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PWA $\mathcal{H}_\infty$ controller synthesis for uncertain PWA slab systems

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***Abstract*— This paper presents a new piecewise-affine (PWA) $\mathcal{H}_\infty$ controller synthesis method for uncertain PWA slab systems. The synthesis problem is formulated as a set of Linear Matrix Inequalities (LMIs), which can then be solved efficiently using available software. The proposed synthesis methodology is applied to a circuit example.**

I. INTRODUCTION

PWA systems represent a powerful modeling framework for complex dynamical systems involving nonlinear phenomena. In fact, a broad range of nonlinear systems are either already PWA or can be accurately approximated by PWA systems. These include, but are not limited to, dead-zones, saturations, relays, and hysteresis. With the emergence of promising new methods for stability analysis [10], [5], [6], [14], state feedback controller synthesis [5], [11], [15], [16] and output feedback controller synthesis [12], [13], [8] for continuous-time PWA systems, this class of hybrid systems has become increasingly attractive for control purposes. In terms of convex formulations of state feedback controller synthesis, Hassibi [5] has shown that the PWL $\mathcal{H}_\infty$ controller synthesis problem for a class of PWA systems can be cast as an LMI. Later, Rodrigues and Boukas [17] have shown that PWL $\mathcal{H}_\infty$ controller synthesis for PWA slab systems with input and output constraints can also be cast as an LMI. However, these methods do not take into account model uncertainties.

Although stability analysis and controller synthesis for continuous-time piecewise-affine systems have received a great deal of attention, it is only recently that the robustness of these systems has been studied. Johansson [6] developed a method for performance analysis of piecewise-linear (PWL) systems while an *a posteriori* analysis method was developed by Rodrigues [16] for PWA systems acted upon by norm-bounded noise. Feng [3], [4] is probably the first to examine the synthesis of stabilizing and $\mathcal{H}_\infty$ controllers for uncertain PWA systems. However, this problem is in general not convex and can only be transformed into an LMI by assuming a special structure for the controller gain matrix. Recently, reference [18] suggested a convex optimization approach for PWA controller synthesis using PWA slab differential inclusions. To the best of our knowledge, this reference is the only convex approach to PWA controller synthesis in the literature.

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Given that PWA systems form a diverse and complex class of systems, formal synthesis methods must be targeted to systems with additional structure in order to cast the synthesis as a convex problem. In this sense, the work in this paper departs considerably from previous work on $\mathcal{H}_\infty$ controller synthesis for uncertain PWA systems and offers an interesting complementary approach. In fact, rather than imposing additional structure on the controller or using differential inclusions, in this paper we will focus instead on adding structure to the systems themselves by considering slab partitions of the state space. Although not the most general class of PWA systems, PWA slab systems represent an important subclass because many practical system models are either already in PWA slab form or can be approximated by this form with high degree of accuracy. Moreover, even for some systems that do not fall into this category, recent methods have been developed using backstepping techniques to transform the synthesis problem into the class of PWA slab systems [7], [19].

In summary, the work presented in this paper provides a systematic convex formulation of the PWA $\mathcal{H}_\infty$ controller synthesis problem for uncertain PWA slab systems as a set of LMIs. Efficient interior-point algorithms, implemented in available software packages, can then be used to solve the LMI constraints. The outline of the paper is as follows. First, the class of uncertain PWA systems is described and the control design problem is formulated. Then, the new PWA $\mathcal{H}_\infty$ controller synthesis method is developed. Finally, the proposed methodology will be applied to a circuit example and conclusions will be drawn.

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II. Problem Definition

A. Class of Uncertain PWA Systems

Consider a PWA system with dynamics described by

$$\begin{aligned}\dot{x}(t) = &(A_i + \Delta A_i)x(t) + (a_i + \Delta a_i) \\ &+(B_i + \Delta B_i)u(t) + (B_{w_i} + \Delta B_{w_i})w(t),\end{aligned} \tag{1}$$

for $x(t) \in \mathcal{R}_i$, where $u(t) \in \mathbb{R}^{n_u}$ is the input vector and $w(t) \in \mathbb{R}^{n_w}$ is the disturbance vector. Matrices $A_i \in \mathbb{R}^{n\times n}$, $a_i \in \mathbb{R}^n$, $B_i \in \mathbb{R}^{n\times n_u}$, $B_{w_i} \in \mathbb{R}^{n\times n_w}$ represent the nominal PWA system, while matrices $\Delta A_i \in \mathbb{R}^{n\times n}$, $\Delta a_i \in \mathbb{R}^n$, $\Delta B_i \in \mathbb{R}^{n\times n_u}$, $\Delta B_{w_i} \in \mathbb{R}^{n\times n_w}$ are the uncertainty terms. The polytopic regions, $\mathcal{R}_i$, $i \in \mathcal{I} = \{1, \ldots, M\}$, partition a subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ such that $\cup_{i=1}^M \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. We assume in this paper that the objective is to find a controller that stabilizes the closed-loop system to a single equilibrium point if no disturbances act in the system, and that provides a bound on the $\mathcal{L}_2$ gain in the presence of disturbances. Therefore, we denote the region in which the desired closed-loop equilibrium point x_{cl} lies as $\mathcal{R}_{i^\star}$. It is assumed that $a_{i^\star} = 0,\ \Delta a_{i^\star} = 0$.

