Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Finite Horizon Optimal Memoryless Control of a Delay in Gaussian Noise: A Simple Counterexample

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Abstract— **In this paper, we investigate control strategies for a scalar, one-step delay system in discrete time, i.e., the system's state is the input delayed by one time unit. We allow control policies that are memoryless functions of noisy measurements of the state of the delay system. We adopt a first order state-space representation for the delay system, where the initial state of the system is a Gaussian and zero mean random variable. In addition, we assume that the measurement noise is drawn from a white and Gaussian process with zero mean and constant variance. Performance evaluation is carried out via a finite-time quadratic cost that combines the second moment of the control signal, and the second moment of the difference between the initial state and the state at the final time. We show that if the time-horizon is one or two then the optimal control is a linear function of the plant's output, while for a sufficiently large horizon a control taking only two values will outperform the optimal linear solution. This paper complements the well known counterexample by Hans Witsenhausen, which showed that the solution to a linear, quadratic and Gaussian optimal control paradigm might be nonlinear. Witsenhausen's counterexample considered an optimization horizon with two time-steps (two stage control). In contrast to Witsenhausen's work, the solution to our counterexample is linear for one and two stages but it becomes nonlinear as the number of stages is increased. Existing tests for linearity of the optimal memoryless control consider only the two-stage problem. The fact that our paradigm leads to non-linear solutions, in the multi-stage case, could not be predicted from prior results. In particular, the fact that the optimal solution for the two stage problem is linear, but the multiple stage might not be, also shows that dynamic programming principles cannot be used for our paradigm. Our paper provides analytical proofs which hold for any number of stages.**

Index Terms— **Decentralized noise cancellation, limited information, estimation**

I. INTRODUCTION

Consider the following discrete-time delay system:

$$X(k+1) = U(k),\ k \geq 0 \tag{1}$$

$$Y(k) = X(k) + V(k),\ k \geq 0 \tag{2}$$

where $W(k)$, $U(k)$, $X(k)$, and $Y(k)$ take values on the reals, and they represent the measurement noise, input, state, and output of the plant, respectively. In addition, we assume that the initial state $X(0)$ is a Gaussian random variable, with zero mean and variance σ_0^2. The measurement noise $\{V(k)\}_{k=0}^{\infty}$ is white, Gaussian, zero mean and with constant variance given by σ_V^2.

This work was supported by NSF Award EECS 0644764 CAREER and AFOSR Award FA95500810120

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In this paper, we will investigate the following problem:

Problem 1.1: Let a positive integer m, and positive real constants σ_0^2 and σ_V^2 be given. Consider that the system described by (1)-(2) accepts a control strategy of the following form:

$$U(k) = \mathcal{F}_k(Y(k)),\ k \in \{0, \dots, m-2\} \tag{3}$$

$$X(m) = \mathcal{F}_{m-1}\left(Y(m-1)\right) \tag{4}$$

where, for each k in the set $\{0, \dots, m-1\}$, $\mathcal{F}_k : \mathbb{R} \rightarrow \mathbb{R}$ is a Lebesgue measurable function. Given a positive real parameter ϱ, we wish to select Lebesgue measurable functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ that minimize the following cost:

$$\mathcal{J}(\{\mathcal{F}_k\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2) \stackrel{def}{=} E[(X(m) - X(0))^2] + \varrho \sum_{k=0}^{m-2} E[U(k)^2] \tag{5}$$

Notice that Problem 1.1 can be viewed as an optimal control problem aimed at the design of a memory element capable of storing a zero mean Gaussian random variable $X(0)$ over multiple time-steps. The memory element must be constructed using a one-step delay and memoryless components $\{\mathcal{F}_k\}_{k=0}^{m-1}$, which are used in a feedback configuration. In addition, the memoryless control has access to noisy measurements of the delay's state. Minimizing the cost function defined in (5) amounts to finding the minimal energy memoryless control that leads to the optimal recovery of $X(0)$ from $Y(m-1)$, in a mean square sense. Our paradigm is identical to the problem of conservation of analog recodings. Indeed if the stored values are subject to noise due to degradation over time, then the problem of what should be recorded in successive copies, so as to increase fidelity, is exactly what we adress here.

Paper organization and overview of main results: The following is the organization of this paper (introduction not included):

- In **Section II** , we derive the optimal solution to Problem 1.1, subject to the constraint that the feedback functions $\{\mathcal{L}_k\}_{k=0}^{m-1}$ are affine. We also show that if m is one or two then affine solutions are optimal.
- In **Section III**, we adopt a class of functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ that take on only two values. Given σ_0 and σ_V, we show that there exist m for which the aforementioned class of functions outperforms the best optimal affine solution and we provide also numerical examples.
- In **Section IV** , we discuss conclusions and open issues.
- The paper ends with **Section V** , where we provide proofs of Lemmas presented in the paper.

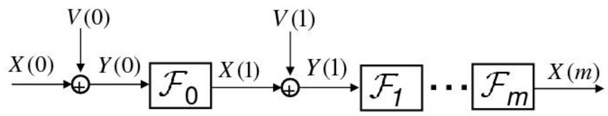

Fig. 1. Alternative interpretation to Problem 1.1.

A. Comparison with related work

The paradigm described in Problem 1.1 is a linear quadratic and Gaussian optimal control problem. We show that, for up to two stages ($m \in \{1, 2\}$), an optimal solution is attained via affine memoryless control. However, affine solutions are not optimal for all m. In fact, we show that a memoryless control strategy taking on only two values may outperform the optimum, within the class of memoryless affine controllers. The fact that a memoryless policy taking on only two values outperforms the best afine control shows that, for m sufficiently large, the optimal solution to Problem 1.1 is nonlinear. In fact, for m larger than two, we do not know the optimal solution to Problem 1.1. This is not surprising, since the similar two stage problem suggested by Hans Witsenhausen [1] remains open after exactly forty years after its publications. Moreover, it was reported in [6] that the discretized version of Witsenhausen's counter-example is NP-complete. This fact has motivated the numerical studies in [7], [8], [9], [10].

The work in [2], [3], considered the case where a linear information pattern is defined by a directed graph. Using the notion of partially nested information structure, the authors of [2], [3] characterize when the optimal solution can be found, while bounds are derived when the optimal is unknown. In [18], it is shown that, if the induced norm is used then, the linear controllers are optimal.

