. These numbers demonstrate a significant improvement in the performance of the control system. It is to be noted that $J(\mathbf{K}_0) < J_i(\mathbf{K}_0)$ and $J(\mathbf{K}_k) < J_i(\mathbf{K}_k)$, which confirm the result of Theorem 1. Furthermore, it can be easily observed in this example that the property:

$$J(\mathbf{K}_k) \simeq J_i(\mathbf{K}_k) \quad (40)$$

holds for the fine-tuned controller $\mathbf{K}_k$. It is worth mentioning that although there is a noticeable discrepancy between $J(\mathbf{K}_0)$ and $J_i(\mathbf{K}_0)$ as a result of choosing a small i, Algorithm 1 still arrives at a controller which is satisfactorily close to a locally optimal one.

It is desired now to demonstrate the effectiveness of a *parameter-dependent* static controller in a set-up where the uncertain parameter α can be measured on-line and be used in the controller accordingly. For this purpose, consider two different degrees 1 and 2 for the polynomial gain of the static controller $\mathbf{K}_k$. The results of Algorithm 1 for these two cases are given in rows 3 and 4 of Table I. It can be observed from these results that an optimal parameter-dependent static controller outperforms its constant static counterpart, as expected.

Example 2: Let $\mathcal{S}$ be an uncertain fourth-order system with the state-space matrices $A = \alpha_1\alpha_2^2\tilde{A},\ B = \alpha_1\tilde{B} + \alpha_2^2 I_4$ and $C = \alpha_2^2\tilde{C} + \alpha_1 I_4$, where

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} -0.6957 & 0.2674 & -0.0031 & -0.0257 \\ 0.6457 & -0.3634 & 0.0019 & -0.1042 \\ 0.1407 & 0.5206 & -0.8307 & -0.2681 \\ 0.0797 & -0.0139 & 0.0197 & -0.9133 \end{bmatrix}, \\ \tilde{B} &= \begin{bmatrix} 0.4518 & -0.2859 & -0.0059 & 0.0861 \\ 0.8478 & -0.6839 & -0.0039 & 0.0889 \\ 0.5211 & -0.1206 & -0.2406 & -0.7968 \\ 0.2219 & -0.1109 & -0.1109 & 0.2407 \end{bmatrix}, \\ \tilde{C} &= \begin{bmatrix} -0.3790 & -0.2632 & 0.0073 & 0.0076 \\ 0.4781 & 0.6674 & 0.0224 & -0.1912 \\ -0.5010 & -0.2988 & 0.3827 & 0.2004 \\ 0.0076 & 0.4705 & 0.0983 & -0.3630 \end{bmatrix} \end{aligned} \quad (41)$$

and where α_1 and α_2 are the uncertain variables belonging to the region $\mathcal{D} = \{(\alpha_1, \alpha_2) | \alpha_1^2 + \alpha_2^2 = 1\}$. It can be verified that the system $\mathcal{S}$ is stable under the unity feedback controller (this can be deduced from Example 1 in [12], after some manipulations). Consider the cost function (5) with the parameters $X_0 = I_2,\ Q(\alpha) = R = I_2,\ f(\boldsymbol{\alpha}) = 1,\ \forall \boldsymbol{\alpha} \in \mathcal{D}$. Algorithm 1 is utilized to improve the performance of the unity feedback controller, by setting the degree of $P(\boldsymbol{\alpha})$ to 2. The results obtained are summarized in Table II. One can observe that a noticeable improvement has been achieved after only one iteration, and that the algorithm has converged in 10 iterations, resulting in $\%57$ improvement in the control performance. The optimal controller obtained is as follows:

$$u[\kappa] = \begin{bmatrix} 0.1341 & -0.0332 & 0.0048 & 0.0361 \\ -0.0884 & 0.0323 & 0.0341 & 0.0096 \\ -0.0739 & -0.0421 & 0.1458 & 0.0837 \\ 0.0060 & -0.0338 & -0.0273 & 0.1787 \end{bmatrix} y[\kappa] \quad (42)$$

On the other hand, one can employ the grid technique proposed in Example 1 to obtain:

$$\begin{aligned} J(\mathbf{K}_0) = 16.0996, \quad J_2(\mathbf{K}_0) = 20.5719, \\ J(\mathbf{K}_k) = 8.4898, \quad J_2(\mathbf{K}_k) = 8.8265 \end{aligned} \quad (43)$$

These values point out that although the approximation of the rational Lyapunov function with a polynomial of degree 2 is not acceptable for the initial controller, the algorithm works remarkably well.

TABLE I

THE RESULTS OBTAINED BY USING ALGORITHM 1 FOR DESIGNING AN OPTIMAL PARAMETER-DEPENDENT STATIC CONTROLLER

	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
static gain of degree 0	29.3820	6.7712	5.4733	5.4550	5.4550	5.4550
static gain of degree 1	29.3820	6.7440	5.4265	5.4079	5.4079	5.4079
static gain of degree 2	29.3820	6.7430	5.4246	5.4059	5.4059	5.4059

TABLE II

THE RESULTS OBTAINED BY USING ALGORITHM 1 FOR EXAMPLE 2

	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$
static gain of degree 0	20.5719	13.1489	10.4558	9.4193	9.0164	8.8760	8.8407	8.8312	8.8277	8.8268	8.8265

V. CONCLUSIONS

This paper deals with high-performance robust control synthesis for uncertain discrete-time LTI systems whose state-space matrices are polynomials in term of the uncertainty variables. Given a pre-defined LQ performance index associated with the uncertainty region, it is aimed to design a LQ robust optimal static controller, whose gain can be a polynomial with respect to the uncertain variables. An exact formulation for this problem is explored, which requires solving two hierarchies of SDP problems due to the non-convexity of the original problem. In the case when an initial robust stabilizing controller is available, it is shown how another approach can be developed based on the same techniques, which is noticeably simple (compared to the case when there is no initial controller). Two illustrative examples are provided to shed light on the main contribution of the paper.

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