Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Transverse Function control of a class of non-invariant driftless systems. Application to vehicles with trailers

Pascal Morin and Claude Samson
INRIA
2004, Route des Lucioles
06902 Sophia-Antipolis Cedex, France
E-mail address: *Pascal.Morin@inria.fr, Claude.Samson@inria.fr*

***Abstract*— The paper addresses the stabilization of reference trajectories for a class of nonlinear driftless systems. The proposed method is based on the so-called Transverse Function approach, a control design method initially developed by the authors for driftless systems invariant with respect to a Lie group operation. The present work shows how the approach can be generalized to a larger class of systems, not necessarily invariant. This possibility is illustrated with the control of unicycle-type (or car-like) vehicles with an arbitrary number of trailers, and with simulation results in the case of two trailers.**

I. INTRODUCTION

The problems here addressed concern the stabilization of reference trajectories for driftless systems of the form

$$\dot{x} = \sum_{i=1}^{m} X_i(x)u_i \qquad (1)$$

with $X_1, \ldots, X_m$ smooth vector fields (v.f.) on some manifold, x the state, and $u = (u_1, \ldots, u_m)'$ the control vector. It is assumed that $m < n := \dim(x)$. This inequality is commonly attached to kinematic models of nonholonomic mechanical systems (wheeled robots, rolling spheres, etc). It is well known that some of the problems associated with the feedback control of System (1) are particularly difficult in this case. First, given any equilibrium $(x, u) = (x_0, 0)$, the associated linearized system is neither controllable nor asymptotically stabilizable. Then, it follows from Brockett's theorem [1] that when the vectors $X_1(x_0), \ldots, X_m(x_0)$ are linearly independent, x_0 cannot be asymptotically stabilized by using smooth pure state feedback $u(x)$. Other classes of feedbacks (time-varying periodic feedback [2], [3], hybrid feedback [4], [5],...) have been proposed to circumvent this difficulty, but with mitigated success in practice due to unsolved robustness problems. For a certain number of systems the asymptotic stabilization of non-stationary reference trajectories is less difficult, and many feedback control methods have also been proposed for this problem (see e.g. [6], [7]). They usually rely on "persistent excitation" conditions which allow to exploit the controllability properties of the linearized error system. However, an important obstruction to the asymptotic stabilization of admissible reference trajectories has been proved by Lizárraga in [8], the essence of which is that some *a priori* knowledge about the properties of these trajectories is required. A consequence is that given a causal trajectory tracking controller, there always exists a reference trajectory which this controller cannot asymptotically stabilize. Such an obstruction does not hold for linear systems.

In [9], we have proposed a new control approach for driftless systems. It relies on the "transverse function" concept [10]. Contrary to more classical methods, the primary objective of the approach is *practical* stabilization, i.e. stabilization of the system's state in a "small" neighborhood of the reference state. One of its most noticeable feature is that it allows for the construction of feedback controllers ensuring the practical stabilization of *any* reference trajectory, i.e. it does not have to be admissible. In addition, the ultimate tracking error can be made arbitrarily small by a proper choice of the control parameters. This approach has mostly been developed under the assumption that the state space is a Lie group and that the control v.f. $X_1, \ldots, X_m$ are invariant with respect to the group operation. However, many physical systems do not possess this property. The main contribution of the present paper is to extend the control solution proposed in [9] to a class of non-invariant systems. The control of unicycle-type (or car-like) vehicles with an arbitrary number of trailers is used to illustrate the proposed method.

The paper is organized as follows. Section II presents the notation and recalls the basics of the transverse function approach. The main results are presented in Section III. First, the calculation of transverse functions is addressed for a class of systems which are feedback equivalent to systems on a Lie group. Then, stabilizing feedback laws are proposed for an encompassing class of non-invariant systems. In Section IV, the results of Section III are applied to unicycle-type or car-like vehicles with trailers. Finally, simulation results for a unicycle with two trailers are reported in Section V. The choice of some of the control degrees of freedom is also briefly discussed in this last section. The proofs of the presented results are given in the appendix.

II. NOTATION AND RECALLS

A. *Vectors, manifolds, differential geometry*

The transpose of a vector $x \in \mathbb{R}^n$ is denoted as x', its i-th component as x_i, and its euclidean norm as $|x|$. The i-th vector of the canonical basis of $\mathbb{R}^n$ is denoted

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

as b_i, i.e. $b_i' x = x_i \ \forall x$. The $m \times m$ identity matrix is denoted as I_m. The tangent space at q of a manifold M is denoted as $T_q M$. Given a family $X^1 = \{X_1, \ldots, X_m\}$ of smooth v.f. on a manifold M, we denote by $\text{Lie}(X^1)$ the Lie algebra of v.f. generated by $X_1, \ldots, X_m$, i.e. $\text{Lie}(X^1) = \text{span}\{X_i, [X_i, X_j], [X_i, [X_j, X_k]], \ldots\}$, and by $\text{Lie}(X^1)(q)$ the subspace of $T_q M$ equal to $\text{span}\{X(q); X \in \text{Lie}(X^1)\}$. Recall that the family X^1 satisfies the so-called "Lie Algebra Rank Condition" at $q \in M$ if $\text{Lie}(X^1)(q) = T_q M$. The following notation is used repeatedly in the sequel. Given a family $X^1 = \{X_1, \ldots, X_m\}$ of smooth v.f. on M and a vector $\xi \in \mathbb{R}^m$, we denote by $X^1(q)\xi$ the tangent vector $\sum_{i=1}^m X_i(q)\xi_i \in T_q M$.

B. Systems on Lie groups

Let G denote a connected Lie group of dimension n. The unit element of G is denoted as e, i.e. $\forall g \in G \ : \ ge = eg = g$. The inverse g^{-1} of $g \in G$ is the (unique) element in G such that $gg^{-1} = g^{-1}g = e$. The left (resp. right) translation operator on G is denoted as L (resp. R), i.e. $\forall (\sigma, \tau) \in G^2 \ : \ L_\sigma(\tau) = R_\tau(\sigma) = \sigma\tau$. A v.f. X on G is left-invariant iff $\forall (\sigma, \tau) \in G^2, \ dL_\sigma(\tau)X(\tau) = X(\sigma\tau)$, with df denoting the differential of a mapping f. The Lie algebra –of left-invariant v.f.– of the group G is denoted as $\mathfrak{g}$. The adjoint representation of G is denoted as Ad, i.e. $\forall \sigma \in G, \ \text{Ad}(\sigma) := dI_\sigma(e)$, with $I_\sigma : G \to G$ defined by $I_\sigma(g) := \sigma g \sigma^{-1}$. If $X \in \mathfrak{g}$, $\exp(tX)$ is the solution at time t of $\dot{g} = X(g)$ with the initial condition $g(0) = e$. A driftless control system $\dot{g} = \sum_{i=1}^m X_i(g)\xi_i$ on G is said to be left-invariant on G if the control v.f. X_i are left-invariant.

