Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Control of the Hopf Bifurcation in the Takens-Bogdanov bifurcation

Francisco Armando Carrillo Navarro and Fernando Verduzco González

***Abstract*— It is a well-known result that in a versal deformation of the Takens-Bogdanov bifurcation is possible to find dynamical systems that undergo saddle-node, homoclinic and Hopf bifurcations. In this document a nonlinear control system in the plane is considered, whose nominal vector field undergoes the Takens-Bogdanov bifurcation, and then the idea is to design a scalar control law such that the closed-loop system undergoes the called controllable Hopf bifurcation.**

Key words: Takens-Bogdanov bifurcation, versal deformation, Hopf bifurcation, Normal Forms Theory, Central Manifold Theory.

I. INTRODUCTION

One of the goals about the control of bifurcations is to establish *a priori* the creation or elimination of stationary states, like critical points, limit cycles, torus and strange attractors, with their respective stability characteristics. Even though 20 years ago began the study of control of bifurcations, only has been systematized the control of codimension one bifurcations: Hopf, saddle-node, transcritic and pitchfork. See [1], [2], [5], [9], [10]. Few papers are related with the control of codimension two bifurcation, see [6], [7].

In this paper we began a systematic study to control the codimension two bifurcation called Takens-Bogdanov or double-cero. The Takens-Bogdanov bifurcation happens when the linear part of the dynamical system has a double-zero eigenvalue and the rest of the eigenvalues have real part different of zero. In [8] and [3], Takens and Bogdanov, respectively, both found of independent way, a versal deformation of this bifurcation, that is, they found a two-parametric family which contains all the possible perturbations of the original system. They demonstrated that around the mentioned bifurcation point, the system undergoes the saddle-node and the homoclinic as well as the Hopf bifurcation, see [4] and [11]. We will say that we have controlled the Takens-Bogdanov bifurcation when it is possible to design control laws that allow us to cross all the possible dynamic scenes that exist around this bifurcation point.

The idea of this work is to design a control law such that our feedback nonlinear control system represents a perturbation of the open-loop system that undergoes the Takens-Bogdanov bifurcation, and such that our feedback system undergoes a controllable Hopf bifurcation.

F.A. Carrillo N. and F. Verduzco G. are with Mathematics Department, Sonora University, 83000, Hermosillo, Sonora, Mexico. carrillo@gauss.mat.uson.mx; verduzco@gauss.mat.uson.mx

II. HOPF BIFURCATION

Theorem 1: (Hopf Bifurcation Theorem) Suppose that the system $\dot{x} = f(x,\mu)$, $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}$, has an equilibrium point (x_0, μ_0) such that

(H1) $D_x f(x_0)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts.

(H2) Let $\lambda(\mu)$, $\bar{\lambda}(\mu)$ be the eigenvalues of $D_x f(x_0, \mu_0)$ which are imaginary at $\mu = \mu_0$, such that

$$d = \frac{d}{d\mu}\left(Re(\lambda(\mu))\right)|_{\mu=\mu_0} \neq 0. \tag{1}$$

Then there is a unique three-dimensional center manifold passing through $(x_0, \mu_0) \in \mathbb{R}^n \times \mathbb{R}$ and a smooth system of coordinates for which the Taylor expansion of degree three on the center manifold, in polar coordinates, is given by

$$\begin{aligned} \dot{r} &= (d\mu + lr^2)r, \\ \dot{\theta} &= \omega + c\mu + br^2. \end{aligned}$$

If $l \neq 0$, then there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of $\lambda(\mu_0)$, $\bar{\lambda}(\mu_0)$ agreeing to second order with the paraboloid $\mu = -\frac{l}{d}r^2$, see Figure 1. If $l < 0$, then these periodic solutions are stable, while if $l > 0$, they are repelling limit cycles.

The quantities d and l we will be called *cross speed* and *first Lyapunov coefficient*, respectively.

There is a formulae to find in cartesian coordinates for bidimensional systems, the first Lyapunov coefficient l (see [4]). Let us consider the system

$$\dot{x} = Jx + F(x),$$

where $J = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$, $F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix}$ with $F(0) = 0$ and $DF(0) = 0$. Then

$$l = \frac{1}{16\omega}(R_1 + \omega R_2), \tag{2}$$

where

$$\begin{aligned} R_1 &= [F_{1x_1x_2}(F_{1x_1x_1} + F_{1x_2x_2}) - F_{2x_1x_2}(F_{2x_1x_1} + \\ &\quad F_{2x_2x_2}) - F_{1x_1x_1}F_{2x_1x_1} + F_{1x_2x_2}F_{2x_2x_2}]|_{x=0}\,, \\ R_2 &= [F_{1x_1x_1x_1} + F_{1x_1x_2x_2} + F_{2x_1x_1x_2} + F_{2x_2x_2x_2}]|_{x=0}\,. \end{aligned}$$

Observe that for different signs of d and l we have four possible stages or directions of the Hopf bifurcation, see Figure 2.