II. Optimal affine memoryless control

In this section, we find the solution of Problem 1.1 with the constraint that the functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ are affine. The section starts with Proposition 2.1, which solves a special case Problem 1.1, with the initial noise dropped, when the number of stages is equal to 2. Lemma 2.2 solves Problem 1.1 given that the functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ are affine, as a problem with constraints and gives the optimal affine functions, which solve Problem 1.1. Lemmas 2.3 and 2.4 are supporting results for Lemma 2.2. The main result of the section is given in Theorem 2.5, in which the optimal cost of Problem 1.1 is computed under the constraint that the functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ are affine. In subsection II-A, we show that for the number of stages equal to 2, the affine functions solve Problem 1.1 among all measurable functions.

Proposition 2.1: For σ_X^2 and σ_W^2 positive numbers, let X and W be zero mean Gaussian random variables with variance σ_X^2 and σ_W^2 respectively. For a positive real number σ^2, define the optimal cost:

$$J^* \stackrel{def}{=} \min_{\mathcal{G}_0,\mathcal{G}_1} E\left[(X - Z(1))^2\right] \tag{6}$$

$$\text{s.t.} E\left[Z(0)^2\right] \leq \sigma^2 \tag{7}$$

where the search space of $\mathcal{G}_0$ and $\mathcal{G}_1$ is the space of Lebesque measurable functions and $Z(0)$ and $Z(1)$ are random variables defined as $Z(0) \stackrel{def}{=} \mathcal{G}_0(X)$ and $Z(1) \stackrel{def}{=} \mathcal{G}_1(Z(0) + W)$ respectively. Then the following hold:

$$J^* = \sigma_X^2 \left(1 - \frac{\sigma^2}{\sigma^2 + \sigma_W^2}\right)$$

and the functions $\mathcal{G}_0^*$ and $\mathcal{G}_1^*$, which minimize the cost, are linear and

$$\mathcal{G}_0^*(x) = \frac{\sigma}{\sigma_X}x \text{ and } \mathcal{G}_1^*(x) = \frac{\sigma_X \cdot \sigma}{\sigma^2 + \sigma_W^2}x$$

or

$$\mathcal{G}_0^*(x) = -\frac{\sigma}{\sigma_X}x \text{ and } \mathcal{G}_1^*(x) = -\frac{\sigma_X \cdot \sigma}{\sigma^2 + \sigma_W^2}x$$

Proof: See Appendix. We note that a similar problem was solved by Bansal and Basar in [7] although their problem had no constraint, but the cost function consisted from sum of the cost function in this proposition and the constraint multiplied by some positive constant. Our proof was inspired by the proof in [7]. ■

Lemma 2.2: Let all parameters defining Problem 1.1 be given. Adopt the following class of affine memoryless control strategies:

$$U(k) = \lambda(k)Y(k) + \beta(k), \ k \in \{0, \ldots, m-2\} \tag{8}$$

$$X(m) = \lambda(m-1)Y(m-1) + \beta(m-1) \tag{9}$$

where $\{\lambda(k)\}_{k=0}^{m-1}$ and $\{\beta(k)\}_{k=0}^{m-1}$ are real numbers. In addition, consider the following cost:

$$\mathcal{C}_A\left(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2\right) \stackrel{def}{=} E[(X(m) - X(0))^2] \tag{10}$$

which must be computed with the control (8) applied to (1)-(2). Given real a positive constant γ, define the following optimal cost:

$$\mathcal{C}_A^*\left(m, \sigma_0^2, \sigma_V^2\right) \stackrel{def}{=} \min_{\{(\lambda(k),\beta(k))\}_{k=0}^{m-1}} \mathcal{C}_A\left(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2\right) \tag{11}$$

$$\text{s.t.} \quad \sum_{k=0}^{m-2} E[U(k)^2] \leq (m-1)\gamma\sigma_V^2 \tag{12}$$

The following holds:

$$\mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) = \sigma_0^2\left(1 - \frac{\sigma_0^2}{\sigma_0^2 + \sigma_V^2}\frac{\gamma^{m-1}}{(1+\gamma)^{m-1}}\right) \tag{13}$$

In order to prove the Lemma 2.2, we need the following two lemmas.

Lemma 2.3: Let all parameters and cost function defining Lemma 2.2 be given. Given the positive numbers $\left\{\sigma_i^2\right\}_{i=1}^{m-1}$, define the optimal cost:

$$\mathcal{C}_\sigma^*\left(m, \sigma_0^2, \sigma_V^2, \left\{\sigma_i^2\right\}_{i=1}^{m-1}\right) \stackrel{def}{=} \min_{\{(\lambda(k), \beta(k))\}_{k=0}^{m-1}} \mathcal{C}_A\left(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2\right) \tag{14}$$

$$\text{s.t.} \quad E[U(k)^2] = \sigma_{k+1}^2, \quad k \in \{0 \ldots m-2\} \tag{15}$$

Then the following holds:

$$\mathcal{C}_\sigma^*\left(m, \sigma_0^2, \sigma_V^2, \left\{\sigma_i^2\right\}_{i=1}^{m-1}\right) = \sigma_0^2\left(1 - \prod_{i=0}^{m-1} \frac{\sigma_i^2}{\sigma_i^2 + \sigma_V^2}\right) \tag{16}$$

$$E[X(m)^2] = \sigma_0^2 \prod_{i=0}^{m-1} \frac{\sigma_i^2}{\sigma_i^2 + \sigma_V^2}$$

and the optimum is reached by selecting the following affine functions:

$$\beta(k) = 0, k \in \{0 \ldots m-1\} \tag{17}$$

$$\lambda(k) = \sqrt{\frac{\sigma_{k+1}^2}{\sigma_k^2 + \sigma_V^2}}, \quad k \in \{0 \ldots m-2\} \tag{18}$$

$$\lambda(m-1) = \prod_{i=0}^{m-1} \sqrt{\frac{\sigma_i^2}{\sigma_i^2 + \sigma_V^2}} \cdot \sqrt{\frac{\sigma_0^2}{\sigma_{m-1}^2 + \sigma_V^2}} \tag{19}$$

Proof: See Appendix ■

Lemma 2.4: Let $\{\alpha_i\}_{i=1}^n$ be positive real numbers. Consider the following cost function:

$$C\left(\{\alpha_i\}_{i=1}^n\right) \stackrel{def}{=} \prod_{i=1}^n \frac{\alpha_i}{1 + \alpha_i}$$

Given a positive real number P, define the following optimal cost:

$$\mathcal{C}^* \stackrel{def}{=} \max_{\{\alpha_i\}_{i=1}^n} C\left(\{\alpha_i\}_{i=1}^n\right)$$

$$\text{subject to:} \quad \begin{array}{l} \sum_{i=1}^n \alpha_i \leq P \\ \alpha_i \geq 0, i \in \{1, \ldots n\} \end{array}$$

Then the following hold:

$$\mathcal{C}^* = \frac{\left(\frac{P}{n}\right)^n}{\left(1 + \frac{P}{n}\right)^n}$$

$$\alpha_i^* = \frac{P}{n}, i \in \{1, \ldots, n\}$$

where $\{\alpha_i^*\}_{i=1}^n$ are the optimal values of $\{\alpha_i\}_{i=1}^n$ for which the problem is solved.