Let $X = \{X_1, \ldots, X_n\}$ denote a basis of $\mathfrak{g}$. If $(g_a(t), \xi_a(t))$ and $(g_b(t), \xi_b(t))$ $(t \geq 0)$ are two solutions to $\dot{g} = X(g)\xi = \sum_{i=1}^n X_i(x)\xi_i$, then (omitting the time index)

$$\frac{d}{dt}(g_a g_b^{-1}) = X(g_a g_b^{-1})\text{Ad}^X(g_b)(\xi_a - \xi_b) \qquad (2)$$

with Ad^X the expression of the Ad operator in the basis X, i.e. the (invertible) matrix-valued function defined by $\forall \sigma \in G, \ \forall \xi \in \mathbb{R}^n, \ \text{Ad}(\sigma)X(e)\xi = X(e)\text{Ad}^X(\sigma)\xi$. According to this definition, $\text{Ad}^X(e) = I_n$. We have also

$$\frac{d}{dt}(g_a^{-1} g_b) = X(g_a^{-1} g_b)(\xi_b - \text{Ad}^X(g_b^{-1} g_a)\xi_a) \qquad (3)$$

C. Transverse Functions

Definition and general characterization Let $X^1 = \{X_1, \ldots, X_m\}$ denote a family of smooth v.f. $X_1, \ldots, X_m$ on a n-dimensional manifold M, $\mathbb{T}^p$ denote the p-dimensional torus, and H denote another manifold. A smooth function $f : \mathbb{T}^p \times H \longrightarrow M$ is *transverse to* X^1 if, for any $(\alpha, \xi) \in \mathbb{T}^p \times H$, the vectors

$$X_1(f(\alpha, \xi)), \ldots, X_m(f(\alpha, \xi)), \frac{\partial f}{\alpha_1}(\alpha, \xi), \ldots, \frac{\partial f}{\alpha_p}(\alpha, \xi)$$

span $T_{f(\alpha,\xi)}M$. Note that p, the dimension of $\mathbb{T}^p$, must be at least equal to $(n - m)$. This definition is a slight generalization of the original definition given in [10], for which H was the empty set, i.e. $\mathbb{T}^p \times H = \mathbb{T}^p$. Given smooth functions $f^\varepsilon : \mathbb{T}^p \times H \longrightarrow M$ defined for $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 > 0$, we say that (f^ε) is a *family of functions transverse to* X^1 if $\forall \varepsilon \in (0, \varepsilon_0)$, f^ε is transverse to X^1. Given $q_0 \in M$ such that $\text{Lie}(X^1)(q_0) = T_{q_0}M$, the "transverse function theorem" given in [10] ensures the existence of a family of functions transverse to X^1, with $H = \emptyset$ and $\max_\alpha \text{dist}(f^\varepsilon(\alpha), q_0) \to 0$ as $\varepsilon \to 0$, where "dist" denotes any distance locally defined in the neighborhood of q_0. When these complementary properties are satisfied we say that the family (f^ε) is *centered on* q_0.

The case of invariant v.f. on Lie groups When $M = G$ is a Lie group and $X_1, \ldots, X_m$ are independent elements[1] of the Lie algebra $\mathfrak{g}$, stronger results can be obtained (see [9] for details). First, provided that $\text{Lie}(X^1)(e) = T_e G \approx \mathfrak{g}$, functions transverse to X^1 can be defined on $\mathbb{T}^{n-m}$, i.e. with the minimal value $(n - m)$ of p and $H = \emptyset$. An expression of such functions f^ε is given in [9]. It defines a family (f^ε) of functions transverse to X^1 and centered on e. Finally, in the case of Lie groups the transversality property can also be expressed as follows. Let $X^2 = \{X_{m+1}, \ldots, X_n\}$ denote a family of elements of $\mathfrak{g}$ such that $X = \{X_1, \ldots, X_n\}$ forms a basis of $\mathfrak{g}$. Given any smooth function $f : \mathbb{T}^{n-m} \times H \longrightarrow G$ and any smooth curve $(\alpha, \xi)(.)$ on $\mathbb{T}^{n-m} \times H$, one has

$$\dot{f}(\alpha, \xi) = X(f(\alpha, \xi)) \left(A_\alpha(\alpha, \xi)\dot{\alpha} + A_\xi(\alpha, \xi)\dot{\xi} \right) \qquad (4)$$

for some smooth matrix-valued functions A_α, A_ξ. By denoting $A_\alpha^1 \in \mathbb{R}^{m \times n-m}$ and $A_\alpha^2 \in \mathbb{R}^{(n-m) \times n-m}$ the components of the following block decomposition of A_α:

$$A_\alpha(\alpha, \xi) = \begin{pmatrix} A_\alpha^1(\alpha, \xi) \\ A_\alpha^2(\alpha, \xi) \end{pmatrix}$$

one easily verifies that f is transverse to X^1 iff $A_\alpha^2(\alpha, \xi)$ is invertible $\forall (\alpha, \xi) \in \mathbb{T}^{n-m} \times H$.

III. MAIN RESULTS

For systems (left-invariant) on a Lie group G, transverse functions can be used to design stabilizing feedback laws for *arbitrary* (i.e. not necessarily feasible) reference trajectories [9]. Let us briefly recall how this can be done. Consider a left-invariant system

$$\dot{g} = \sum_{i=1}^m X_i(g)u_i = X(g)Cu \qquad (5)$$

with $C = (I_m \mid 0_{m \times (n-m)})'$ and $X = \{X_1, \ldots, X_n\}$ a basis of $\mathfrak{g}$. Let g_r denote any smooth reference trajectory on G. One can decompose the time derivative $\dot{g}_r$ on the basis X, i.e. $\dot{g}_r = X(g_r)v_r$ with v_r a smooth $\mathbb{R}^n$ valued function. Let $\tilde{g} := g_r^{-1}g$ denote the tracking error between g and g_r. It follows from (3) that

$$\dot{\tilde{g}} = X(\tilde{g})(Cu - \text{Ad}^X(\tilde{g}^{-1})v_r)$$

Now, let $z := \tilde{g}f(\alpha, \xi)^{-1}$ with $(\alpha, \xi)(.)$ a smooth curve on $\mathbb{T}^{n-m} \times H$. It follows from (2) that

$$\begin{aligned} \dot{z} = X(z)\text{Ad}^X(f(\alpha, \xi))\big(&\bar{C}(\alpha, \xi)\bar{u} \\ &- A_\xi(\alpha, \xi)\dot{\xi} - \text{Ad}^X(\tilde{g}^{-1})v_r\big) \end{aligned} \qquad (6)$$

[1] a property equivalent to $X_1(e), \ldots, X_m(e)$ being independent.

with $\bar{u}' := (u', \dot{\alpha}') = (u_1, \ldots, u_m, \dot{\alpha}_1, \ldots, \dot{\alpha}_{n-m})$ and

$$\bar{C}(\alpha,\xi) := (C \mid -A_\alpha(\alpha,\xi)) = \begin{pmatrix} I_m & -A_\alpha^1(\alpha,\xi) \\ 0 & -A_\alpha^2(\alpha,\xi) \end{pmatrix}$$

When f is a transverse function, the matrix $\bar{C}(\alpha,\xi)$ is invertible for any (α,ξ) so that the feedback

$$\bar{u} = \bar{C}(\alpha,\xi)^{-1}\big(A_\xi(\alpha,\xi)\dot{\xi} + \mathrm{Ad}^X(\tilde{g}^{-1})v_r + \mathrm{Ad}^X(f(\alpha,\xi)^{-1})\bar{v}(z)\big) \quad (7)$$

transforms Eq. (6) into $\dot{z} = X(z)\bar{v}(z)$. Therefore, any asymptotic stabilizer $\bar{v}(z)$ of $z = e$ for this system yields a feedback $\bar{u}(g, g_r, v_r, \alpha, \xi)$ which makes the tracking error $\tilde{g}$ converge to the image set of the function f. Clearly, the design of such a stabilizer is not a difficult task, since $X(z)\bar{v}(z) = \sum_{i=1}^n X_i(z)v_i(z)$ and $X_1(z), \ldots, X_n(z)$ is a basis of T_zG for all z. When $f(\alpha,\xi) = f^\varepsilon(\alpha)$ with (f^ε) a family of functions transverse to X^1 and centered on e, the convergence of z to e thus ensures the ultimate boundedness of the tracking error by $\max_\alpha \mathrm{dist}(f^\varepsilon(\alpha), e)$ *independently* of the reference trajectory g_r. Moreover, the ultimate bound can be made arbitrarily small via the choice of ε.