Following [5], [6], [9], each cell is constructed as the intersection of a finite number (p_i) of half spaces

$$\mathcal{R}_i = \{x \in \mathbb{R}^n \mid E_i x + e_i \succ 0\}, \tag{2}$$

where $E_i \in \mathbb{R}^{p_i\times n}$, $e_i \in \mathbb{R}^{p_i}$, and $\succ$ represents an element-wise inequality. Each polytopic cell has a finite number of facets and vertices. Any two cells sharing a common facet will be called *level-1* neighboring cells. Let $\mathcal{N}_i = \{$*level*-1 neighboring cells of $\mathcal{R}_i\}$. It is assumed that vectors $H_{ij} \in \mathbb{R}^n$ and scalars h_{ij} exist such that the facet boundary between cells $\mathcal{R}_i$ and $\mathcal{R}_j$ is contained in the hyperplane described by $\{x \in \mathbb{R}^n | H_{ij}^T x + h_{ij} = 0\}$, for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$. A parametric description of the boundaries can then be obtained as [5]

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{F_{ij}s + f_{ij} \mid s \in \mathbb{R}^{n-1}\}, \tag{3}$$

where $F_{ij} \in \mathbb{R}^{n\times(n-1)}$ is a full rank matrix whose columns span the null space of H_{ij}^T, and $f_{ij} \in \mathbb{R}^n$ is given by

$$f_{ij} = -H_{ij}(H_{ij}^T H_{ij})^{-1} h_{ij}.$$

A slab is a special case of a polyhedron, and is defined as follows.

Definition 1: A slab is defined as

$$\mathcal{S} = \{x \in \mathbb{R}^n \mid h_1 < H^T x < h_2\}, \tag{4}$$

where $H \in \mathbb{R}^n$ and $h_1, h_2 \in \mathbb{R}$. □

If $\mathcal{R}_i$ from (2) is a slab $\mathcal{S}$ defined as in (4), we have

$$E_i = \begin{bmatrix} H^T \\ -H^T \end{bmatrix}, \quad e_i = \begin{bmatrix} -h_1 \\ h_2 \end{bmatrix}. \tag{5}$$

Moreover, each region $\mathcal{R}_i$ from (2), can be outer approximated by a degenerate ellipsoid $\mathcal{E}_i$, such that $\mathcal{E}_i \subseteq \mathcal{R}_i$,

$$\mathcal{E}_i = \{x \in \mathbb{R}^n \mid \|L_i x + l_i\| < 1\}, \tag{6}$$

where

$$\begin{cases} L_i = 2H^T/(h_2 - h_1) \\ l_i = -(h_2 + h_1)/(h_2 - h_1) \end{cases}. \tag{7}$$

Definition 2: A PWA slab system is a PWA system for which the regions are slabs. □

This paper focuses on PWA slab systems. Note that for these systems, the ellipsoidal covering (6) is exact, *i.e.*, $\mathcal{E}_i \subseteq \mathcal{R}_i$ and $\mathcal{R}_i \subseteq \mathcal{E}_i$. The use of slab regions is the key element enabling the proof of the main theorem of this paper, stating controller synthesis as a set of LMIs.

B. Design Objective

The objective is to design a PWA state feedback control law that stabilizes the uncertain PWA system (1) to the origin without disturbances and that verifies an $\mathcal{H}_\infty$ performance criterion in the presence of disturbances. In that regard, it is assumed that there is a measured output according to

$$y(t) = (C_i + \Delta C_i)x(t) \tag{8}$$

for $x(t) \in \mathcal{R}_i$, where $y(t) \in \mathbb{R}^p$ is the output vector, $C_i \in \mathbb{R}^{p\times n}$ and $\Delta C_i \in \mathbb{R}^{p\times n}$. The PWA state feedback control law is

$$u(t) = K_i x(t) + k_i, \tag{9}$$

for $x(t) \in \mathcal{R}_i$. Substituting (9) into (1) and using (8) yields the closed-loop system

$$\begin{cases} \dot{x}(t) = [(A_i + \Delta A_i) + (B_i + \Delta B_i)K_i]\, x(t) \\ \qquad + [(a_i + \Delta a_i) + (B_i + \Delta B_i)k_i] \\ \qquad + (B_{w_i} + \Delta B_{w_i})w(t) \\ y(t) = (C_i + \Delta C_i)x(t) \end{cases} \tag{10}$$

which can be rewritten as

$$\begin{cases} \dot{x}(t) = \bar{A}_i^{cl} x(t) + \bar{a}_i^{cl} + \bar{B}_{w_i} w(t) \\ y(t) = \bar{C}_i x(t) \end{cases} \tag{11}$$

where $\bar{A}_i^{cl} = (A_i + \Delta A_i) + (B_i + \Delta B_i)K_i$, $\bar{a}_i^{cl} = (a_i + \Delta a_i) + (B_i + \Delta B_i)k_i$, $\bar{B}_{w_i} = B_{w_i} + \Delta B_{w_i}$, and $\bar{C}_i = C_i + \Delta C_i$.