Proof: See Appendix ■

Proof: Lemma 2.2 The initial optimization problem :

$$\mathcal{C}_A^*\left(m, \sigma_0^2, \sigma_V^2\right) \stackrel{def}{=} \min_{\{(\lambda(k), \beta(k))\}_{k=0}^{m-1}} \mathcal{C}_A\left(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2\right)$$

$$\sum_{k=0}^{m-2} E[U(k)^2] \leq (m-1)\gamma\sigma_V^2$$

is equivalent to the following optimization problem:

$$\mathcal{C}_A^*\left(m, \sigma_0^2, \sigma_V^2\right) \stackrel{def}{=} \min_{\left\{\sigma_i^2\right\}_{i=1}^{m-1}} \mathcal{C}_\sigma^*\left(m, \sigma_0^2, \sigma_V^2, \left\{\sigma_i^2\right\}_{i=1}^n\right)$$

$$\text{s.t.} \sum_{i=1}^{m-1} \sigma_i^2 \leq (m-1)\gamma\sigma_V^2$$

Taking into consideration that σ_i^2's are the variances of some random variables so they must be positive, the results of Lemma 2.2 follow directly from Lemma 2.3 and Lemma 2.4. ■

Theorem 2.5: Let all parameters defining Problem 1.1 be given, with m, the number of stages greater than 3. Denote by $\mathcal{J}_A^*\left(m, \sigma_0^2, \sigma_V^2\right)$ as the solution to Problem 1.1 subject to affine strategies of the form (8). The following statements hold:

$$\mathcal{J}_A^*\left(m, \sigma_0^2, \sigma_V^2\right) = \min_{\gamma \geq 0} \mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) + (m-1)\varrho\gamma\sigma_V^2 \tag{20}$$

where $\mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right)$ is given by (13). Consider the following conditions:

(a) $\varrho = \frac{\sigma_0^4}{(\sigma_0^2 + \sigma_V^2)\sigma_V^2} \frac{\gamma^{m-2}}{(1+\gamma)^m}$
(b) $m < \gamma + 2$

If ϱ and m are chosen such that there exists a positive real number $\gamma_{\varrho,m}$, such that conditions (a) and (b) are satisfied, then $\gamma_{\varrho,m}$ minimizes the problem from equation (20), otherwise $\gamma = 0$ minimizes the problem in equation (20).

Proof: The first statement of the theorem , i.e. equation 20, is immediate. From Lemma 2.2, we know that:

$$\mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) = \sigma_0^2\left(1 - \frac{\sigma_0^2}{\sigma_0^2 + \sigma_V^2} \frac{\gamma^{m-1}}{(1+\gamma)^{m-1}}\right)$$

Define:

$$f(\gamma) \stackrel{def}{=} \mathcal{C}_A^*\left(m, \gamma, \sigma_0^2, \sigma_V^2\right) + \varrho(m-1)\gamma\sigma_V^2$$

Then, the derivative of the function $f(\gamma)$ is:

$$\frac{\partial f(\gamma)}{\partial \gamma} = -(m-1)\frac{\sigma_0^4}{(\sigma_0^2 + \sigma_V^2)} \frac{\gamma^{m-2}}{(1+\gamma)^m} + \varrho(m-1)\sigma_V^2$$

The function $\frac{\gamma^{m-2}}{(1+\gamma)^m}$ has a single stationary point, which is a point of maximum at $\frac{m-2}{2}$, for $\gamma > 0$. This means that $\frac{\partial f}{\partial \gamma}$ is decreasing until $\gamma = \frac{m-2}{2}$ and increasing afterwords. We note that $\lim_{\gamma \to \infty} \frac{\gamma^{m-2}}{(1+\gamma)^m} = 0$, and also $\frac{\gamma^{m-2}}{(1+\gamma)^m} = 0$, for $\gamma = 0$. If exists a positive γ_1, such that $\frac{\partial f}{\partial \gamma}(\gamma_1) = 0$, then there exists a γ_2 such that $\frac{\partial f}{\partial \gamma}(\gamma_2) = 0$, with $\gamma_1 = \gamma_2$ if and only if they are both equal to $\frac{m-2}{2}$. Let $\gamma_1 \leq \frac{m-2}{2}$ and $\gamma_2 \geq \frac{m-2}{2}$. Then, on $[0, \gamma_1]$, the function $f(\gamma)$ is increasing, since its derivative is positive, on $[\gamma_1, \gamma_2]$, f is decresing, and on $[\gamma_2, \infty)$, f is increasing. This means that the function $f(\gamma)$ has 2 points of local minimum, one for $\gamma = 0$ and the second one is $\gamma = \gamma_2$, so in order to compute the minimum, one has to compute $f(0)$ and $f(\gamma_2)$ and take the minimum between these two.

If there exists a positive $\gamma_{\varrho,m}$, which satisfies conditions (a) and (b) from the statement of the theorem, it means that $\frac{\partial f}{\partial \gamma}(\gamma) = 0$ has two solutions. Let $\gamma_1 \le \frac{m-2}{2} \le \gamma_2$, be the solutions. Since $\gamma_{\varrho,m} > m-2$, it follows that $\gamma_{\varrho,m} = \gamma_2$.

$$f(\gamma_{\varrho,m}) = \sigma_0^2\left(1 - \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\left(\frac{\gamma_{\varrho,m}}{\gamma_{\varrho,m}+1}\right)^{m-1}\right) + \frac{m-1}{\gamma_{\varrho,m}+1}\frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\left(\frac{\gamma_{\varrho,m}}{\gamma_{\varrho,m}+1}\right)^{m-1} < \sigma_0^2 = f(0)$$

where the inequality takes place because $m < \gamma_{\varrho,m} + 2$.