The extension of this approach to systems which are not invariant on a Lie group raises two issues: 1) the calculation of transverse functions and 2) the design of stabilizing control laws. In the following subsections, these two issues are addressed for some classes of systems

A. Transverse functions for feedback equivalent systems

Consider two driftless systems $\Sigma, \bar{\Sigma}$ on possibly different manifolds $M, \bar{M}$:

$$\Sigma: \quad \dot{q} = \sum_{i=1}^m Y_i(q)w_i\,, \qquad \bar{\Sigma}: \quad \dot{\bar{q}} = \sum_{i=1}^m \bar{Y}_i(\bar{q})\bar{w}_i \quad (8)$$

Let us recall (see e.g. [11]) that Σ is *feedback equivalent* to $\bar{\Sigma}$ on an open subset $\mathcal{O} \subset M$ if there exists a diffeomorphism $\chi : \mathcal{O} \times \mathbb{R}^m \longrightarrow \bar{\mathcal{O}} \times \mathbb{R}^m \subset \bar{M} \times \mathbb{R}^m$ of the form

$$\begin{pmatrix} \bar{q} \\ \bar{w} \end{pmatrix} = \chi(q,w) = \begin{pmatrix} \Phi(q) \\ \Psi(q)w \end{pmatrix} \quad (9)$$

which transforms Σ into $\bar{\Sigma}$, i.e. such that

$$\forall w, \quad d\Phi(q)\sum_{i=1}^m Y_i(q)w_i = \sum_{i=1}^m \bar{Y}_i(\Phi(q))\Psi_i(q)w \quad (10)$$

with Ψ_i the i-th component of the function Ψ. Recall that d is the differentiation operator. Relation (10) can also be written, with the notation of Section II-A, in the more compact form

$$\forall w, \quad d\Phi(q)Y^1(q)w = \bar{Y}^1(\Phi(q))\Psi(q)w \quad (11)$$

with $Y^1 = \{Y_1, \ldots, Y_m\}$ and $\bar{Y}^1 = \{\bar{Y}_1, \ldots, \bar{Y}_m\}$. Note also that the property of feedback equivalence is symmetric, i.e. if Σ is feedback equivalent to $\bar{\Sigma}$ on $\mathcal{O}$, then $\bar{\Sigma}$ is feedback equivalent to Σ on $\bar{\mathcal{O}} = \Phi(\mathcal{O})$.

Proposition 1 *Assume that Σ is feedback equivalent to $\bar{\Sigma}$ on $\mathcal{O}$ and that $f : \mathbb{T}^p \times H \longrightarrow \mathcal{O}$ is a function transverse to Y^1. Then $\bar{f} := \Phi(f)$ is transverse to $\bar{Y}^1$.*

When one of the two systems $\Sigma, \bar{\Sigma}$ is an invariant system on a Lie group, one obtains complementary properties.

Proposition 2 *Assume that Σ is feedback equivalent to the left-invariant system* (5) *on $\mathcal{O}$, so that* (11) *is satisfied with $\bar{Y}^1 = X^1 = \{X_1, \ldots, X_m\}$. Let $(\bar{f}^\varepsilon)$ denote a family of functions transverse to X^1 and centered on e, and let $q_0 \in \mathcal{O}$. Then,*

1) *there exists $\varepsilon_0 > 0$ such that $\Phi(q_0)\bar{f}^\varepsilon(\alpha) \in \bar{\mathcal{O}}$ for all $\alpha \in \mathbb{T}^{n-m}$ and all $\varepsilon \in (0, \varepsilon_0)$,*
2) $$f^\varepsilon(\alpha, q_0) := \Phi^{-1}(\Phi(q_0)\bar{f}^\varepsilon(\alpha)) \quad (12)$$
defines a family of functions transverse to Y^1 and centered on q_0.

Proposition 2 shows how the property of feedback equivalence to an invariant system can be used to design a family of transverse functions centered on any point $q_0 \in \mathcal{O}$ and which depend smoothly on q_0. The following section illustrates a possible application of this result.

B. Control laws for a class of non-invariant systems

In this section, we show how to extend the design of stabilizing control laws based on the transverse function approach to a class of systems of the form

$$\begin{cases} \dot{g} &= X(g)C_g(\xi)w \\ \dot{\xi} &= C_\xi(\xi)w \end{cases} \quad (13)$$

with g an element of a Lie group G, $X = \{X_1, \ldots, X_n\}$ a basis of the group's Lie algebra $\mathfrak{g}$, $\xi \in \mathbb{R}^N$, $w \in \mathbb{R}^m$ the control input, and C_g, C_ξ smooth matrix-valued functions of adequate dimensions. We denote by n the dimension of the state space, so that $\dim(G) = n_0 := n - N$.

System (13) is a particular case of a system Σ in (8), with

$$q = (g,\xi) \quad \text{and} \quad Y_i(q) = \begin{pmatrix} X(g)C_g(\xi)b_i \\ C_\xi(\xi)b_i \end{pmatrix}$$

The class of systems of the form (13), which generalizes the class on invariant systems (5) (for which $N = 0$), is of interest to model the kinematics of articulated nonholonomic mechanical systems, where g corresponds (typically) to the system's "situation" (position and orientation), and ξ to a vector of "shape" variables [12].

Let $\bar{X}$ and $\mathrm{Ad}^{\bar{X}}$ denote the mapping defined by

$$\forall u = (u^1, u^2) \in \mathbb{R}^{n_0} \times \mathbb{R}^N,\ \forall q = (g,\xi) \in G \times \mathbb{R}^N$$
$$\bar{X}(q)u = \begin{pmatrix} X(g)u_1 \\ u_2 \end{pmatrix}, \quad \mathrm{Ad}^{\bar{X}}(q)u = \begin{pmatrix} \mathrm{Ad}^X(g)u^1 \\ u^2 \end{pmatrix} \quad (14)$$

We note that $\{\bar{X}_i(q) := \bar{X}(q)b_i\,,\ i = 1, \ldots, n\}$ is a basis of left-invariant v.f. for the Lie group $G \times \mathbb{R}^N$ endowed with the group law inherited from the group law on G and the vector addition on $\mathbb{R}^n$, i.e.

$$q_1q_2 = \begin{pmatrix} g_1 \\ \xi_1 \end{pmatrix}\begin{pmatrix} g_2 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} g_1g_2 \\ \xi_1 + \xi_2 \end{pmatrix},\ q^{-1} = \begin{pmatrix} g \\ \xi \end{pmatrix}^{-1} = \begin{pmatrix} g^{-1} \\ -\xi \end{pmatrix}$$

and $\mathrm{Ad}^{\bar{X}}$ is the associated Ad operator. With this notation, System (13) can also be written as

$$\dot{q} = \bar{X}(q)C(\xi)w\,, \quad \text{with} \quad C(\xi) := \begin{pmatrix} C_g(\xi) \\ C_\xi(\xi) \end{pmatrix}$$

The possible dependence of C on ξ accounts for the possible non-invariance of this system on the group $G \times \mathbb{R}^N$.