Finally, the following *a priori* assumptions adapted from previous work in the robust PWL control literature [3] are made for the uncertainty terms:

$$\begin{aligned} \Delta A_i^T \Delta A_i &\le U_{A_i}^T U_{A_i}, \\ \Delta a_i \Delta a_i^T &\le U_{a_i} U_{a_i}^T, \\ \Delta B_i^T \Delta B_i &\le U_{B_i}^T U_{B_i}, \\ \Delta B_i \Delta B_i^T &\le U_{B_i} U_{B_i}^T, \\ \Delta B_{w_i} \Delta B_{w_i}^T &\le U_{B_{w_i}} U_{B_{w_i}}^T, \\ \Delta C_i^T \Delta C_i &\le U_{C_i}^T U_{C_i}. \end{aligned} \tag{12}$$

The controller synthesis method will be described in the next section.

III. Controller Synthesis

This section presents the main result of the paper. We begin by stating one lemma and one definition to be used in the ensuing development. The PWA $\mathcal{H}_\infty$ controller synthesis method is then presented.

Lemma 1: [1] Let X and Y be real constant matrices of compatible dimensions. Then the following equation

$$X^T Y + Y^T X \le \epsilon X^T X + \epsilon^{-1} Y^T Y$$

holds for any $\epsilon > 0$. □

Definition 3: The $\mathcal{L}_2$ gain of the closed-loop system (11) is defined as

$$\sup_{\|w\|_2 \neq 0} \frac{\|y\|_2}{\|w\|_2},$$

where the $\mathcal{L}_2$ norm of an unknown time-varying signal $w(t)$ is defined by

$$\|w(t)\|_2^2 = \int_0^\infty [w(t)^T w(t)] dt,$$

and the supremum is taken over all nonzero trajectories of the system, starting from the state $x(0) = 0$. If the $\mathcal{L}_2$ gain of a system is less than some constant $\gamma > 0$, the system is said to have disturbance attenuation by a factor of at least γ. □

A. PWA Controller Synthesis

This section will present a PWA controller synthesis method as a theorem. The proof of the theorem will use the following result.

Lemma 2: If $M \ge 0$ then, for any matrix B with appropriate dimensions,

$$BMB^T \le \text{trace}(M) BB^T$$

Proof: It suffices to show that $\text{trace}(M)I - M \ge 0$. This is true because the eigenvalues $\lambda_i\,(\beta I - M)$ are equal to $\beta - \lambda_i(M)$ for any β. Therefore, since the trace is the sum of all eigenvalues and since $M \ge 0$, the eigenvalues of $[\text{trace}(M)I - M]$ are all greater than or equal to zero, which finishes the proof. □

Theorem 1: Consider the uncertain PWA slab system (1) with the PWA state feedback (9). Let the index of the region holding the desired equilibrium point for the closed-loop system be $i^\star$. If $Q = Q^T > 0$, $\alpha > 0$, $\eta > 0$, $\mu_i < 0$, $i = 1, \ldots, M$, $\epsilon_j > 0$, $j = 1, \ldots, 17$, inequality (17) and $k_{i^\star} = 0$ are verified for $i = i^\star$, and inequality (18) and $1 - l_i^2 < 0$ are verified for $i \neq i^\star$, where

$$\begin{aligned} \Omega_i = {} & A_i Q + B_i Y_i + Q A_i^T + Y_i^T B_i^T \\ & + \alpha Q + (\epsilon_1^{-1} + \epsilon_2^{-1}) I_{(n)} \\ & + \eta[(1 + \epsilon_6^{-1}) B_{w_i} B_{w_i}^T + (1 + \epsilon_6) U_{B_{w_i}} U_{B_{w_i}}^T] \end{aligned} \tag{13}$$