If exists γ_ϱ such that condition (a) is satisfied, but for none of these γ_ϱ condition (b) is not satisfied, then

$$f(\gamma_{\varrho,m}) = \sigma_0^2\left(1 - \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\left(\frac{\gamma_{\varrho,m}}{\gamma_{\varrho,m}+1}\right)^{m-1}\right) + \frac{m-1}{\gamma_{\varrho,m}+1}\frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\left(\frac{\gamma_{\varrho,m}}{\gamma_{\varrho,m}+1}\right)^{m-1} < \sigma_0^2 = f(0)$$

If no positive γ exists, such that such that condition (a) is satisfied, it means that $\frac{\partial f}{\partial \gamma}(\gamma) = 0$, has no solution, and since $\frac{\partial f}{\partial \gamma}(\gamma)$ is continuous in γ and $\frac{\partial f}{\partial \gamma}(0) > 0$, for all $\gamma \ge 0$, which implies that f is increasing for $\gamma \ge 0$ and $f(\gamma) \ge f(0)$, for all positive γ. □ ■

Remark 2.1: From Lemmas 2.2, 2.3 and 2.4 we can compute the Lagrange multiplier of the constraint problem in Lemma 2.2. The Lagrange multiplier for a fixed γ has the value $\lambda = \frac{\sigma_0^4}{(\sigma_0^2+\sigma_V^2)\sigma_V^2}\frac{\gamma^{m-2}}{(1+\gamma)^m}$. So, whenever the condition on ϱ, m and $\gamma_{\varrho,m}$ from theorem 2.5 is satisfied, then ϱ is actually the Lagrange multiplier of the problem in Lemma 2.2, for the optimal γ.

A. The optimal solution to Problem 1.1 is linear for $m = 2$

Proposition 2.6: Let all the parameters defining Problem 1.1 be given and assume that $m = 2$. Given a positive real constant γ, let $\mathcal{F}_0$ be Lebesgue measurable function satisfying: $E[U(0)^2] \le \gamma\sigma_V^2$. The following holds:

$$E[(X(2) - X(0))^2] \ge \mathcal{C}_A^*\left(2, \gamma, \sigma_0^2, \sigma_V^2\right) \tag{21}$$

where $\mathcal{C}_A^*(2, \gamma, \sigma_0^2, \sigma_V^2)$ is given by (13).

Proof: Let:

$$\widetilde{X}(1) = E[X(0)|Y(0)] = \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}Y(0)$$

The cost function can be written:

$$\begin{aligned}
&E[(X(2) - X(0))^2] = E[(X(2) - \widetilde{X}(1) + \widetilde{X}(1) - X(0))^2] \\
&= E[(X(2) - \widetilde{X}(1))^2] + E[(\widetilde{X}(1) - X(0))^2] \\
&+ 2E[(X(2) - \widetilde{X}(1))(\widetilde{X}(1) - X(0))] \\
&= E[(X(2) - \widetilde{X}(1))^2] + E[(\widetilde{X}(1) - X(0))^2] \\
&= E[(X(2) - \widetilde{X}(1))^2] + \frac{\sigma_0^2\sigma_V^2}{\sigma_0^2+\sigma_V^2}
\end{aligned}$$

The cross term is zero because of the orthogonality principle. We note that $\widetilde{X}(1)$ is a linear function of $Y(0)$, which means that $Y(0)$ can be written as a linear function of $\widetilde{X}(0)$ and also $X(1) = U(0)$ is a function of $\widetilde{X}(0)$. Then, by proposition 2.1:

$$\begin{aligned}
&E[(X(2) - X(0))^2] = E[(X(2) - \widetilde{X}(1))^2] + \frac{\sigma_0^2\sigma_V^2}{\sigma_0^2+\sigma_V^2} \\
&\ge \frac{\sigma_0^4}{\sigma_V^2+\sigma_0^2}\left(1 - \frac{\sigma_V^2\gamma}{\sigma_V^2+\sigma_V^2\gamma}\right) + \frac{\sigma_0^2\sigma_V^2}{\sigma_0^2+\sigma_V^2} \\
&= \sigma_0^2\left(1 - \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\frac{\gamma}{1+\gamma}\right) = \mathcal{C}_A^*\left(2, \gamma, \sigma_0^2, \sigma_V^2\right)
\end{aligned}$$

The inequality can be reached with equality by selecting the functions $\mathcal{F}_0$ and $\mathcal{F}_1$ like in Lemma 2.2, for $m = 2$ ■

III. TWO VALUED MEMORYLESS CONTROL

In this section, we show that, in general affine functions do not solve Problem 1.1. In Lemma 3.1, we compute the cost when are functions $\{\mathcal{F}_k\}_{k=0}^{m-1}$ take only 2 values. The main result in this section is Theorem 3.2, in which we show that the two valued functions reach a lower cost than affine functions. The section ends with some numerical results, which show that two valued functions can be better than affine functions.

Lemma 3.1: Let the parameters in Problem 1.1 be given. Given the positive real numbers $\left\{\sigma_i^2\right\}_{i=1}^m$, define the following class of functions $\{\mathcal{F}_i\}_{i=0}^{m-1}$, which take only two values:

$$\mathcal{F}_i(x) = \sigma_{i+1} sgn(x), i \in \{0, \ldots, m-1\} \tag{22}$$

Adopt the functions $\{\mathcal{F}_i\}_{i=0}^{m-1}$ as control strategies for Problem 1.1, as it follows:

$$U(k) = \mathcal{F}_k\left(Y(k)\right),\ k \in \{0, \ldots m-2\}$$

$$X(m) = \mathcal{F}_{m-1}\left(Y(m-1)\right)$$

Consider the following cost:

$$\mathcal{C}_L(\{\sigma_k^2\}_{k=1}^m, \sigma_0^2, \sigma_V^2) \stackrel{def}{=} E\left[(X(m) - X(0))^2\right]$$

Then, the following holds:

$$\mathcal{C}_L(\{\sigma_k^2\}_{k=1}^m, \sigma_0^2, \sigma_V^2) = \sigma_0^2 + \sigma_m^2 - 4\frac{\sigma_m\sigma_0^2}{\sqrt{2\pi(\sigma_0^2+\sigma_V^2)}}\prod_{i=1}^{m-1}\left(2P(V(i) \le \sigma_i) - 1\right)$$