Let us now consider a (smooth) reference trajectory $q_r = (g_r, \xi_r)$ and denote v_r the smooth function such that $\dot{g}_r = X(g_r)v_r$. Let $\tilde{q} := (\tilde{g}, \tilde{\xi}) := (g_r^{-1}g, \xi - \xi_r)$ denote the tracking error between q and q_r.

Proposition 3 *Consider System* (13) *and a reference trajectory* (g_r, ξ_r) *to be stabilized. Assume that*

1) *For any* $i = 1, \dots, N$*, the* i*-th row* $C_{\xi,i}(\xi)$ *of the matrix* $C_\xi(\xi)$ *only depends on* $\xi_{i+1}, \dots, \xi_N$*,*
2) *For any* $\bar{\xi} := (e, \xi)$ *there exists a family* $(f^\varepsilon(.,\bar{\xi})) = (f_g^\varepsilon(.,\bar{\xi}), f_\xi^\varepsilon(.,\bar{\xi}))$ *of functions transverse to* Y^1 *and centered on* $\bar{\xi}$*, with* f^ε *depending smoothly on* ξ*,*
3) *The time-functions* ξ_r*,* $\dot{\xi}_r$*, and* v_r *are bounded,*

For any ε*, let*

$$\begin{aligned} z = \begin{pmatrix} z_g \\ z_\xi \end{pmatrix} &:= \begin{pmatrix} \tilde{g} f_g^\varepsilon(\alpha, \bar{\xi}_r)^{-1} \\ \xi - f_\xi^\varepsilon(\alpha, \bar{\xi}_r) \end{pmatrix} \\ \bar{C}(\alpha, \xi_r) &:= \Big(C(f_\xi^\varepsilon(\alpha, \bar{\xi}_r)) \mid -A_\alpha(\alpha, \xi_r) \Big) \end{aligned} \tag{15}$$

with A_α *the matrix-valued function defined by*

$$\dot{f}^\varepsilon(\alpha, \bar{\xi}_r) = \bar{X}(f^\varepsilon(\alpha, \bar{\xi}_r))(A_\alpha(\alpha, \xi_r)\dot{\alpha} + A_\xi(\alpha, \xi_r)\dot{\xi}_r) \tag{16}$$

Finally, let $\bar{v}(z) = (\bar{v}_g(z_g), \bar{v}_\xi(z_\xi))$*, with* $\frac{\partial \bar{v}_\xi}{\partial z_\xi}(0)$ *a diagonal matrix, denote a smooth feedback law that makes* $z = (e, 0)$ *an exponentially stable equilibrium of the system* $\dot{z} = \bar{X}(z)\bar{v}(z)$*. Then,*

1) the matrix $\bar{C}(\alpha, \xi_r)$ *is invertible for any* (α, ξ_r)*,*
2) the "extended control" $\bar{w} := (w', \dot{\alpha}')'$ *given by*

$$\bar{w} = \bar{C}(\alpha, \xi_r)^{-1}\Big(A_\xi(\alpha, \xi_r)\dot{\xi}_r + \begin{pmatrix} \mathrm{Ad}^X(\tilde{g}^{-1})v_r \\ 0 \end{pmatrix} + \mathrm{Ad}^{\bar{X}}(f^\varepsilon(\alpha, \bar{\xi}_r)^{-1})\bar{v}(z)\Big) \tag{17}$$

makes $z = (e, 0)$ *an asymptotically stable equilibrium of the controlled system. Moreover, the tracking error* $\tilde{q}$ *converges to a neighborhood of* $(e, 0)$ *whose size can be made as small as desired by choosing* ε *small enough.*

Let us comment on this result. Assumption 1) specifies the class of systems to which the proposition results apply. It is used in the stability analysis of the closed-loop system. Assumption 2) concerns the existence of transverse functions and is much weaker. Indeed, the main result in [10] guarantees the existence of a family $(f^\varepsilon(.,\bar{\xi}))$ of functions transverse to Y^1 and centered on $\bar{\xi}$ provided that System (13) satisfies the Lie algebra rank condition at this point. However, the existence of a family of functions depending *smoothly* on ξ is not proved in [10]. This complementary property relies on additional assumptions. The next section illustrates with the example of the N-trailer system how Proposition 2 can be used to derive such a family of transverse functions. As for the third assumption of the proposition, it is clearly little restrictive in practice. The stabilizing control law (17) is essentially the same as the control law (7) used in the case of a left-invariant system, except for the additional condition upon $\frac{\partial \bar{v}_\xi}{\partial z_\xi}(0)$ induced by the possible non-invariance of the system.

The proof of Proposition 3 only establishes the *local* asymptotic stability of $z = e$. Global stability can be obtained for specific Lie groups, as shown below for $SE(2) \approx \mathbb{R}^2 \times \mathbb{S}^1$, whose group law is defined by

$$g_1 g_2 = \begin{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + R(\theta_1)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ \theta_1 + \theta_2 \end{pmatrix} \tag{18}$$

with $g_i = (x_i, y_i, \theta_i) \in \mathbb{R}^2 \times \mathbb{S}^1$ and $R(\theta)$ the matrix of rotation in the plane of angle θ. The corresponding unit element is $e = (0, 0, 0)$. For the sake of conciseness, we denote by $|g|$ the norm of the element of $\mathbb{R}^3$ associated to g by identifying $\theta \in \mathbb{S}^1$ with an element of $(-\pi, \pi]$.

Proposition 4 *Suppose that* $G = SE(2)$ *and that Assumptions 1, 2, and 3 of Proposition 3 are verified. Let* $\bar{v}(z,t) = (\bar{v}_g(z_g,t), \bar{v}_\xi(z_\xi))$ *denote a bounded feedback law with* $\bar{v}_\xi(z_\xi) = -\tau_\xi(|z_\xi|)z_\xi$ *and* $z_g' X(z_g)\bar{v}_g(z_g,t) \leq -\tau_g(|z_g|)|z_g|^2$ *for some strictly positive continuous functions* τ_g, τ_ξ*. Define* $\tilde{g}_f^{-1} := (h^\varepsilon(\alpha, \bar{\xi}_r), -\tilde{\theta})$ *with* $h^\varepsilon(\alpha, \bar{\xi}_r)$ *the first two components of* $f_g^\varepsilon(\alpha, \bar{\xi}_r)^{-1}$*, and* $\tilde{\theta}$ *the last component of* $\tilde{g}$*. Then, the "extended control"* $\bar{w} := (w', \dot{\alpha}')'$ *given by*

$$\bar{w} = \bar{C}(\alpha, \xi_r)^{-1}\Big(A_\xi(\alpha, \xi_r)\dot{\xi}_r + \begin{pmatrix} \mathrm{Ad}^X(\tilde{g}_f^{-1})v_r \\ 0 \end{pmatrix} + \mathrm{Ad}^{\bar{X}}(f^\varepsilon(\alpha, \bar{\xi}_r)^{-1})\bar{v}(z,t)\Big) \tag{19}$$

makes $z = (e, 0)$ *a globally asymptotically stable equilibrium of the controlled system.*

Remark: The dependence of $\bar{v}_g(z_g, t)$ upon t is an additional degree of freedom the usefulness of which is illustrated in Section V. Note also that the small difference between the control expressions (17) and (19) has its importance.