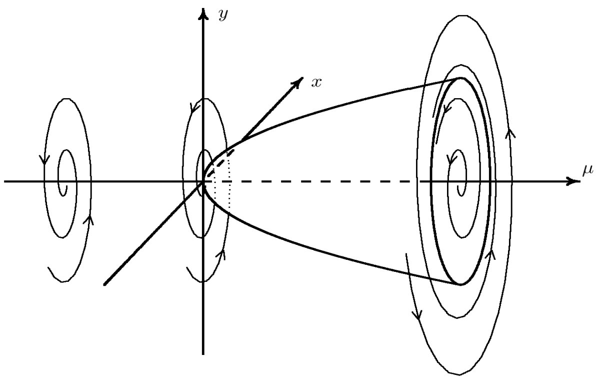

Fig. 1. One-parametric family of periodic orbits results of the Hopf bifurcation, at a non hiperbolic equilibrium x_0 and a bifurcation value $\mu_0 = 0$, $d > 0$ and $l < 0$.

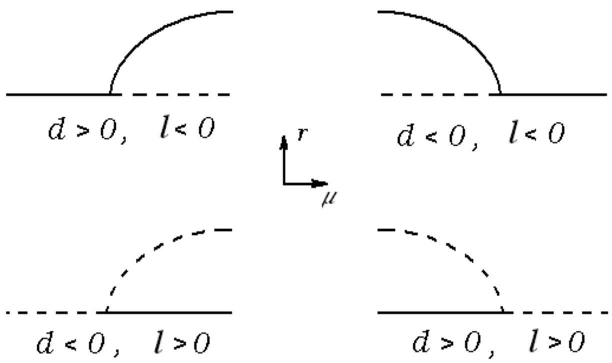

Fig. 2. Four possible directions of the Hopf bifurcation. The solid line represents behavior stable while the broken line unstable.

III. Controllable Hopf Bifurcation

Let us consider a nonlinear control system

$$\dot{x} = F(x) + G(x)u \tag{3}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, F and G sufficiently smooth. Suppose that there exists x_0 such that $F(x_0) = 0$ and $DF(x_0)$ has two imaginary eigenvalues, and the rest have negative real part.

Definition 2: (Controllable Hopf bifurcation). If there exists a control law

$$u = u(x, \mu, \gamma), \tag{4}$$

where $\mu \in \mathbb{R}$ is an artificial parameter of bifurcation, and $\gamma \in \mathbb{R}^k$, for some integer k, is an artificial vector of control parameters, such that the closed-loop system (3-4) undergoes a Hopf bifurcation when $\mu = 0$ at $x = x_0$, and besides it is possible to establish *a priori* any of the four possible directions of the bifurcation, by the manipulation of γ, then we are going to say that system (3) undergoes a **controllable Hopf bifurcation** at $x = x_0$ when $\mu = 0$.

Both parameters cross speed and first Lyapunov coefficient, we will be called the ***controllability coefficients*** of the controllable Hopf bifurcation, because they control the four possible directions of the Hopf bifurcation.

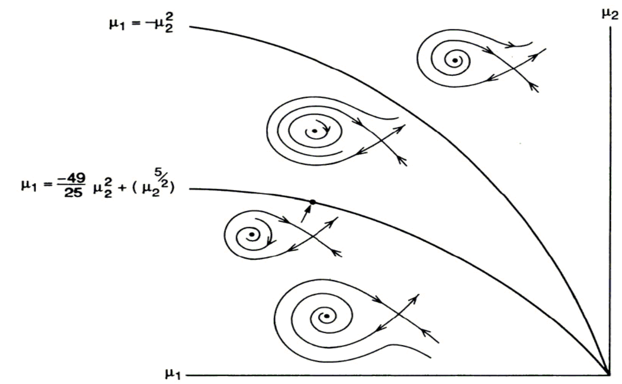

Fig. 3. Takens-Bogdanov bifurcation diagram and the corresponding phase portraits

In other words, system (3) undergoes a controllable Hopf bifurcation if it is possible to design a control law such that be possible to establish a priori the sign of the controllability coefficients d and l.