Proof: See Appendix ■

In Problem 1.1, there is no constraint on the final function $\mathcal{F}_{m-1}$. In the problem given in Lemma 3.1, the cost can be minimized with respect to σ_m. Define the following cost:

$$\mathcal{C}_L^*(\{\sigma_k^2\}_{k=1}^{m-1}, \sigma_0^2, \sigma_V^2) \stackrel{def}{=} \min_{\sigma_m}\mathcal{C}_L\left(\{\sigma_k^2\}_{k=1}^m, \sigma_0^2, \sigma_V^2\right) \tag{23}$$

Since the cost function defined in Lemma 3.1 is a quadratic function of σ_m, it is straightforward that:

$$\mathcal{C}_L^*\left(\{\sigma_k\}_{k=1}^{m-1}, \sigma_0^2, \sigma_V^2\right) = \sigma_0^2 - \frac{4}{2\pi}\sigma_0^2\prod_{i=1}^{m-1}\left(2P(V(i) \le \sigma_i) - 1\right)^2 \cdot \frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}$$

Theorem 3.2: Let all the parameters defining Problem 1.1 be given. There exists a positive real number ϱ, an integer m and measurable nonlinear functions $\{\mathcal{F}_i\}_{i=0}^{m-1}$ such that:

$$\mathcal{J}(\{\mathcal{F}_k\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2) <$$
$$\mathcal{C}_A(\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}, \sigma_0^2, \sigma_V^2) + \varrho \sum_{k=0}^{m-2} E[U(k)^2]$$

for any choice of $\{\lambda(k)\}_{k=0}^{m-1}, \{\beta(k)\}_{k=0}^{m-1}$

Proof: Choose a positive real number γ, big enough such that $\left(2Q\left(\sqrt{\gamma}\right)-1\right)^2 > \frac{\gamma}{1+\gamma}$, where $Q(x)$ is the cumulative distribution function of a normal random variable with zero mean and unit variance. Note that such γ always exists, for example take any real number greater than 1.5. Then, there exists m_0, such that for any integer m greater than m_0:

$$\frac{\left(2Q\left(\sqrt{\gamma}\right)-1\right)^{2(m-1)}}{\left(\frac{\gamma}{1+\gamma}\right)^{m-1}} > \frac{2\pi}{4}$$

Choose an integer m, such that, $m \geq m_0$ and $m < \gamma+2$. To prove that such pair of γ and m exists, pick $m = \lfloor \gamma+1 \rfloor$. Then by noticing that:

$$\lim_{\gamma\to\infty}\left(\frac{\gamma}{1+\gamma}\right)^{\gamma+1} = e^{-1}$$
$$\lim_{\gamma\to\infty}\left(2Q(\sqrt{(\gamma)})-1\right)^{2(\gamma+1)} = 1$$

and $e > \frac{2\pi}{4}$, it follows that for a big enough γ and $m = \lfloor \gamma+1 \rfloor$, the pair (γ, m) satisfies both conditions $m \geq m_0$ and $m < \gamma+2$.

Choose $\mathcal{F}_k(x) = \sqrt{\gamma\sigma_V^2}sgn(x), k \in \{0, \dots m-2\}$. Choose $\mathcal{F}_{m-1} = \sigma_m sgn(x)$. Choose σ_m^2 in order to minimize the cost defined in (23) for which $\sigma_k^2 = \gamma\sigma_V^2, k \in \{0, \dots m-2\}$. It is clear that by this choice of functions, $\sum_{k=0}^{m-2} E\left[U(k)^2\right] = (m-1)\gamma\sigma_V^2$. For this choice of functions, the cost becomes:

$$E\left[(X(0)-X(m))^2\right] = \sigma_0^2$$
$$-\frac{4}{2\pi}\sigma_0^2\prod_{i=1}^{m-1}\left(2P\left(V(i)\leq\sigma_i\right)-1\right)^2\frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}$$
$$= \sigma_0^2 - \frac{4}{2\pi}\prod_{i=1}^{m-1}\left(2P\left(\frac{V(i)}{\sigma_V}\leq\frac{\sigma_i}{\sigma_V}\right)-1\right)^2\frac{\sigma_0^4}{\sigma_0^2+\sigma_V^2}$$
$$= \sigma_0^2 - \frac{4}{2\pi}\prod_{i=1}^{m-1}\left(2P\left(\frac{V(i)}{\sigma_V}\leq\sqrt{\gamma}\right)-1\right)^2\frac{\sigma_0^4}{\sigma_0^2+\sigma_V^2}$$
$$= \sigma_0^2 - \frac{4}{2\pi}\left(2Q\left(\sqrt{\gamma}\right)-1\right)^{2(m-1)}\frac{\sigma_0^4}{\sigma_0^2+\sigma_V^2}$$

It follows then, that for $m \geq m_0$:

$$E\left[(X(0)-X(m))^2\right] =$$
$$= \sigma_0^2\left(1-\frac{4}{2\pi}\left(2Q\left(\sqrt{\gamma}\right)-1\right)^{2(m-1)}\frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\right)$$
$$< \sigma_0^2\left(1-\frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}\frac{\gamma^{m-1}}{(1+\gamma)^{m-1}}\right) = \mathcal{C}_A^*(m,\gamma,\sigma_0^2,\sigma_V^2)$$

Pick $\varrho = \frac{\sigma_0^4}{\left(\sigma_0^2+\sigma_V^2\right)\sigma_V^2}\frac{\gamma^{m-2}}{(1+\gamma)^m}$. We note that with thes ϱ and m, the conditions of theorem 2.5 are satisfied, so the cost of problem 1.1 subject to affine strategies of the form (8) is given by equation (20). Since $m > m_0$, for the nonlinear functions chosen above, it follows that:

$$\mathcal{C}_A^*\left(m,\gamma,\sigma_0^2,\sigma_V^2\right) + \varrho(m-1)\gamma\sigma_V^2 >$$
$$> \sigma_0^2 - \frac{4}{2\pi}\sigma_0^2\left(2Q\left(\sqrt{\gamma}\right)-1\right)^{2(m-1)}\frac{\sigma_0^2}{\sigma_0^2+\sigma_V^2}$$
$$+\varrho(m-1)\gamma\sigma_V^2$$

This shows that, the nonlinear cost is smaller than the linear cost, so the optimum of problem 1.1 is reached in general by nonlinear functions rahter then affine functions. ■

We can also provide a simple numerical example, by solving numerically the problems defined in Theorem 2.5 and in Lemma 2.2. Let $\sigma_0 = 1$, $\sigma_V = 0.4$, $m = 7$ and $\varrho = 0.0858$. Then by running a numerical algorithm for optimization we obtain $\gamma_{opt} = 6$. The cost with linear functions is 0.9485, while the cost with the two-valued functions chosen like in the Theorem 3.2 is 0.8222. Using then the γ_{opt}, we can compute the Lagrange multiplier, and we obtain the result equal to 0.0858, which is the same as the value of ϱ. One important issue, is that the proof theorem 3.2 gives means how to provide counterexamples. Most of the work done on the Witsenhausen problem restricts itself only to some of the values of the problem parameters in order to give valid counterexamples.