IV. Application to the N-trailer system

With the notation of Fig. 1, the kinematic model of a unicycle with N trailers is given by

$$\left\{ \begin{aligned} \dot{x} &= \nu_1 \cos\theta \\ \dot{y} &= \nu_1 \sin\theta \\ \dot{\theta} &= \nu_1 \frac{\tan\varphi_1}{\ell_0} \\ \dot{\varphi}_i &= \nu_1 \frac{\frac{\tan\varphi_{i+1}}{\ell_i} - \frac{\sin\varphi_i}{\ell_{i-1}}}{\prod_{k=1}^{i}\cos\varphi_k} \quad (i = 1, \dots, N-1) \\ \dot{\varphi}_N &= \nu_2 \end{aligned} \right. \tag{20}$$

with (x, y) the vector of coordinates of P_0 in the fixed frame $\{0, \vec{\imath}, \vec{\jmath}\}$, and ν_1, ν_2 the control inputs, related to the linear and angular velocities v_1, v_2 of the front unicycle by the equations $\nu_1 = v_1 \prod_{k=1}^{N} \cos\varphi_k$ and $\nu_2 = v_2 - \frac{v_1}{\ell_{N-1}} \sin\varphi_N$.

System (20) is well defined on the configuration space $\mathbb{R}^2 \times \mathbb{S}^1 \times (-\pi/2, \pi/2)^N$. The following lemma, the proof

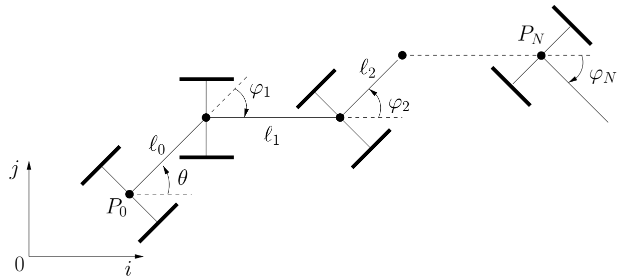

Fig. 1. Unicycle with N trailers

of which is straightforward (and omitted), states that, via a change of variables on $\varphi_1, \ldots, \varphi_N$, System (20) can be transformed into an equivalent system the expression of which is simpler.

Lemma 1 *There exists a diffeomorphism* $\Upsilon : (-\pi/2, \pi/2)^N \times \mathbb{R}^2 \longrightarrow \mathbb{R}^N \times \mathbb{R}^2$ *of the form*

$$\begin{pmatrix} \xi \\ w \end{pmatrix} = \Upsilon(\varphi, \nu) = \begin{pmatrix} \Gamma(\varphi) \\ \Delta(\varphi)\nu \end{pmatrix}$$

which transforms System (20) *into the system*

$$\left\{ \begin{array}{lcl} \dot{x} & = & w_1 \cos\theta \\ \dot{y} & = & w_1 \sin\theta \\ \dot{\theta} & = & w_1 \xi_1 \\ \dot{\xi}_i & = & w_1 \xi_{i+1} \quad (i = 1, \ldots, N-1) \\ \dot{\xi}_N & = & w_2 \end{array} \right. \tag{21}$$

It well known [13], [14] that System (20) is feedback equivalent to the $(N+3)$-d chained system (whose equations are recalled below) on the set $\mathcal{O} = \mathbb{R}^2 \times (-\pi/2; \pi/2)^{N+1}$. While System (21) is reminiscent of the chained system, its advantage w.r.t. the chained system is that it is feedback equivalent to System (20) on the larger domain $\mathbb{R}^2 \times \mathbb{S}^1 \times (-\pi/2, \pi/2)^N$. In particular, contrary to the transformation into a chained system, the diffeomorphism Υ does not introduce limitations on the orientation variable θ. For this reason it is preferable to base the control design on the state equations (21) rather than on the locally equivalent chained system.

A. Calculation of transverse functions

System (20) is feedback equivalent to the $(N+3)$-d chained system on $\mathcal{O} = \mathbb{R}^2 \times (-\pi/2; \pi/2)^{N+1}$, and so is System (21) by the transitivity of the feedback equivalence property. Recall that the n-d chained system is defined as

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & u_1 \\ \dot{x}_2 & = & u_2 \\ \dot{x}_i & = & u_1 x_{i-1} \quad (i = 3, \ldots, n) \end{array} \right. \tag{22}$$

This is a system invariant on the Lie group $\mathbb{R}^n$ endowed with the group law $(x, y) \longmapsto xy$ defined by

$$(xy)_i = \left\{ \begin{array}{ll} x_i + y_i & \text{if } i = 1, 2 \\ x_i + y_i + \sum_{j=2}^{i-1} \frac{y_1^{i-j}}{(i-j)!} x_j & \text{otherwise} \end{array} \right. \tag{23}$$

with the unit element $e = 0$. A family $(\bar{\bar{f}}^\varepsilon)$ of functions transverse to the v.f. X_1^c, X_2^c of the chained system and centered on e is defined by (see [9, Sec. VII] for details)

$$\bar{\bar{f}}^\varepsilon(\alpha) = \bar{\bar{f}}^\varepsilon_{n-2}(\alpha_{n-2}) \ldots \bar{\bar{f}}^\varepsilon_1(\alpha_1), \quad \varepsilon \in (0, +\infty) \tag{24}$$

with, for $i = 1, \ldots, n-2$,

- $\bar{\bar{f}}^\varepsilon_i(\alpha_i) = \exp(\varepsilon\beta_{i,1} \sin(\alpha_i) X_1^c + \varepsilon^i \beta_{i,2} \cos(\alpha_i) X_{i+1}^c)$
- $X_{i+2}^c := [X_{i+1}^c, X_1^c] = b_{i+2}$,
- $\beta_{i,1}, \beta_{i,2}$ adequately chosen non-zero parameters.

A second family, derived from the first one, is defined by

$$\bar{f}^\varepsilon(\alpha) := \bar{\bar{f}}^\varepsilon(\alpha^\star)^{-1} \bar{\bar{f}}^\varepsilon(\alpha), \quad \varepsilon \in (0, +\infty) \tag{25}$$

with $\alpha^\star \in \mathbb{T}^{n-2}$. The pre-multiplication by $\bar{\bar{f}}^\varepsilon(\alpha^\star)^{-1}$ ensures that $e = 0$ belongs to the image of the transverse function, whatever the value of ε. As shown in [15], this feature can in turn be used by choosing $\alpha^\star$ adequately in order to make $\bar{f}^\varepsilon(\alpha)$ converge to zero when tracking a "persistently exciting" feasible trajectory and, subsequently, to make the tracking error converge to zero when z converges to zero. The important point is that this convergence is obtained without having to make ε tend to zero, i.e. without rendering the control law ill-conditioned. By extension of the 4-d chained system case treated in [15], a suitable choice is $\alpha^\star = (-\frac{\pi}{2}, \ldots, -\frac{\pi}{2})'$, with the complementary requirement of equal signs for the parameters $\beta_{i,1}$ $(i = 1, \ldots, n-2)$ and the reference input $u_{r,1}$.