IV. Takens-Bogdanov Bifurcation

Let us consider the dynamical system in the plane $\dot{z} = f(z)$, with $f(0) = 0$ and $J = Df(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. From the normal forms theory, there is a transformation of coordinates, such that, the original system can be expressed up order two, in the form

$$\dot{z} = f_0(z) = \begin{pmatrix} z_2 \\ a_0 z_1^2 + b_0 z_1 z_2 \end{pmatrix}, \tag{5}$$

which is called the truncated normal form of the original system. A versal deformation of this truncated normal form roughly speak, is a dynamical systems which contains to system (5) and a whole "perturbations family" to this truncated normal form. Takens and Bogdanov showed that the family

$$\dot{z} = F(z, \mu) = \begin{pmatrix} z_2 \\ \mu_1 + \mu_2 z_2 + a_0 z_1^2 + b_0 z_1 z_2 \end{pmatrix}, \tag{6}$$

with $\mu = (\mu_1, \mu_2)$ represent a versal deformation of the truncated system (5), see [4], [11]. Can be proved that for $\mu_1 = 0$ and $\mu_2 \neq 0$ the family represent a system which undergoes the saddle-node bifurcation; for $\mu_1 = -\mu_2^2$ the system undergoes the Hopf bifurcation, and for $\mu_1 = -\frac{49}{25}\mu_2^2 + \cdots$ undergoes the homoclinic bifurcation.

We can see the diagram of Takens-Bogdanov bifurcation in the Figure 3

V. Statement of the problem

Let us consider the nonlinear control system

$$\dot{x} = Jx + f(x) + g(x)u \tag{7}$$

with

$$x \in \mathbb{R}^2, \text{ such that } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; u \in \mathbb{R}, \; J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$f(x) = f_2(x) + \mathcal{O}(|x|^3)$, where

$$f_2(x) = \begin{pmatrix} f_{11}x_1^2 + f_{12}x_1x_2 + f_{13}x_2^2 \\ f_{21}x_1^2 + f_{22}x_1x_2 + f_{23}x_2^2 \end{pmatrix},$$

$$g(x) = b + Mx + \mathcal{O}(|x|^2), \text{ where } b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

and $M = (m_{ij})_{2\times 2}$.

Our goal in this document is to design a control law $u = u(x, \mu, \delta)$ where $\mu = (\mu_1, \mu_2)$ representing the artificial vector of bifurcation parameters, while $\delta = (\delta_1, \delta_2, \delta_3)$ representing the artificial vector of control parameters, such that, the closed-loop system undergoes a controllable Hopf bifurcation. That is, the idea is to design a control law u sucht that, the family of systems move on the curve of Hopf bifurcation points $\mu_1 = -\mu_2^2$ of the Figure 3, and be possible to control them.

VI. Control Design

In this part we will design a control law $u(x, \mu, \delta)$ such that the nonlinear control system (7), become into a new system equivalent to the versal deformation of the Takens-Bogdanov bifurcation (6). So by manipulating of artificial vector of bifurcation parameters μ, we can control the emergence or elimination of closed orbits, and by manipulating of artificial vector of control parameters δ, we can control the stability of such periodic orbits.

A. First Coordinate Transformation

Let us consider the change of coordinates

$$x = P(y + H(y)) \tag{8}$$

into the system (7), where $y = (y_1, y_2)^T$,

$$P = \begin{pmatrix} b_2 & b_1 \\ 0 & b_2 \end{pmatrix}, \tag{9}$$

$$H(y) = y^T \mathcal{H} y, \tag{10}$$

where $\mathcal{H} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$, with $H_i = \begin{pmatrix} h_{i1} & \frac{1}{2}h_{i2} \\ \frac{1}{2}h_{i2} & h_{i3} \end{pmatrix}$ for $i = 1, 2$.

Observe that

$$\begin{aligned} \dot{y} &= [P(I + 2y^T\mathcal{H})]^{-1}\dot{x} \\ &= (I + 2y^T\mathcal{H})^{-1}P^{-1}[JP(y + y^T\mathcal{H}y + \\ &\quad f(P(y + y^T\mathcal{H}y) + g(P(y + y^T\mathcal{H}y)u] \end{aligned}$$

but

$$(I + 2y^T\mathcal{H})^{-1} = I - 2y^T\mathcal{H} + \cdots,$$

$$f(P(y + y^T\mathcal{H}y)) = f(Py) + \mathcal{O}(|y|^3)$$

and

$$g(P(y + y^T\mathcal{H}y)) = b + MPy + \mathcal{O}(|y|^2)$$

then

$$\begin{aligned} \dot{y} &= (I - 2y^T\mathcal{H} + \cdots)P^{-1}[JP(y + y^T\mathcal{H}y + \\ &\quad f(P(y + y^T\mathcal{H}y)) + g(P(y + y^T\mathcal{H}y))u]. \end{aligned}$$