and

$$\begin{aligned} \bar{\Omega}_i = {} & A_i Q + B_i Y_i + Q A_i^T + Y_i^T B_i^T \\ & + \alpha Q + (\epsilon_1^{-1} + \epsilon_2^{-1}) I_{(n)} \\ & + \mu_i[(1 + \epsilon_3^{-1}) a_i a_i^T + (1 + \epsilon_3) U_{a_i} U_{a_i}^T] \\ & + \mu_i[a_i k_i^T B_i^T + B_i k_i a_i^T] \\ & + \mu_i[\epsilon_7 k_{max}^2 U_{a_i} U_{a_i}^T + \epsilon_7^{-1} B_i B_i^T] \\ & + \mu_i[\epsilon_8 k_{max}^2 a_i a_i^T + \epsilon_8^{-1} U_{B_i} U_{B_i}^T] \\ & + \mu_i[\epsilon_9 k_{max}^2 U_{a_i} U_{a_i}^T + \epsilon_9^{-1} U_{B_i} U_{B_i}^T] \\ & + \mu_i k_{max}^2 [(1 + \epsilon_{10}^{-1}) B_i B_i^T + (1 + \epsilon_{10}) U_{B_i} U_{B_i}^T] \\ & + \mu_i l_i^2 (1 - l_i^2)^{-1} [(1 + \epsilon_{11}^{-1}) a_i a_i^T + (1 + \epsilon_{11}) U_{a_i} U_{a_i}^T] \\ & + \mu_i l_i^2 (1 - l_i^2)^{-1} [a_i k_i^T B_i^T + B_i k_i a_i^T] \\ & + \mu_i l_i^2 (1 - l_i^2)^{-1} [\epsilon_{12} k_{max}^2 U_{a_i} U_{a_i}^T + \epsilon_{12}^{-1} B_i B_i^T] \\ & + \mu_i l_i^2 (1 - l_i^2)^{-1} [\epsilon_{13} k_{max}^2 a_i a_i^T + \epsilon_{13}^{-1} U_{B_i} U_{B_i}^T] \\ & + \mu_i l_i^2 (1 - l_i^2)^{-1} [\epsilon_{14} k_{max}^2 U_{a_i} U_{a_i}^T + \epsilon_{14}^{-1} U_{B_i} U_{B_i}^T] \\ & + \mu_i l_i^2 (1 - l_i^2)^{-1} k_{max}^2 * \\ & \qquad [(1 + \epsilon_{15}^{-1}) B_i B_i^T + (1 + \epsilon_{15}) U_{B_i} U_{B_i}^T] \\ & + l_i (1 - l_i^2)^{-1} [a_i L_i Q + Q L_i^T a_i^T] \\ & + \epsilon_4 U_{a_i} U_{a_i}^T + \epsilon_{16} B_i B_i^T + \epsilon_{17} U_{B_i} U_{B_i}^T \\ & + \eta[(1 + \epsilon_6^{-1}) B_{w_i} B_{w_i}^T + (1 + \epsilon_6) U_{B_{w_i}} U_{B_{w_i}}^T] \end{aligned} \tag{14}$$

and

$$\Gamma_i = \mu_i^{-1}(1 - l_i^2)^{-1} + (\epsilon_4^{-1} + k_{max}^2 \epsilon_{16}^{-1} + k_{max}^2 \epsilon_{17}^{-1}) l_i^2 (1 - l_i^2)^{-2} \tag{15}$$

and if, furthermore,

$$\begin{bmatrix} k_{max}^2 & k_i \\ k_i^T & 1 \end{bmatrix} > 0, \ i = 1, \ldots, M, \tag{16}$$

then the $\mathcal{L}_2$ gain of the closed-loop system (11) is less than $\gamma = \eta^{-0.5}$ and the system is exponentially stable with a decay rate of at least α in the absence of disturbances. □

Proof: The proof is given for the case where $i \neq i^\star$. The proof for the case where $i = i^\star$ is similar with the important difference that $a_{i^\star} = 0$, $\Delta a_{i^\star} = 0$, $k_{i^\star} = 0$

$$\begin{bmatrix} \Omega_i & Y_i^T U_{B_i}^T & QU_{A_i}^T & QC_i^T & QU_{C_i}^T \\ U_{B_i}Y_i & -\epsilon_2^{-1}I_{(n)} & 0 & 0 & 0 \\ U_{A_i}Q & 0 & -\epsilon_1^{-1}I_{(n)} & 0 & 0 \\ C_iQ & 0 & 0 & -(1+\epsilon_5^{-1})^{-1}I_{(p)} & 0 \\ U_{C_i}Q & 0 & 0 & 0 & -(1+\epsilon_5)^{-1}I_{(p)} \end{bmatrix} < 0, \tag{17}$$

$$\begin{bmatrix} \bar{\Omega}_i & Y_i^T U_{B_i}^T & QU_{A_i}^T & QC_i^T & QU_{C_i}^T & QL_i^T \\ U_{B_i}Y_i & -\epsilon_2^{-1}I_{(n)} & 0 & 0 & 0 & 0 \\ U_{A_i}Q & 0 & -\epsilon_1^{-1}I_{(n)} & 0 & 0 & 0 \\ C_iQ & 0 & 0 & -(1+\epsilon_5^{-1})^{-1}I_{(p)} & 0 & 0 \\ U_{C_i}Q & 0 & 0 & 0 & -(1+\epsilon_5)^{-1}I_{(p)} & 0 \\ L_iQ & 0 & 0 & 0 & 0 & -\Gamma_i^{-1} \end{bmatrix} < 0, \tag{18}$$