Now, the mapping χ which transforms System (21) into the $(N+3)$-d chained system is a diffeomorphism from $\mathcal{O} \times \mathbb{R}^2$ onto $\bar{\mathcal{O}} \times \mathbb{R}^2 = \mathbb{R}^{N+3} \times \mathbb{R}^2$. Therefore, it follows from Proposition 2 that for any $q_0 = (x_0, y_0, \theta_0, \xi_0) \in \mathcal{O}$, the functions $f^\varepsilon(., q_0)$ given by (12), with $\varepsilon \in (0 + \infty)$ and $\bar{f}^\varepsilon$ given by (25), define a family of functions transverse to the v.f. Y_1, Y_2 of System (21) and centered on q_0.

B. Stabilizing feedback laws

Let $X(g) = \{X_1(g), X_2(g), X_3(g)\}$ denote the following "canonical basis" of left-invariant v.f. on $SE(2)$:

$$X_1(g) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \; X_2(g) = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \; X_3(g) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

With this notation, it is straightforward to verify that System (21) is of the form (13) with

$$C_g(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \xi_1 & 0 \end{pmatrix}, \quad C_\xi(\xi) = \begin{pmatrix} \xi_2 & 0 \\ \vdots & \vdots \\ \xi_n & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly, C_ξ satisfies Assumption 1 of Proposition 3. By this proposition, the feedback control (17) associated with the family $(f^\varepsilon(., \bar{\xi}_r))$ of functions transverse to Y^1 and centered on $\bar{\xi}_r$, the determination of which was specified in the previous subsection, asymptotically stabilizes $z = (e, 0)$.

V. SIMULATION RESULTS FOR A UNICYCLE WITH TWO TRAILERS

The 1-trailer system, or car-like system, is studied in some detail in [15]. The 2-trailer system comes as the next example in increased complexity to which the results of the present study apply. Physically, this system may either take the form of a unicycle pulling two trailers, or a car (truck) pulling a single trailer. For the simulation results presented below, the control objective is to have the last trailer track a reference frame whose situation g_r varies with a velocity v_r according to the relation $\dot{g}_r = X(g_r)v_r$. The motion of the reference frame is compatible with the kinematic equations of the trailer system provided that $v_{r,2}(t) = 0$ ($\forall t$). Then, in view of (21), perfect tracking implies that the reference "shape" variables should be $\xi_{r,1} = \frac{v_{r,3}}{v_{r,1}}$ and $\xi_{r,2} = \frac{\dot{\xi}_{r,1}}{v_{r,1}} = \frac{\dot{v}_{r,3}v_{r,1}-v_{r,3}\dot{v}_{r,1}}{v_{r,1}^3}$. For the simulations, we have regularized these expressions in order to avoid divisions by zero when $v_{r,1} = 0$ by setting $\xi_{r,1} = v_{r,1}\frac{v_{r,3}}{v_{r,1}^2+\delta}$ and $\xi_{r,2} = v_{r,1}\frac{\dot{v}_{r,3}v_{r,1}-v_{r,3}\dot{v}_{r,1}}{v_{r,1}^4+\delta}$, with δ a small positive number. The time-derivative $\dot{\xi}_r$ is then well defined provided that $v_{r,3}$ and $v_{r,1}$ are twice differentiable.

The matrix-valued functions involved in the transformation of the kinematic model (20) of a unicycle with two trailers ($N = 2$) into the system (21) are

$$\Gamma(\varphi) = \begin{pmatrix} \frac{\tan\varphi_1}{l_0} \\ (\frac{\tan\varphi_2}{l_1} - \frac{\sin\varphi_1}{l_0})\frac{1}{l_0(\cos\varphi_1)^3} \end{pmatrix}$$

$$\Delta(\varphi) = \begin{pmatrix} 1 & 0 \\ \Delta_{2,1}(\varphi) & \frac{1}{l_0 l_1(\cos\varphi_1)^3(\cos\varphi_2)^2} \end{pmatrix}$$

with

$$\Delta_{2,1}(\varphi) = \left(\frac{3(\sin\varphi_1)\tan\varphi_2}{l_1} - \frac{1+2(\sin\varphi_1)^2}{l_0}\right)\frac{\Gamma_2(\varphi)}{(\cos\varphi_1)^2}$$

The matrix-valued functions Φ involved in the transformation of the system (21) with $N = 2$ into the 5-d chained system, and thus in the calculation of the transverse functions via relation (12), is

$$\Phi(q) = \left(q_1, \frac{q_5 + 3q_4^2\tan q_3}{(\cos q_3)^4}, \frac{q_4}{(\cos q_3)^3}, \tan q_3, q_2\right)'$$

The parameters of the function (24) involved in the transverse function expression have been chosen as $\varepsilon = 1, \beta_{1,1} = 0.14, \beta_{1,2} = 3, \beta_{2,1} = 0.4, \beta_{2,2} = 0.8, \beta_{3,1} = 1, \beta_{3,2} = 0.4$.

There remains to specify the choice of $\bar{v} = (\bar{v}_g, \bar{v}_\xi)$ in (19). The function $\bar{v}_\xi$ is defined as in Proposition 4, with $\tau_\xi(|z_\xi|) = 5/(1 + 0.1|z_\xi|)$. Similarly, $\bar{v}_g$ can be chosen as $\bar{v}_g(z_g) = -\tau_g(|z_g|)z_g$ with $\tau_g = \tau_\xi$. Motivated by [15], another choice has been made. We have assumed that $\frac{\nu_1}{\cos\varphi_1}$ (the longitudinal velocity of the first trailer) and $\dot{\varphi}_2$ are the physical control variables whose amplitude should be kept as small as possible –without preventing the practical stabilization of the reference frame. This led to define the extended control vector $\bar{\bar{w}} := (\frac{\nu_1}{\cos\varphi_1}, \dot{\varphi}_2, \dot{\alpha}')'$, and $D(\alpha, \xi_r)$ the matrix such that, in view of (19), $\bar{\bar{w}} = D(\alpha, \xi_r)\bar{v}_g$ when $z_\xi, \bar{v}_\xi, v_r$, and $\dot{\xi}_r$ are identically equal to zero. Then, we have set

$$\bar{v}_g(z_g, t) = -\frac{(z_g' W_2 z_g)Q(t)^{-1}X(z_g)'z_g}{z_g' X(z_g)Q(t)^{-1}X(z_g)'z_g}\frac{1}{1 + a|z_g|}$$

with $Q(t) = (D'W_1D)_{|(\alpha(t),\xi_r(t))}$. The above expression of $\bar{v}_g$ implies that, along any solution of the controlled system such that the above mentioned variables are identically equal to zero, one has $\frac{d}{dt}|z_g|^2 = -2z_g'W_2z_g$ (≤ 0) with $\bar{\bar{w}}'W_1\bar{\bar{w}}$ being minimized at each time-instant. As explained in [15], this way of calculating $\bar{v}_g$ also allows for the reduction of the number of transient maneuvers when the distance between the last trailer and the reference frame is initially large. For these simulations, we have set $W_1 = diag\{10, 1, 0.01, 0.1, 0.1\}$, $W_2 = I_3$, and $a = 0.1$.

The trailers "lengths" are $l_0 = 2$ and $l_1 = 1$ (meters). A single reference trajectory presenting different properties at different times is used. The values of the associated reference frame velocity v_r are given in the following table.