Now we consider

$$u = \mu_1 + v \tag{11}$$

then

$$\dot{y} = \mu_1 P^{-1}b + \overline{J}y + \overline{f}_2(y) + \mathcal{O}(\mu_1|y|^2) + \overline{g}(y)v, \tag{12}$$

where

$$\overline{J} = P^{-1}JP + \mu_1[P^{-1}MP - 2b^T(P^{-1})^T\mathcal{H}], \tag{13}$$

$$\begin{aligned} \overline{f}_2(y) &= P^{-1}JPy^T\mathcal{H}y + P^{-1}f_2(Py) - \\ &\quad 2y^T\mathcal{H}P^{-1}JPy, \end{aligned} \tag{14}$$

$$\begin{aligned} \overline{g}(y) &= P^{-1}b + [P^{-1}MP - 2b^T(P^{-1})^T\mathcal{H}]y \\ &\quad + \mathcal{O}(|y|^2), \end{aligned} \tag{15}$$

It is not difficult to see that $P^{-1}b = (0, 1)^T = e_2$ and $P^{-1}JP = J$.

Lemma 3: If $b_2 \neq 0$, then there exists H given by (10), such that $P^{-1}MP - 2b^T(P^{-1})^T\mathcal{H} \equiv 0$.

Proof: If we define the coefficients

$$\begin{aligned} h_{12} &= -\frac{-m_{11}b_2 + b_1m_{21}}{b_2} \\ h_{13} &= -\frac{1}{2}\frac{-b_1m_{11}b_2 + b_1^2m_{21} - m_{12}b_2^2 + b_2b_1m_{22}}{b_2^2} \\ h_{22} &= m_{21} \\ h_{23} &= \frac{1}{2}\frac{b_1m_{21} + m_{22}b_2}{b_2} \end{aligned}$$

then $P^{-1}MP - 2b^T(P^{-1})^T\mathcal{H} \equiv 0$. □

From lemma 3, we have that

$$\overline{J} = J, \tag{16}$$

$$\overline{g}(y) = e_2 + \mathcal{O}(|y|^2), \tag{17}$$

and

$$\dot{y} = \mu_1e_2 + Jy + \overline{f}_2(y) + \mathcal{O}(\mu_1|y|^2) + (e_2 + \mathcal{O}(|y|^2))v \tag{18}$$

B. Second Coordinate Transformation

From (18), we can see that the next step of the control design is to transform the part $\overline{f}(y) + (e_2 + \mathcal{O}(|y|^2))v$ in a vector of the form

$$\nu = \begin{pmatrix} 0 \\ \kappa_1z_2 + \kappa_2z_1^2 + \kappa_3z_1z_2 \end{pmatrix},$$

with κ_1, κ_2, κ_3 constants. Now then, from the normal form theory we consider

$$y = z + h(z), \tag{19}$$

$$v = \mu Lz + z^TKz, \tag{20}$$

where $z = (z_1, z_2)^T$, $\mu = (\mu_1, \mu_2)$, and

$$h(z) = \begin{pmatrix} 0 \\ c_3z_2^2 \end{pmatrix}, \tag{21}$$

$$L = \begin{pmatrix} 0 & 2c_3 \\ 0 & \delta_1 \end{pmatrix}, \tag{22}$$

$$K = \begin{pmatrix} q_1 & \frac{1}{2}q_2 \\ \frac{1}{2}q_2 & q_3 \end{pmatrix}. \tag{23}$$

Thus,

$$\begin{aligned}\dot{z} &= (I + Dh(z))^{-1}[\mu_1 e_2 + J(z + h(z)) + \overline{f}(z) + \\ &\quad (e_2 + \mathcal{O}(|z|^2))(\mu L z + z^T K z)] \\ &= \mu_1 e_2 + \widetilde{J} z + \widetilde{f}_2(z) + \mathcal{O}(\mu |z|^2),\end{aligned}$$

where

$$\widetilde{J} z = Jz - \mu_1 Dh(z) e_2 + \mu L z e_2, \tag{24}$$
$$\widetilde{f}_2(z) = Jh(z) + \overline{f}_2(z) + z^T K z e_2. \tag{25}$$

From (24) it is not difficult to see that $\widetilde{J} z = \begin{pmatrix} z_2 \\ \mu_2 \delta_1 z_2 \end{pmatrix}$, and $\widetilde{f}_2(z) = z^T F z$, where