and the $\mathcal{S}-$procedure cannot be used for $i = i^\star$ because the equilibrium point is in the closure of $\mathcal{R}_{i^\star}$. Using a quadratic candidate Lyapunov function

$$V(x) = x^T P x, \tag{19}$$

sufficient conditions for the $\mathcal{L}_2$ gain of system (11) being less than $\gamma > 0$ and for exponential stability with a decay rate of at least α in the absence of disturbances are

$$P = P^T > 0 \tag{20}$$

and

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} < -\alpha x^T P x + \gamma^2 w^T w - y^T y. \tag{21}$$

These conditions can be rewritten as $\gamma > 0$, $\alpha > 0$, $P = P^T > 0$ and

$$\begin{bmatrix} x \\ w \\ 1 \end{bmatrix}^T \begin{bmatrix} \Xi_i & P\bar{B}_{w_i} & P(\bar{a}_i^{cl}) \\ \bar{B}_{w_i}^T P & -\gamma^2 I & 0 \\ (\bar{a}_i^{cl})^T P & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ w \\ 1 \end{bmatrix} < 0,\ x \in \mathcal{R}_i, \tag{22}$$

where $\Xi_i = P(\bar{A}_i^{cl}) + (\bar{A}_i^{cl})^T P + \alpha P + \bar{C}_i^T \bar{C}_i$. This condition can be relaxed using the $\mathcal{S} - procedure$ [2] with multipliers $\lambda_i < 0$ and the ellipsoid description (6), yielding the sufficient conditions $\gamma > 0$, $\alpha > 0$, $P = P^T > 0$ and

$$\begin{bmatrix} \Xi_i + \lambda_i L_i^T L_i & P\bar{B}_{w_i} & (P\bar{a}_i^{cl} + \lambda_i L_i^T l_i) \\ \bar{B}_{w_i}^T P & -\gamma^2 I & 0 \\ (P\bar{a}_i^{cl} + \lambda_i L_i^T l_i)^T & 0 & -\lambda_i(1 - l_i^T l_i) \end{bmatrix} < 0. \tag{23}$$

Using Schur complement twice, inequality (23) is equivalent to $1 - l_i^T l_i < 0$ and

$$\begin{aligned} &P(\bar{A}_i^{cl}) + (\bar{A}_i^{cl})^T P + \alpha P + \lambda_i L_i^T L_i \\ &\quad + \lambda_i^{-1}(P\bar{a}_i^{cl} + \lambda_i L_i^T l_i)(1 - l_i^T l_i)^{-1}(P\bar{a}_i^{cl} + \lambda_i L_i^T l_i)^T \\ &\quad + \bar{C}_i^T \bar{C}_i + \gamma^{-2} P\bar{B}_{w_i}\bar{B}_{w_i}^T P < 0. \end{aligned} \tag{24}$$

Expanding the first two terms of (24) yields

$$\begin{aligned} &PA_i + PB_iK_i + A_i^T P + K_i^T B_i^T P + \alpha P + \lambda_i L_i^T L_i \\ &\quad + P\Delta A_i + \Delta A_i^T P + P\Delta B_i K_i + K_i^T \Delta B_i^T P \\ &\quad + \lambda_i^{-1}(P\bar{a}_i^{cl} + \lambda_i L_i^T l_i)(1 - l_i^T l_i)^{-1}(P\bar{a}_i^{cl} + \lambda_i L_i^T l_i)^T \\ &\quad + \bar{C}_i^T \bar{C}_i + \gamma^{-2} P\bar{B}_{w_i}\bar{B}_{w_i}^T P < 0. \end{aligned} \tag{25}$$

Applying Lemma 1 (the order of the factors is arbitrary),

$$\begin{cases} P\Delta A_i + \Delta A_i^T P \leq \epsilon_1 \Delta A_i^T \Delta A_i + \epsilon_1^{-1} PP \\ P\Delta B_i K_i + K_i^T \Delta B_i^T P \leq \\ \qquad \epsilon_2 K_i^T \Delta B_i^T \Delta B_i K_i + \epsilon_2^{-1} PP \end{cases}$$

with constants $\epsilon_j > 0,\ j = 1, 2$. The left-hand side (L.H.S) of inequality (25) can then be bounded as

$$\begin{aligned} &L.H.S. \leq PA_i + PB_iK_i + A_i^T P + K_i^T B_i^T P \\ &\quad + \alpha P + \lambda_i L_i^T L_i + (\epsilon_1^{-1} + \epsilon_2^{-1})PP \\ &\quad + \epsilon_1 \Delta A_i^T \Delta A_i + \epsilon_2 K_i^T \Delta B_i^T \Delta B_i K_i \\ &\quad + \lambda_i^{-1}(P\bar{a}_i^{cl} + \lambda_i L_i^T l_i)(1 - l_i^T l_i)^{-1}(P\bar{a}_i^{cl} + \lambda_i L_i^T l_i)^T \\ &\quad + \bar{C}_i^T \bar{C}_i + \gamma^{-2} P\bar{B}_{w_i}\bar{B}_{w_i}^T P. \end{aligned} \tag{26}$$