In [7], Bansal and Basar made a classification of the problems with 2-stages with respect to the cost function to be minimized. They show that the Witsenhausen counter example falls into one category of problems, which they define. We can show that the cost function to be minimized in this paper can be put in a form similar to the form of Bansal and Basar, where the solution in nonlinear. Just like in the proof of lemma 2.3, let $\widetilde{X}(m-1) = E[X(0)|Y(m-2)]$, then by the same arguments like in the proof of lemma 2.3:

$$E[(X(m)-X(0))^2]$$
$$= E[(X(0)-\widetilde{X}(m-1))^2] + E[(\widetilde{X}(m-1)-X(m)^2]$$

We note that this cost can be compared to the cost in the Witsenhausen problem. It is true that is not exactly the Witsenhausen cost since $\widetilde{X}(m-1)$ is a function of $X(0)$, but also of $V(k), k = 0, \dots m-2$.

IV. CONCLUSIONS

In this paper, we have discussed the finite horizon optimal memoryless control of a delay in gaussian noise with quadratic cost and showed that in general, affine functions are not optimal for this kind of problems. By writing the cost in a different way, we showed that the qudratic cost can be written as a cost which resembles the cost in the Witsenhausen problem [1]. We showed that the class of functions which take only two values are better than the affine functions in some cases. This led to the belief that, just like in the numerical studies over the Witsenhausen problem, the optimal functions for this given problem are functions which take only discrete values. For future work, a more thorough investigation, both analytical and numerical, of the functions taking discrete values is necessary.

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V. APPENDIX

A. Proof of Propostion 2.1

We use in the proof notions from information theory, like mutual information and differential entropy. By taking $\mathcal{G}_0$ and $\mathcal{G}_1$ to be both identically zero, then the cost function will be σ_X^2. Since the cost function is positive then, we can restrict our search only to functions which give a bounded cost. This is required, so no problem will appear with respect to the existence of differential entropy. The differential entropy exists for random values with finte second moment, as it was proved in [17].

$$\begin{aligned} I(X, Z(1)) &\leq I(Z(0), Z(0)+W) \\ I(Z(0), Z(0)+W) &\leq \frac{1}{2}\log\left(1+\frac{\sigma^2}{\sigma_W^2}\right) \end{aligned}$$

The first inequality is because X, $Z(0)$, $Z(0)+W$ and $Z(1)$ form a Markov chain. The second inequality is because of the definition of the channel capacity.

$$\begin{aligned} &h\left(X|Z(1)\right) = h\left(X\right) - I(X, Z(1)) \\ &h\left(X-Z(1)|Z(1)\right) = h\left(X\right) - I(X, Z(1)) \\ &h\left(X-Z(1)\right) \geq h\left(X\right) - I(X, Z(1)) \\ &h\left(X-Z(1)\right) \geq h\left(X\right) - \frac{1}{2}\log\left(1+\frac{\sigma^2}{\sigma_W^2}\right) \\ &h\left(X-Z(1)\right) \geq \frac{1}{2}\log\left(2\pi e\sigma_X^2\right) - \frac{1}{2}\log\left(1+\frac{\sigma^2}{\sigma_W^2}\right) \\ &\frac{1}{2}\log\left(2\pi e Var[X-Z(1)]\right) \\ &\qquad \geq \frac{1}{2}\log\left(2\pi e\sigma_X^2\right) - \frac{1}{2}\log\left(1+\frac{\sigma^2}{\sigma_W^2}\right) \\ &\log\left(Var[X-Z(1)]\right) \geq \log\left(\sigma_X^2\right) - \log\left(1+\frac{\sigma^2}{\sigma_W^2}\right) \\ &Var[X-Z(1)] \geq \frac{\sigma_X^2\sigma^2}{\sigma^2+\sigma_W^2} \end{aligned}$$

The first inequality appears due to the fact that conditioning reduces entropy, the second inequality was proved in the beginning, the third, firth and fifth inequalityies appear because of the properties of the differential entropy with respect to the variance of the random variables. The last two inequalities are obtained from the fifth one, through direct manipulation. All the inequalities can be reached with equalities if the random variables X, $Z(0)$, $Z(0)+W$ and $Z(1)$ are Gaussian. X is Gaussian from the problem and the other can be gaussian if $\mathcal{G}_0$ and $\mathcal{G}_1$ are affine functions. Once we established the fact that $\mathcal{G}_0$ and $\mathcal{G}_1$ are affine, by straightforward computation, we get that the functions are in fact linear and the results of the proposition follow.

B. Proof of Lemma 2.3

One easy way to show it, it to notice that the affine functions at each step with $k \in \{0, \ldots m-2\}$ act only as scale factors. Because of linearity the values of the $\lambda(k)$'s $k \in \{0, \ldots m-2\}$ appear immediately the way they are written in equation (18), $\lambda(m-1)$ can be computed using the fact that $Y(m-1)$ is gaussian, so $X(m) = E[X(0)|Y(m-1)]$, and with the values of $\beta(k) = 0, \forall k$. It is true that the values of $\lambda(k)$ are not unique. It is straightforward to show that if we take a even number of parameters $\lambda(k)$, when $\beta(k) = 0$ and flip their sign the value of the cost function remains the same. But in order to be more rigorous, we will prove the claims in the lemma by induction. First we show that for a general m, there is no k for which $\lambda(k) = 0$.

Assume that exists such a k, then $U(k) = \beta(k)$, which will be just a constant. Then all the $Y(l), l \geq k+1$ will be independent of $X(0)$, which will make $X(m)$ independent of $X(0)$. The cost function becomes $E[(X(0)-X(m))^2] = E[X(0)^2] + E[X(m)^2] \geq \sigma_0^2$ but the parameters assumed above give a value of the cost function less then σ_0^2, which means that the λ's are always non zero.