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \text{ with } F_i = \begin{pmatrix} \widetilde{f}_{i1} & \frac{1}{2}\widetilde{f}_{i2} \\ \frac{1}{2}\widetilde{f}_{i2} & \widetilde{f}_{i3} \end{pmatrix}, \ i = 1, 2,$$

and

$$\begin{aligned}\widetilde{f}_{11} &= h_{21} + b_2 f_{11} - b_1 f_{21}, \\ \widetilde{f}_{12} &= m_{21} - 2h_{11} + f_{12} b_2 + 2 f_{11} b_1 - b_1 f_{22} - \frac{2 f_{21} b_1}{b_2}, \\ \widetilde{f}_{13} &= c_3 - m_{11} + \frac{1}{2} m_{22} + f_{12} b_1 + f_{13} b_2 - f_{23} b_1 \\ &\quad + \frac{3}{2}\frac{b_1 m_{21}}{b_2} + \frac{(f_{11} - f_{22}) b_1^2}{b_2} - \frac{f_{21} b_1^3}{b_2^2}, \\ \widetilde{f}_{21} &= q_1 + b_2 f_{21}, \\ \widetilde{f}_{22} &= q_2 + f_{22} b_2 + 2 f_{21} b_1 - 2h_{21}, \\ \widetilde{f}_{23} &= q_3 - m_{21} + f_{22} b_1 + f_{23} b_2 + \frac{f_{21} b_1^2}{b_2},\end{aligned}$$

we need only to transform the vector $\widetilde{f}_2(z)$ in the form given by the next

Lemma 4: If $b_2 \neq 0$, then there are h, L and K as given by (21), (22) and (23) respectively, such that

$$\widetilde{f}_2(z) = \begin{pmatrix} 0 \\ \delta_2 z_1^2 + \delta_3 z_1 z_2 \end{pmatrix}, \tag{26}$$

where δ_2 and δ_3 are constants.

Proof: If we define

$$\begin{aligned}h_{11} &= -\frac{1}{2}\frac{f_{22} b_2 b_1 + 2 f_{21} b_1^2 - f_{12} b_2^2 - 2 f_{11} b_2 b_1 - b_2 m_{21}}{b_2} \\ h_{21} &= f_{21} b_1 - f_{11} b_2 \\ c_3 &= \frac{2 b_2^2 m_{11} - 2 f_{12} b_2^2 b_1 - 3 b_2 b_1 m_{21} - 2 b_2 f_{11} b_1^2}{2 b_2^2} \\ &\quad + \frac{2 f_{21} b_1^3 + 2 f_{22} b_2 b_1^2 - 2 f_{13} b_2^3 - b_2^2 m_{22} + 2 b_1 f_{23} b_2^2}{2 b_2^2} \\ q_1 &= -f_{21} b_2 + \delta_2 \\ q_2 &= -2 f_{11} b_2 - f_{22} b_2 + \delta_3 \\ q_3 &= \frac{m_{21} b_2^2 - f_{23} b_2^3 - b_1 f_{22} b_2^2 - f_{21} b_2 b_1^2}{b_2^2}\end{aligned}$$

then we obtain $\widetilde{f}_2(z) = \begin{pmatrix} 0 \\ \delta_2 z_1^2 + \delta_3 z_1 z_2 \end{pmatrix}$. □

From lemma 4, finally we succeeded that the nonlinear control system (7) becomes into the system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ \mu_1 + \mu_2 \delta_1 z_2 + \delta_2 z_1^2 + \delta_3 z_1 z_2 \end{pmatrix} + \mathcal{O}(\mu |z|^2) \tag{27}$$

Our goal is to prove that under certain conditions, this system undergoes the controllable Hopf bifurcation.

VII. Study of the local dynamics

The fixed points of (27) are $z_0 = \left(\pm\sqrt{-\frac{\mu_1}{\delta_2}}, 0\right)$, and the Jacobian matrix of this system is

$$\mathcal{J}(z) = \begin{pmatrix} 0 & 1 \\ 2\delta_2 z_1 + \delta_3 z_2 & \mu_2 \delta_1 + \delta_3 z_1 \end{pmatrix} \tag{28}$$

with eigenvalues given by

$$\lambda_{1,2}(z) = \frac{1}{2}\left[(\mu_2 \delta_1 + \delta_3 z_1) \pm \sqrt{(\mu_2 \delta_1 + \delta_3 z_1)^2 + 4(2\delta_2 z_1 + \delta_3 z_2)}\right] \tag{29}$$