Defining $\mu_i = \lambda_i^{-1}$, $\eta = \gamma^{-2}$ and $Q = P^{-1}$, then pre-multiplying by Q^T and post-multiplying by Q, where $Q = Q^T$, the sufficient conditions become $\mu_i < 0$, $\eta > 0$, $\alpha > 0$, $1 - l_i^T l_i < 0$, $Q > 0$, and

$$\begin{aligned} &A_iQ + B_iK_iQ + QA_i^T + QK_i^T B_i^T \\ &\quad + \alpha Q + \mu_i^{-1} QL_i^T L_i Q + (\epsilon_1^{-1} + \epsilon_2^{-1})I_{(n)} \\ &\quad + \epsilon_1 Q\Delta A_i^T \Delta A_i Q + \epsilon_2 QK_i^T \Delta B_i^T \Delta B_i K_i Q \\ &\quad + \mu_i(\bar{a}_i^{cl} + \mu_i^{-1} QL_i^T l_i)(1 - l_i^T l_i)^{-1}(\bar{a}_i^{cl} + \mu_i^{-1} QL_i^T l_i)^T \\ &\quad + Q\bar{C}_i^T \bar{C}_i Q + \eta \bar{B}_{w_i}\bar{B}_{w_i}^T < 0. \end{aligned} \tag{27}$$

Following a similar procedure as the one used by Rodrigues and Boyd [15], it can be concluded that (27) is equivalent to

$$
\begin{aligned}
&A_iQ + B_iK_iQ + QA_i^T + QK_i^TB_i^T \\
&\quad +\alpha Q + \mu_i(\bar{a}_i^{cl})(\bar{a}_i^{cl})^T + (\epsilon_1^{-1} + \epsilon_2^{-1})I_{(n)} \\
&\quad +\epsilon_1 Q\Delta A_i^T\Delta A_i Q + \epsilon_2 QK_i^T\Delta B_i^T\Delta B_iK_iQ \\
&\quad +\mu_i^{-1}(1-l_i^2)^{-1}(\mu_i l_i\bar{a}_i^{cl} + QL_i^T)(\mu_i l_i\bar{a}_i^{cl} + QL_i^T)^T \\
&\quad +\eta\bar{B}_{w_i}\bar{B}_{w_i}^T + Q\bar{C}_i^T\bar{C}_iQ < 0,
\end{aligned} \tag{28}
$$

using the fact that $l_i = l_i^T$ is a scalar for slab systems. Expanding terms, using Lemma 1, Lemma 2, assumptions (12), and $k_i^Tk_i < k_{max}^2$, which can be written as (16), changing variables to $Y_i = K_iQ$, the constraints (20) and (21) are guaranteed to be satisfied if