It is a standard computation to show that the claims hold for $m = 1$ and for $m = 2$. For $m = 1$, $X(1) = E[X(0)|Y(0)]$, due to the Gaussianity of $X(0)$ and the noise, and the result is immediate. The results for $m = 2$ are found also in the proof for proposition 2.6. Assume that the claim holds for $m \geq 2$. We need to prove that it holds also for $m+1$. Let it be the $m+1$ stage problem. Let $\widetilde{X}(m)$ the best affine estimator of $X(0)$ given $Y(m-1)$. By the properties of the affine estimators $\widetilde{X}(m)$ is an affine function of $Y(m-1)$ and $E[\widetilde{X}(m)] = E[X(0)] = 0$. Since all the $\lambda(k) \neq 0, k \in \{0 \ldots m-2\}$ it follows that $\widetilde{X}(m)$ is an invertible affine function of $Y(m-1)$. This means that $X(m)$, being an affine function of $Y(m-1)$, is an affine function of $\widetilde{X}(m)$. Then by orthogonality principle we can write the cost function:

$$
\begin{aligned}
&E[(X(0)-X(m+1)^2] = \\
&= E[(X(0)-\widetilde{X}(m)+\widetilde{X}(m)-X(m+1)^2] \\
&= E[(X(0)-\widetilde{X}(m))^2] + E\left[(\widetilde{X}(m)-X(m+1)^2\right] \\
&\quad + 2E[(X(0)-\widetilde{X}(m))(\widetilde{X}(m)-X(m+1)] \\
&= E[(X(0)-\widetilde{X}(m))^2] + E[(\widetilde{X}(m)-X(m+1)^2]
\end{aligned}
$$

The value $E[(\widetilde{X}(m)-X(m+1)^2]$ can be bounded from below using Proposition 2.1, since $X(m)$ is an affine function of $\widetilde{X}(m)$ and $E[X(m)^2] = \sigma_m^2$. We know that $\widetilde{X}(m)$ is the best affine estimator of $X(0)$ given $Y(m-1)$. Then using the orthogonality principle we get:

$$
\begin{aligned}
E[X(0)^2] &= E[(X(0)-\widetilde{X}(m)+\widetilde{X}(m))^2] \\
&= E[(X(0)-\widetilde{X}(m))^2] + E[\widetilde{X}(m)^2] \\
&\quad + 2E[(X(0)-\widetilde{X}(m))\widetilde{X}(m)] \\
&= E[(X(0)-\widetilde{X}(m))^2] + E[\widetilde{X}(m)^2]
\end{aligned}
$$

Looking back at the inital cost:

$$
\begin{aligned}
&E[(X(0)-X(m+1)^2] = \\
&= E[(X(0)-\widetilde{X}(m))^2] + E[(\widetilde{X}(m)-X(m+1)^2] \\
&\geq E[(X(0)-\widetilde{X}(m))^2] + E[\widetilde{X}(m)^2]\left(1-\frac{\sigma_m^2}{\sigma_m^2+\sigma_V^2}\right) \\
&= E[X(0)^2]\left(1-\frac{\sigma_m^2}{\sigma_m^2+\sigma_V^2}\right) + E[(X(0)-\widetilde{X}(m))^2] \\
&- E[(X(0)-\widetilde{X}(m))^2]\left(1-\frac{\sigma_m}{\sigma_m+\sigma_V}\right) \\
&= \sigma_0^2\left(1-\frac{\sigma_m^2}{\sigma_m^2+\sigma_V^2}\right) + E[(X(0)-\widetilde{X}(m))^2]\frac{\sigma_m^2}{\sigma_m^2+\sigma_V^2} \\
&\geq \sigma_0^2\left(1-\prod_{i=0}^{m-1}\frac{\sigma_i^2}{\sigma_i^2+\sigma_V^2}\right)\frac{\sigma_m^2}{\sigma_m^2+\sigma_V^2} \\
&+ \sigma_0^2\left(1-\frac{\sigma_m^2}{\sigma_m^2+\sigma_V^2}\right) = \sigma_0^2\left(1-\prod_{i=0}^{m}\frac{\sigma_i^2}{\sigma_i^2+\sigma_V^2}\right)
\end{aligned}
$$

The first inequality takes place due to the fact that $X(m)$ is an affine function of $\widetilde{X}(m)$ and $E[X^2(m)] = \sigma_m^2$, so the second term can be lower bounded using Proposition 2.1 and the second inequality appears due to the induction. Both inequalities can be reached with equality by selecting the paramters $\lambda(k), k \in \{0, \ldots m-2\}$ and $\beta(k), k \in \{0, \ldots m-2\}$ for the m stage problem and the values for $\lambda(m-1)$, $\lambda(m)$, $\beta(m-1)$, $\beta(m)$ and $E[X(m+1)]$ follow from Proposition 2.1 and straightforward computation.□

C. Proof of Lemma 2.4

First we show that the optimization problem is equivalent to the following problem:

$$\max C\left(\{\alpha_i\}_{i=1}^n\right)$$

where the maximum is subject to the following constraints:

$$
\begin{aligned}
&\sum_{i=1}^n \alpha_i \leq P \\
&\alpha_i \geq \epsilon, i = 1, \ldots n
\end{aligned}
$$

for some $\epsilon > 0$.
The cost function is positive for any choice of positive $\alpha_i \geq 0$ and is zero if $\exists i, s.t.\alpha_i = 0$. Choose any $\alpha_i > 0$ such that $\sum_{i=1}^n \alpha_i \leq P$. For this choice, let $\prod_{i=1}^n \frac{\alpha_i}{\alpha_i+1} = \bar{\epsilon} > 0$. Then for any $k \in \{1, \ldots, n\}$

$$\prod_{i=1}^n \frac{\alpha_i}{\alpha_i+1} \leq \frac{\alpha_k}{\alpha_k+1} \leq \alpha_k$$