Let us denote the two branches of fixed points by

$$z_0^+ = \left(\sqrt{-\frac{\mu_1}{\delta_2}}, 0\right) \quad \text{and} \quad z_0^- = \left(-\sqrt{-\frac{\mu_1}{\delta_2}}, 0\right),$$

and we will made the local dynamic around the *negative branch* z_0^-, the analysis for the *positive branch* z_0^+ is totally similar. The matrix (28) evaluated in z_0^- takes the form

$$\mathcal{J}(z_0^-) = \begin{pmatrix} 0 & 1 \\ -2\delta_2\sqrt{-\frac{\mu_1}{\delta_2}} & \mu_2 \delta_1 - \delta_3\sqrt{-\frac{\mu_1}{\delta_2}} \end{pmatrix},$$

and we know that matrix $\mathcal{J}(z_0^-)$ has a pair of pure imaginary eigenvalues $\lambda_{1,2}^-$ if the tr $\left[\mathcal{J}(z_0^-)\right] = 0$ and det $\left[\mathcal{J}(z_0^-)\right] > 0$. Since tr $\left[\mathcal{J}(z_0^-)\right] = \mu_2 \delta_1 - \delta_3\sqrt{-\frac{\mu_1}{\delta_2}}$ and det $\left[\mathcal{J}(z_0^-)\right] = 2\delta_2\sqrt{-\frac{\mu_1}{\delta_2}}$, then if

$$\mu_2 = \frac{\delta_3\sqrt{-\frac{\mu_1}{\delta_2}}}{\delta_1}, \tag{30}$$

we obtain $\lambda_{1,2}^- = \pm\sqrt{-2\delta_2\sqrt{-\frac{\mu_1}{\delta_2}}}$, thus, we might expect that the curve (30) with $\delta_2 > 0$ and $\mu_1 < 0$ is a bifurcation curve on which z_0^- undergoes a controllable Hopf bifurcation. To verify the above, we need to calculate the controllability coefficients.

A. The controllability coefficients

We next examine the change of stability of the fixed points z_0^- on (30), with $\mu_1 < 0$ and $\delta_2 > 0$. Associated eigenvalues with the linearization about this curve of fixed points are

$$\lambda_{1,2}^- = \pm i\sqrt{2\delta_2\sqrt{-\frac{\mu_1}{\delta_2}}}$$

If we see μ_2 as a parameter, then using (29) we obtain the cross speed

$$d = \frac{d}{d\mu_2}\Re\mathrm{e}\lambda_{1,2}^{-}\Big|_{\mu_2=\frac{\delta_3\sqrt{-\frac{\mu_1}{\delta_2}}}{\delta_1}} = \frac{1}{2}\delta_1. \quad (31)$$

Therefore, a controllable Hopf bifurcation occurs on $\mu_2 = \frac{\delta_3\sqrt{-\frac{\mu_1}{\delta_2}}}{\delta_1}$.

Next, we check the stability of the bifurcating periodic orbits, for it we need to calculate the first Lyapunov coefficient l, which is given by derivatives of the nonlinear functions on the normal form of the system (27).

First we moved the fixed point to the origin. Let

$$\begin{aligned}\overline{z}_1 &= z_1 + \sqrt{-\frac{\mu_1}{\delta_2}}\\ \overline{z}_2 &= z_2\end{aligned}$$

be and from (27) we have

$$\begin{pmatrix}\dot{\overline{z}}_1\\ \dot{\overline{z}}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -2\delta_2\sqrt{-\frac{\mu_1}{\delta_2}} & 0\end{pmatrix}\begin{pmatrix}\overline{z}_1\\ \overline{z}_2\end{pmatrix} + \begin{pmatrix}0\\ \delta_2\overline{z}_1^2 + \delta_3\overline{z}_1\overline{z}_2\end{pmatrix}, \quad (32)$$

then we put the linear part of (32) in normal form via the linear transformation

$$\begin{pmatrix}\overline{z}_1\\ \overline{z}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ \sqrt{2\delta_2\sqrt{-\frac{\mu_1}{\delta_2}}} & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

under which (32) becomes

$$\begin{pmatrix}\dot{\varphi}_1\\ \dot{\varphi}_2\end{pmatrix} = \begin{pmatrix}0 & -\Phi\\ \Phi & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \begin{pmatrix}L_1(\varphi_1,\varphi_2)\\ L_2(\varphi_1,\varphi_2)\end{pmatrix} \quad (33)$$

where

$$\Phi = \sqrt{2\delta_2\sqrt{-\frac{\mu_1}{\delta_2}}},$$

$$L_1(\varphi_1,\varphi_2) = \frac{\delta_2\varphi_2^2}{\Phi} + \delta_3\varphi_1\varphi_2 \quad \text{and} \quad L_2(\varphi_1,\varphi_2) = 0,$$

and implement (2), we obtain

$$l = \frac{\delta_3}{16\sqrt{-\frac{\mu_1}{\delta_2}}}, \quad (34)$$

and we can conclude that the sign of l directly depending from the sign of δ_3.

We can look the dynamics of the negative branch in the Figure 4.