$t \in (s)$	$v_r = (m/s, rad/s, m/s)'$	properties
$[0, 5)$	$(0, 0, 0)'$	ad,npe
$[5, 10)$	$(1, 0, 0)'$	ad,pe
$[10, 20)$	$(-1, 0, 0)'$	ad,pe
$[20, 25)$	$(1, 0, 0.314)'$	ad,pe
$[25, 30)$	$(-1, 0, -2\sin(2t))'$	ad,pe
$[30, 35)$	$(0, -1, 0)'$	nad
$[35, 40)$	$(0, 0, 0)'$	ad,npe
$[40, 45)$	$(2, -0.5, -0.5\sin(3t) + 0.3)'$	nad
$[45, 50)$	$(0, 0, 0)'$	ad,npe

In this table, the abbreviations used to describe the properties of each part of the reference trajectory are: *ad* and *nad* for admissible and non-admissible respectively, according to whether $v_{r,2}$ is or is not equal to zero; and *pe* and *npe* for persistently exciting and non-persistently exciting respectively, according to whether $v_{r,1}$ is or is not equal to zero.

Figures 2 and 3 show the time evolution of the position errors $(\tilde{g}_1, \tilde{g}_2)$ and orientation error $\tilde{g}_3$ respectively. From these figures, one can observe *i)* the uniform boundedness of the tracking errors whatever the properties of the reference trajectory, *ii)* the convergence of the tracking errors to zero when the reference trajectory is admissible and persistently exciting, *iii)* the automatic (resp. non-systematic) production of maneuvers when the reference trajectory is "strongly" (resp. "weakly") non-admissible. The other figures are attempts to visualize the vehicle's motion in the plane during different phases of the reference trajectory.

APPENDIX

Proof of Proposition 1: By definition of the property of transversality, one has to show that the vectors

$$\bar{Y}_1(\bar{f}(\alpha,\xi)), \ldots, \bar{Y}_m(\bar{f}(\alpha,\xi)), \frac{\partial\bar{f}}{\alpha_1}(\alpha,\xi), \ldots, \frac{\partial\bar{f}}{\alpha_p}(\alpha,\xi)$$

span $T_{\bar{f}(\alpha,\xi)}\bar{M}$ for any (α, ξ). This is equivalent to the surjectivity, for any (α, ξ), of the mapping

$$\eta : \bar{w} = (\bar{w}^1, \bar{w}^2) \longmapsto \bar{Y}^1(\bar{f}(\alpha,\xi))\bar{w}^1 + d_\alpha\bar{f}(\alpha,\xi)\bar{w}^2$$

with d_α the operator of differentiation w.r.t. α. By the property of diffeomorphism of χ, $\Psi(q)$ is invertible for any q. Therefore, the property of transversality is also equivalent to the surjectivity, for any (α, ξ), of the mapping

$$\bar{\eta} : \bar{w} \longmapsto \bar{Y}^1(\bar{f}(\alpha,\xi))\Psi(\Phi^{-1}(\bar{f}(\alpha,\xi)))\bar{w}^1 + d_\alpha \bar{f}(\alpha,\xi)\bar{w}^2$$

We deduce from (11) and the definition of $\bar{f}(=\Phi(f))$ that

$$\bar{\eta}(\bar{w}) = d\Phi(f(\alpha,\xi))(Y^1(f(\alpha,\xi))\bar{w}^1 + d_\alpha f(\alpha,\xi)\bar{w}^2)$$

The proof then follows from the fact that f is transverse to Y^1 and $d\Phi(q)$ is invertible for any q.

Proof of Proposition 2: Using the fact that $\Phi(q_0)$ belongs to the open set $\bar{\mathcal{O}}$, Property 1) is a direct consequence of the definition of a family of transverse functions centered on e, since this implies that $\max_\alpha \text{dist}(\bar{f}^\varepsilon(\alpha), e)$ tends to zero as ε tends to zero. Property 2) then follows by application of Proposition 1, using the fact that on a Lie group, for any fixed g_0 and any transverse function $\bar{f}$, $g_0\bar{f}$ is also a transverse function.

Proof of Proposition 3: Property 1) is a direct consequence of (16) and the property of transversality of the functions $f^\varepsilon(.,\bar{\xi})$ w.r.t. Y^1.

It follows from (3) that

$$\dot{\tilde{g}} = X(\tilde{g})(C_g(\xi)w - \text{Ad}^X(\tilde{g}^{-1})v_r)$$

so that, from (2),

$$\dot{z} = \bar{X}(z)\text{Ad}^{\bar{X}}(f^\varepsilon(\alpha,\xi_r))\big(C(\xi)w - A_\alpha(\alpha,\xi_r)\dot{\alpha} - A_\xi(\alpha,\xi_r)\dot{\xi}_r - \left(\begin{smallmatrix}\text{Ad}^X(\tilde{g}^{-1})v_r\\ 0\end{smallmatrix}\right)\big) \quad (26)$$

This equation can be rewritten as

$$\dot{z} = \bar{X}(z)\text{Ad}^{\bar{X}}(f^\varepsilon)\big(\bar{C}\bar{w} - A_\xi\dot{\xi}_r - \left(\begin{smallmatrix}\text{Ad}^X(\tilde{g}^{-1})v_r\\ 0\end{smallmatrix}\right) + (C(\xi) - C(f_\xi^\varepsilon))w\big) \quad (27)$$

where the arguments α, ξ_r for $f^\varepsilon, \bar{C}, A_\alpha$, and A_ξ have been omitted to lighten the notation. Applying the control law (17) yields the closed loop system

$$\dot{z} = \bar{X}(z)\bar{v} + \bar{X}(z)\text{Ad}^{\bar{X}}(f^\varepsilon)(C(\xi) - C(f_\xi^\varepsilon))w \quad (28)$$

Let us show that the linearization of this system at $z = e$ is exponentially stable. By an abuse of notation, we use the same symbols to denote z and X and their expressions in a system of local coordinates around e mapping e to $0 \in \mathbb{R}^n$. From (17), one has $w = w_0 + O(z)$ with

$$w_0 = (I_m \mid 0)\bar{C}(\alpha,\xi_r)^{-1}\left(A_\xi(\alpha,\xi_r)\dot{\xi}_r + \left(\begin{smallmatrix}\text{Ad}^X((f^\varepsilon)^{-1})v_r\\ 0\end{smallmatrix}\right)\right)$$

the value of the control w at the equilibrium $z = e$ and $O(z)$ a smooth function such $|O(z)| \leq k|z|$ in the neighborhood of $z = 0$. The linearization of System (28) at $z = e$ is thus given by

$$\begin{pmatrix}\dot{z}_g\\ \dot{z}_\xi\end{pmatrix} = \begin{pmatrix}X(0)\frac{\partial \bar{v}_g}{\partial z_g}(0) & X(0)\text{Ad}^X(f_g^\varepsilon)\frac{\partial h_g}{\partial \xi}(f_\xi^\varepsilon, w_0)\\ 0 & \frac{\partial \bar{v}_\xi}{\partial z_\xi}(0) + \frac{\partial h_\xi}{\partial \xi}(f_\xi^\varepsilon, w_0)\end{pmatrix}\begin{pmatrix}z_g\\ z_\xi\end{pmatrix}$$

with $h_g(\xi, w_0) := C_g(\xi)w_0$ and $h_\xi(\xi, w_0) := C_\xi(\xi)w_0$ It follows from Assumption 1) and the assumption upon $\bar{v}_\xi$ that the matrix $\frac{\partial \bar{v}_\xi}{\partial z_\xi}(0) + \frac{\partial h_\xi}{\partial \xi}(f_\xi^\varepsilon, w_0)$ is upper triangular with each element on the diagonal strictly negative. From Assumption 3), all other terms of this matrix are bounded. Therefore the z_ξ sub-system is exponentially stable. Since $\bar{v}_g$ exponentially stabilizes the system $\dot{z}_g = X(z_g)\bar{v}_g$, the associated linearized system $\dot{z} = X(0)\frac{\partial \bar{v}_g}{\partial z_g}(0)$ is also exponentially stable. The exponential stability of the linearized system and the proof of stability of $z = e$ for the original (nonlinear) system follow. The fact that the tracking error $\tilde{q}$ converges to a neighborhood of $(e, 0)$ whose size tends to zero as ε tends to zero is a direct consequence of the convergence of z to e and the fact that $(f^\varepsilon(.,\bar{\xi}))$ is a family of transverse functions centered on $\bar{\xi} = (e, \xi)$.