Choose $\epsilon = \frac{\bar{\epsilon}}{2}$, then if $\alpha_k \leq \epsilon$, then $\prod_{i=1}^n \frac{\alpha_i}{\alpha_i+1} < \bar{\epsilon}$ no matter what are the values of the other α_i, $i \in \{1, \ldots, k-1, k+1, \ldots, n\}$. This shows that the first problem and the second problem are equivalent. Moreover it shows that for the second problem, the inequality constraints $\alpha_i, i \in \{1, \ldots n\}$are inactive. Then the second problem can be solved by solving the equivalent problem:

$$
\begin{aligned}
&\text{maximize} \log \prod_{i=1}^n \frac{\alpha_i}{\alpha_i+1} \\
&\sum_{i=1}^n \alpha_i \leq P \\
&\alpha_i \geq \epsilon \;\; i = 1, \ldots, n
\end{aligned}
$$

which is the same with:

$$
\begin{aligned}
&\text{maximize} \sum_{i=1}^n \log \frac{\alpha_i}{\alpha_i+1} \\
&\sum_{i=1}^n \alpha_i \leq P \\
&\alpha_i \geq \epsilon \;\; i = 1, \ldots, n
\end{aligned}
$$

We note that the optimization function is strictly concave on the maximization domain and the inequality constraints are affine functions, which means that values for $\{\alpha_i\}_{i=1}^n$ which reach the maximum are unique. From the argument of the previous problem the inequality constraints $\alpha_i \geq \epsilon, i \in \{1, \dots n\}$ are inactive, so the Lagrange multipliers for these constraints are 0. Let μ be the Lagrange multiplier of the remaining inequality constraint. Then for the optimization problem the first order optimality conditions can be written:

$$\frac{\partial \sum_{i=1}^n \log \frac{\alpha_i}{\alpha_i+1}}{\partial \alpha_k} + \mu = 0, k = 1, \dots n$$
$$\frac{1}{\alpha_k} - \frac{1}{\alpha_k + 1} + \mu = 0, k = 1, \dots n$$

First we note that $\mu < 0$ and that the inequality constraint is active. Then α_k can be written as a function of μ:

$$\alpha_k = \frac{-1 + \sqrt{1 - \frac{4}{\mu}}}{2}$$

We obtain that the $\alpha_k, k = 1, \dots n$ are equal and:

$$\alpha_k = \frac{P}{n}$$

and the result follows.

D. Proof of Lemma 3.1

Before proving the claim in Lemma 3.1, one needs to prove the following.

$$P(X(m) = \sigma_m) = \frac{1}{2}$$
$$P(U(k) = \sigma_{k+1}) = \frac{1}{2}, \forall k \in \{0, \dots m-2\}$$

One can proove this claim by induction. Due to space constraints, we omit the proof of this claim.

We need to prove that:

$$E[X(m)|Y(m-k-1) < 0] = -\sigma_m \prod_{i=m-k}^{m-1} (2P(V(i) \leq \sigma_i) - 1)$$
$$E[X(m)|Y(m-k-1) > 0] = \sigma_m \prod_{i=m-k}^{m-1} (2P(V(i) \leq \sigma_i) - 1)$$

for $1 \leq k \leq m$. We prove this by induction. For $k = 1$, the proof is done by straightforward computation. Assume that the claim holds for all i, $1 \leq i \leq k$. We need to prove it for $k+1$.

$$\begin{aligned}
&E[X(m)|Y(m-k-2) < 0] \\
&= E[X(m)|Y(m-k-1) < 0, Y(m-k-2) < 0] \\
&\quad \cdot P(Y(m-k-1) < 0 | Y(m-k-2) < 0) \\
&\quad + E[X(m)|Y(m-k-1) > 0, Y(m-k-2) < 0] \\
&\quad \cdot P(Y(m-k-1) > 0 | Y(m-k-2) < 0) \\
&= E[X(m)|Y(m-k-1) < 0] \\
&\quad \cdot P(V(m-k-1) < \sigma_{m-k-1}) \\
&\quad + E[X(m)|Y(m-k-1) > 0] \\
&\quad \cdot P(V(m-k-1) > \sigma_{m-k-1}) \\
&= -\sigma_m \prod_{i=m-k}^{m-1} (2P(V(i) \leq \sigma_i) - 1) \\
&\quad \cdot P(V(m-k-1) < \sigma_{m-k-1}) \\
&\quad + \sigma_m \prod_{i=m-k}^{m-1} (2P(V(i) \leq \sigma_i) - 1) \\
&\quad P(V(m-k-1) > \sigma_{m-k-1}) \\
&= \sigma_m \prod_{i=m-k-1}^{m-1} (2P(V(i) \leq \sigma_i) - 1)
\end{aligned}$$

In the same way, one can show that the claim holds for $E[X(m)|Y(m-k-2) > 0]$. The cost function defined in the lemma is:

$$\begin{aligned}
&E\left[(X(0) - X(m))^2\right] = \sigma_0^2 + \sigma_m^2 - 2E[X(0)X(m)] \\
&E[X(0)X(m)] = E[E[X(0)X(m)|X(0), V(0)]] \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[X(0)X(m)|X(0) = x_0, V(0) = v_0] \\
&\quad \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2} + \frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{-v_0} x_0 E[X(m)|X(0) = x_0, V(0) = v_0] \\
&\quad \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2} + \frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&\quad + \int_{-\infty}^{\infty} \int_{-v_0}^{\infty} x_0 E[X(m)|X(0) = x_0, V(0) = v_0] \\
&\quad \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2} + \frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&= -\sigma_m \prod_{i=1}^{m-1} (2P(V(i) \leq \sigma_i) - 1) \int_{-\infty}^{\infty} \int_{-\infty}^{-v_0} x_0 \\
&\quad \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2} + \frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&\quad + \sigma_m \prod_{i=1}^{m-1} (2P(V(i) \leq \sigma_i) - 1) \int_{-\infty}^{\infty} \int_{-v_0}^{\infty} x_0 \\
&\quad \frac{1}{2\pi\sqrt{\sigma_V^2\sigma_0^2}} e^{-\left(\frac{x_0^2}{2\sigma_0^2} + \frac{v_0^2}{2\sigma_V^2}\right)} dx_0 dv_0 \\
&= 2\sigma_m \prod_{i=1}^{m-1} (2P(V(i) \leq \sigma_i) - 1) \frac{\sigma_0^2}{\sqrt{2\pi(\sigma_0^2 + \sigma_V^2)}}
\end{aligned}$$

It follows then that:

$$\begin{aligned}
&E\left[(X(0) - X(m))^2\right] = \sigma_0^2 + \sigma_m^2 \\
&- 4\sigma_m \prod_{i=1}^{m-1} \left(2P\left(V(i) \leq \sqrt{\sigma_i^2}\right) - 1\right) \frac{\sigma_0^2}{\sqrt{2\pi(\sigma_0^2 + \sigma_V^2)}}
\end{aligned}$$