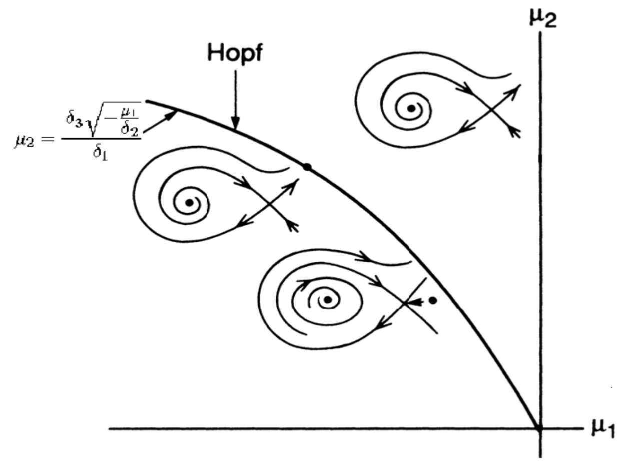

Fig. 4. $\mu_1 < 0$, $\delta_1 > 0$, $\delta_2 > 0$ y $\delta_3 > 0$.

VIII. Main result

Theorem 5: Given the nonlinear control system

$$\dot{x} = Jx + f(x) + g(x)u, \quad (35)$$

where $x \in \mathbb{R}^2$ and the control $u \in \mathbb{R}$. If

$$J = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad g(x) = b + Mx + \cdots,$$

with

$$b = \begin{pmatrix}b_1\\ b_2\end{pmatrix}, \text{ and } b_2 \neq 0,$$

then the feedback control law

$$\begin{aligned}u(x,\mu,\delta) &= \mu_1 + \mu L\left(P^{-1}x - h(P^{-1}x) - H(P^{-1}x)\right)\\ &\quad + x^T\left(P^{-1}\right)^T KP^{-1}x + \mathcal{O}(|x|^3), \quad (36)\end{aligned}$$

where $\mu = (\mu_1, \mu_2)$ is the artificial vector of bifurcation parameters and $\delta = (\delta_1, \delta_2, \delta_3)$ is the artificial vector of control parameters, P, H, h, L and K, are giving by (9), (10), (21), (22) and (23) respectively, is such that the closed-loop system (35)-(36), undergoes a controllable Hopf bifurcation in $\mu_2 = \frac{\delta_3\sqrt{-\frac{\mu_1}{\delta_2}}}{\delta_1}$, with the controllability coefficientes d and l given by (31) and (34) respectively.

IX. An example

We will illustrate the previous result with the following example,

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1^2 - x_2^2 + (x_1 + x_1x_2)u\\ \dot{x}_2 &= x_1x_2 + (1 - x_1 + x_2 + x_2^2)u,\end{aligned} \quad (37)$$

In this case,

$$P = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ H(x) = \begin{pmatrix}-\frac{1}{2}x_1^2 + x_1x_2\\ -x_1^2 - x_1x_2 + \frac{1}{2}x_2^2\end{pmatrix},$$

$$h(x) = \begin{pmatrix}0\\ \frac{3}{2}x_2^2\end{pmatrix},\ L = \begin{pmatrix}0 & 3\\ 0 & \delta_1\end{pmatrix},$$

and

$$K = \begin{pmatrix}\delta_2 & \frac{1}{2}(\delta_3 - 3)\\ \frac{1}{2}(\delta_3 - 3) & -1\end{pmatrix},$$

then, for this system, the control law is given by

$$\begin{aligned}u(x,\mu,\delta) &= \mu_1 + (3\mu_1 + \mu_2\delta_1)(x_2 + x_1^2 + x_1x_2 - 2x_2^2)\\ &\quad + \delta_2x_1^2 + (\delta_3 - 3)x_1x_2 - x_2^2 + \mathcal{O}(|x|^3).\end{aligned}$$

If we consider

$$\mu_1 = -0.0001,\ \mu_2 = -0.019,\ \delta_1 = \delta_2 = 1, \text{ and } \delta_3 = -2,$$

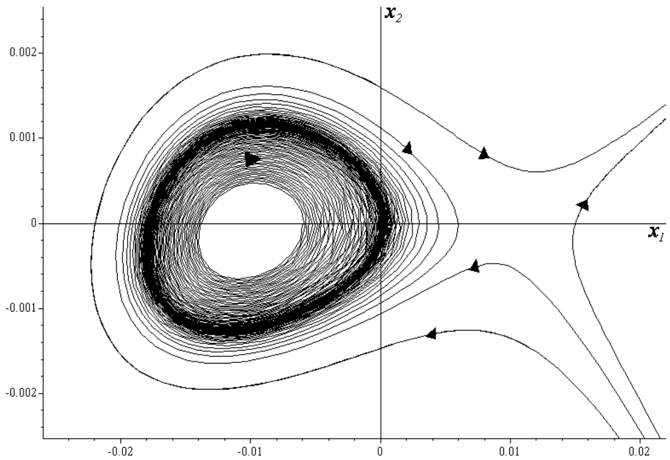

Fig. 5. Supercritical Hopf bifurcation for: $\mu_1 = -0.0001$, $\mu_2 = -0.019$, $\delta_1 = 1$, $\delta_2 = 1$ y $\delta_3 = -2$.