$$
\begin{aligned}
&\begin{bmatrix} k_{max}^2 & k_i \\ k_i^T & 1 \end{bmatrix} > 0, \quad 1 - l_i^2 < 0, \quad \mu_i < 0, \quad \eta > 0, \\
&\alpha > 0, \quad \epsilon_j > 0,\ j = 1,\ldots,17, \quad Q = Q^T > 0, \\
&A_iQ + B_iY_i + QA_i^T + Y_i^TB_i^T \\
&\quad +\alpha Q + (\epsilon_1^{-1} + \epsilon_2^{-1})I_{(n)} \\
&\quad +\epsilon_1 QU_{A_i}^TU_{A_i}Q + \epsilon_2 Y_i^TU_{B_i}^TU_{B_i}Y_i \\
&\quad +\mu_i[(1+\epsilon_3^{-1})a_ia_i^T + (1+\epsilon_3)U_{a_i}U_{a_i}^T] \\
&\quad +\mu_i[a_ik_i^TB_i^T + B_ik_ia_i^T] \\
&\quad +\mu_i[\epsilon_7 k_{max}^2U_{a_i}U_{a_i}^T + \epsilon_7^{-1}B_iB_i^T] \\
&\quad +\mu_i[\epsilon_8 k_{max}^2a_ia_i^T + \epsilon_8^{-1}U_{B_i}U_{B_i}^T] \\
&\quad +\mu_i[\epsilon_9 k_{max}^2U_{a_i}U_{a_i}^T + \epsilon_9^{-1}U_{B_i}U_{B_i}^T] \\
&\quad +\mu_i k_{max}^2[(1+\epsilon_{10}^{-1})B_iB_i^T + (1+\epsilon_{10})U_{B_i}U_{B_i}^T] \\
&\quad +\mu_i l_i^2(1-l_i^2)^{-1}* \\
&\qquad\qquad [(1+\epsilon_{11}^{-1})a_ia_i^T + (1+\epsilon_{11})U_{a_i}U_{a_i}^T] \\
&\quad +\mu_i l_i^2(1-l_i^2)^{-1}[a_ik_i^TB_i^T + B_ik_ia_i^T] \\
&\quad +\mu_i l_i^2(1-l_i^2)^{-1}[\epsilon_{12}k_{max}^2U_{a_i}U_{a_i}^T + \epsilon_{12}^{-1}B_iB_i^T] \\
&\quad +\mu_i l_i^2(1-l_i^2)^{-1}[\epsilon_{13}k_{max}^2a_ia_i^T + \epsilon_{13}^{-1}U_{B_i}U_{B_i}^T] \\
&\quad +\mu_i l_i^2(1-l_i^2)^{-1}[\epsilon_{14}k_{max}^2U_{a_i}U_{a_i}^T + \epsilon_{14}^{-1}U_{B_i}U_{B_i}^T] \\
&\quad +\mu_i l_i^2(1-l_i^2)^{-1}k_{max}^2* \\
&\qquad\qquad [(1+\epsilon_{15}^{-1})B_iB_i^T + (1+\epsilon_{15})U_{B_i}U_{B_i}^T] \\
&\quad +l_i(1-l_i^2)^{-1}[a_iL_iQ + QL_i^Ta_i^T] \\
&\quad +\epsilon_4U_{a_i}U_{a_i}^T + \epsilon_4^{-1}l_i^2(1-l_i^2)^{-2}QL_i^TL_iQ \\
&\quad +\epsilon_{16}B_iB_i^T + \epsilon_{16}^{-1}l_i^2(1-l_i^2)^{-2}k_{max}^2QL_i^TL_iQ \\
&\quad +\epsilon_{17}U_{B_i}U_{B_i}^T + \epsilon_{17}^{-1}l_i^2(1-l_i^2)^{-2}k_{max}^2QL_i^TL_iQ \\
&\quad +\mu_i^{-1}(1-l_i^2)^{-1}QL_i^TL_iQ \\
&\quad +\eta[(1+\epsilon_6^{-1})B_{w_i}B_{w_i}^T + (1+\epsilon_6)U_{B_{w_i}}U_{B_{w_i}}^T] \\
&\quad +Q[(1+\epsilon_5^{-1})C_i^TC_i + (1+\epsilon_5)U_{C_i}^TU_{C_i}]Q \le 0.
\end{aligned} \tag{29}
$$

This last expression can be written in matrix form by successive uses of the Schur complement, yielding the LMI (18), which finishes the proof. □

Based on this result, the PWA $\mathcal{H}_\infty$ controller synthesis problem can be formulated as follows:

Problem 1: Given $\alpha > 0,\ k_{max} > 0,\ \epsilon_j > 0,\ j = 1,\ldots,17$, and $\mu_i < 0$,,

$$
\begin{aligned}
\max \quad & \eta \\
s.t. \quad & \eta > 0,\ (16), \\
& Q = Q^T > 0,\ -Y_i^{lim} \prec Y_i \prec Y_i^{lim}, \\
& (17),\ (13) \text{ for } i = i^\star, \\
& (18),\ (14),\ (15) \text{ and } 1 - l_i^2 < 0 \text{ for } i \neq i^\star, \\
& i = 1,\ldots,M,
\end{aligned}
$$

where $\succ$ and $\prec$ mean component-wise inequalities and Y_{lim} are given vector bounds. □

The new controller synthesis methodology will now be applied to a circuit example.

IV. EXAMPLE

Consider the circuit with dynamics [5]

$$
\begin{cases} \dot{x}_1 = f(x_1) + 0.5x_2 \\ \dot{x}_2 = -0.2x_1 - 0.3x_2 + 0.2u \end{cases} \tag{30}
$$

where x_1 is the capacitor voltage and x_2 is the inductor current. For $x_1 \in (0,1)$, the PWA voltage-current characteristic is given by

$$
f(x_1) = \begin{cases} -4.4164x_1 & ,\ 0.00 < x_1 < 0.11 \\ +1.1747x_1 - 0.615 & ,\ 0.11 < x_1 < 0.46 \\ +0.0039x_1 - 0.0729 & ,\ 0.46 < x_1 < 0.86 \\ -3.0271x_1 + 2.5271 & ,\ 0.86 < x_1 < 1.00 \end{cases} \tag{31}
$$