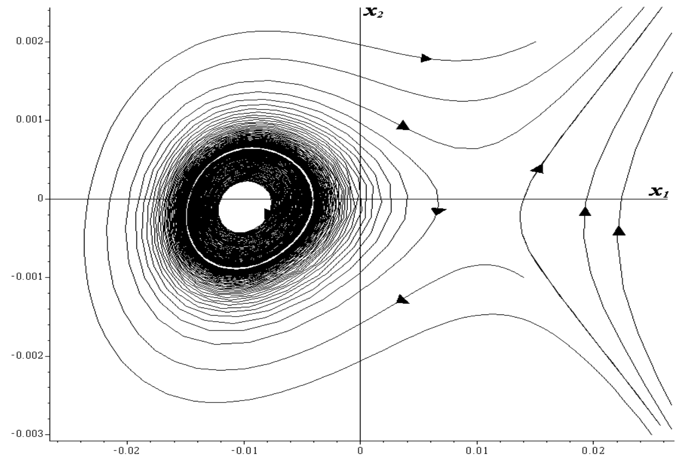

Fig. 6. Subcritical Hopf bifurcation for: $\mu_1 = -0.0001$, $\mu_2 = 0.019$, $\delta_1 = 1$, $\delta_2 = 1$ y $\delta_3 = 2$.

then

$$d = \frac{1}{2} \quad \text{and} \quad l = -\frac{25}{2},$$

and system (37) undergoes a supercritical Hopf bifurcation, where the closed orbit is stable. See Figure 5.

If we define

$$\mu_1 = -0.0001, \; \mu_2 = -0.019, \; \delta_1 = \delta_2 = 1, \; \text{and } \delta_3 = 2,$$

then

$$d = \frac{1}{2} \quad \text{and} \quad l = \frac{25}{2},$$

and system (37) undergoes a subcritical Hopf bifurcation, where the closed orbit is unstable. See Figure 6.

X. CONCLUSIONS

For a nonlinear control system in the plane, whose nominal vector field has in the origin, a double zero eigenvalue, we have designed a scalar control law such that the closed-loop system undergoes a controllable Hopf bifurcation. This work is the begining of a more general analysis about the control of codimension two bifurcations.

REFERENCES

[1] E.H. Abed, J.H. Fu. *Local feedback stabilization and bifurcation control, II. Stationary bifurcation*. Systems & Control Letters. 8 (1987). 467-473.

[2] E.H. Abed, J.H. Fu. *Local feedback stabilization and bifurcation control, I. Hopf bifurcation*. Systems & Control Letters. 7 (1986). 11-17.

[3] R.I. Bogdanov. *Versal deformations of a singular point on the plane in the case of zero eigenvalues*. Functional analysis and its applications 9. pp:144-145. 1975.

[4] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, Dynamical systems, and bifurcations of vector fields*. Springer-Verlag. 1993.

[5] B. Hamzi, W. Kang, J.P. Barbot. *Analysis and control of Hopf bifurcations*. SIAM J. Control Optim. Vol. 42, No. 6. 2200-2220. 2004.

[6] B. Hamzi. *Analysis and Stabilization of Nonlinear Systems with a Zero-Hopf Control Bifurcation*. Proceedings of the 41st IEEE Conference on Decision and Control. 3912-3917. 2002.

[7] B. Hamzi. *Quadratic Stabilization of Nonlinear Control Systems with a Double-Zero Control Bifurcation*. Proceedings of the 5st IFAC Symposium on Nonlinear Control Systems (NOLCOS'2001). pp 161-166. 2001.

[8] F. Takens. *Forced oscillations and bifurcations*. Applications of Global Analysis I. Comm. Math. Inst. Rijksuniversitat Utrecht, 3. pp:1-59. 1974.

[9] F. Verduzco. *Control of codimension one stationary bifurcations*. Int. J. of Bifurcation and Chaos, Vol. 17, No. 2. pp. 575-582. 2007.

[10] F. Verduzco, J. Alvarez. *Hopf bifurcation control: A new approach*. Systems & Control Letters 55. 437-451. 2006.

[11] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Texts in Applied Mathematics, Vol. 2. Springer-Verlag. Second Edition. 2003.