By Definition 2, this is a slab system. Thus, the PWA controller synthesis method developed in the previous section can be applied. The bounding ellipsoids are

$$
\begin{aligned}
L_1 &= \begin{bmatrix} 25.00 & 0 \end{bmatrix}, & l_1 &= 6.50, \\
L_2 &= \begin{bmatrix} 5.26 & 0 \end{bmatrix}, & l_2 &= 0.16, \\
L_3 &= \begin{bmatrix} 5.88 & 0 \end{bmatrix}, & l_3 &= -1.94, \\
L_4 &= \begin{bmatrix} 22.22 & 0 \end{bmatrix}, & l_4 &= -14.56.
\end{aligned} \tag{32}
$$

It is assumed that the system uncertainties verify the upper bounds (12) where

$$
\begin{aligned}
U_{A_i} &= \begin{bmatrix} 0.3 & 0 \\ 0 & 0 \end{bmatrix}, \quad U_{a_i} = \begin{bmatrix} 0.03 \\ 0 \end{bmatrix}, \quad U_{B_i} = \begin{bmatrix} 0 \\ 10^{-6} \end{bmatrix}, \\
U_{B_{w_i}} &= \begin{bmatrix} 0 \\ 10^{-6} \end{bmatrix}, \quad U_{C_i} = \begin{bmatrix} 10^{-6} & 0 \\ 0 & 10^{-6} \end{bmatrix}.
\end{aligned} \tag{33}
$$

The desired closed-loop equilibrium point is

$$
\begin{bmatrix} x_{1_{cl}} \\ x_{2_{cl}} \end{bmatrix} = \begin{bmatrix} 0.300 \\ 0.557 \end{bmatrix}. \tag{34}
$$

Thus, the closed-loop equilibrium lies in region 2, *i.e.*, $i^\star = 2$. For simplicity, a change of coordinates is performed such that the desired closed-loop equilibrium point is transformed to the origin. For the synthesis of the PWA controller, Problem 1 is solved with $\alpha = 0.01$,

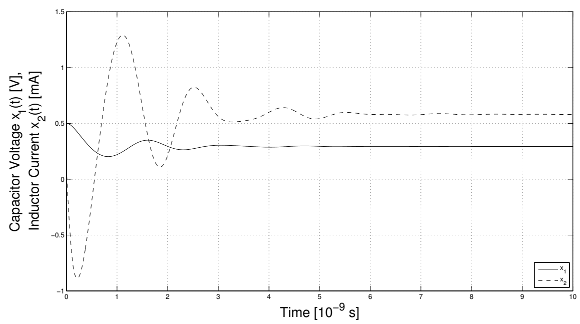

Fig. 1. Plot of states vs. time with disturbances (solid line: capacitor voltage, dashed line: inductor current)

$k_{max} = 9$, $Y_1^{lim} = 50$, $Y_2^{lim} = 0.22$, $Y_3^{lim} = 120$, $Y_4^{lim} = 50$, $\mu_i = -200$,

$$\begin{array}{lll} \epsilon_1 = 1000, & \epsilon_2 = 1000, & \epsilon_3 = 1000, \\ \epsilon_4 = 0.1, & \epsilon_5 = 10, & \epsilon_6 = 1, \\ \epsilon_7 = 1000, & \epsilon_8 = 1000, & \epsilon_9 = 1000, \\ \epsilon_{10} = 10000, & \epsilon_{11} = 1, & \epsilon_{12} = 10, \\ \epsilon_{13} = 10, & \epsilon_{14} = 10, & \epsilon_{15} = 10000, \\ \epsilon_{16} = 1, & \epsilon_{17} = 0.1. & \end{array}$$

The resulting controller gains are given by

$$\begin{array}{ll} K_1 = \begin{bmatrix} +171.44 & +2.06 \end{bmatrix}, & k_1 = -0.1933, \\ K_2 = \begin{bmatrix} -232.27 & -16.15 \end{bmatrix}, & k_2 = 0, \\ K_3 = \begin{bmatrix} -289.51 & -19.84 \end{bmatrix}, & k_3 = -0.1923, \\ K_4 = \begin{bmatrix} -234.24 & -29.53 \end{bmatrix}, & k_4 = +0.0644, \end{array} \quad (35)$$

with $\eta = 1.6303$. The disturbance rejection is thus guaranteed to be at least $\gamma = \eta^{-0.5} = 0.6134$. To simulate the performance of this controller, the system is subjected to a time-varying disturbance given by $w(t) = 5e^{-0.6t} sin(2\pi t)\ V$, which has finite $\mathcal{L}_2$-norm. Figure 1 shows the response of the system. It can be seen that the disturbance is attenuated and the system converges to the desired equilibrium point as the disturbance converges to zero.

V. Conclusions

This paper presented a new method for synthesizing a PWA $\mathcal{H}_\infty$ controller for uncertain PWA slab systems. The problem was formulated as a set of LMIs that can be efficiently solved using available software. The proposed controller synthesis methodology was successfully applied to a circuit example.

VI. Acknowledgements

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