Proof of Proposition 4: In the canonical basis of the Lie algebra of $SE(2)$ (see Section IV-B), the Ad operator is given by the matrix

$$\text{Ad}^X(g) = \begin{pmatrix}R(\theta) & \begin{pmatrix}y\\ -x\end{pmatrix}\\ 0 & 1\end{pmatrix} \quad (29)$$

Since $|\tilde{g}_f^{-1}|$ is bounded by definition, $\text{Ad}^X(\tilde{g}_f^{-1})$ is bounded for any basis X. Using the property of boundedness of $\bar{v}$ and Assumptions 2) and 3) one deduces that the extended control $\bar{w}$ is bounded. Applying the control law (19), one obtains in closed-loop (compare with (28)),

$$\dot{z} = \bar{X}(z)\bar{v} + \bar{X}(z)\text{Ad}^{\bar{X}}(f^\varepsilon)(C(\xi) - C(f_\xi^\varepsilon))w + \bar{X}(z)\text{Ad}^{\bar{X}}(f^\varepsilon)\begin{pmatrix}(\text{Ad}^X(\tilde{g}_f^{-1}) - \text{Ad}^X(\tilde{g}^{-1}))v_r\\ 0\end{pmatrix} \quad (30)$$

This yields the following dynamics of z_ξ:

$$\dot{z}_\xi = \bar{v}_\xi(z_\xi) + (C_\xi(\xi) - C_\xi(f_\xi^\varepsilon))w$$

By using Assumption 1), the expression of $\bar{v}_\xi(z_\xi)$, and the fact that w is bounded (since $\bar{w}$ is bounded), one deduces that z_ξ converges to zero. On the zero-dynamics $z_\xi = 0$, Eq. (30) implies that

$$\dot{z}_g = X(z_g)\bar{v}_g + X(z_g)(\text{Ad}^X(f_g^\varepsilon\tilde{g}_f^{-1}) - \text{Ad}^X(z_g^{-1}))v_r$$

It comes from (29) that

$$X(z_g)(\text{Ad}^X(f_g^\varepsilon\tilde{g}_f^{-1}) - \text{Ad}^X(z_g^{-1}))v_r = \dot{\theta}_r(z_y, -z_x, 0)'$$

with $(z_x, z_y) \in \mathbb{R}^2$ the first components of z_g. Therefore $z_g'\dot{z}_g = z_g'X(z_g)\bar{v}_g = -\tau_g(|z_g|)|z_g|^2$, where the last equality is one of the Proposition's assumptions. The exponential convergence of z_g to zero follows. The stability of the equilibrium $z = (e, 0)$ follows from the fact that, for an adequate choice of $\eta_1, \ldots, \eta_N > 0$, $V(z_g, z_\xi) = |z_g|^2 + \sum_k \eta_k z_{\xi,k}^2$ is a Lyapunov function for the closed-loop system in the neighborhood of $z = (e, 0)$, due to the boundedness of $w = w_0$ at $z = (e, 0)$ (see the proof of Proposition 3).

REFERENCES

[1] R. Brockett, "Asymptotic stability and feedback stabilization," in *Differential Geometric Control Theory*, R. Brockett, R. Millman, and H. Sussmann, Eds. Birkauser, 1983.

[2] C. Samson, "Velocity and torque feedback control of a nonholonomic cart," *Int. Workshop in Adaptative and Nonlinear Control: Issues in Robotics*, 1990, also in LNCIS, Vol. 162, Springer Verlag, 1991.

[3] J.-M. Coron, "Global asymptotic stabilization for controllable systems without drift," *Mathematics of Control, Signals, and Systems*, vol. 5, pp. 295–312, 1992.

[4] O. J. Sørdalen and O. Egeland, "Exponential stabilization of nonholonomic chained systems," *IEEE Trans. on Automatic Control*, vol. 40, pp. 35–49, 1995.

[5] M. K. Bennani and P. Rouchon, "Robust stabilization of flat and chained systems," in *European Control Conference*, 1995, pp. 2642–2646.

[6] C. Samson, "Control of chained systems. application to path following and time-varying point-stabilization," *IEEE Trans. on Automatic Control*, vol. 40, pp. 64–77, 1995.

[7] C. Canudas de Wit, B. Siciliano, and G. Bastin, Eds., *Theory of robot control*. Springer Verlag, 1996.

[8] D. Lizárraga, "Obstructions to the existence of universal stabilizers for smooth control systems," *Mathematics of Control, Signals, and Systems*, vol. 16, pp. 255–277, 2004.

[9] P. Morin and C. Samson, "Practical stabilization of driftless systems on Lie groups: the transverse function approach," *IEEE Trans. on Automatic Control*, vol. 48, pp. 1496–1508, 2003.

[10] ——, "A characterization of the Lie algebra rank condition by transverse periodic functions," *SIAM Journal on Control and Optimization*, vol. 40, no. 4, pp. 1227–1249, 2001.

[11] W. Respondek, "Feedback classification of nonlinear control systems on $\mathbb{R}^2$ and $\mathbb{R}^3$," in *Geometry of feedback and optimal control*, B. Jakubczyk and W. Respondek, Eds. Dekker, 1998, pp. 347–381.

[12] A. Bloch, P. Krishnaprasad, J. Marsden, and R. Murray, "Nonholonomic mechanical systems with symmetry," *Archive for Rational Mechanics and Analysis*, vol. 136, pp. 21–99, 1996.

[13] R. Murray and S. Sastry, "Nonholonomic motion planning: Steering using sinusoids," *IEEE Trans. on Automatic Control*, vol. 38, pp. 700–716, 1993.

[14] O. J. Sørdalen, "Conversion of the kinematics of a car with n trailers into a chained form," in *IEEE Conf. on Robotics and Automation*, 1993, pp. 382–387.

[15] P. Morin and C. Samson, "Trajectory tracking for nonholonomic systems. Theoretical background and applications," INRIA, Research Report 6464, 2008, available at https://hal.inria.fr/inria-00260694.

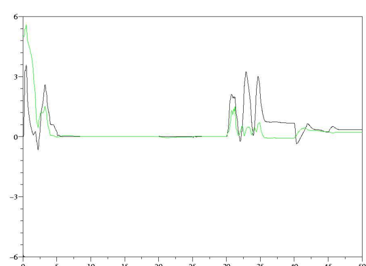

Fig. 2. Position tracking errors $\tilde{g}_{1,2}$ and vs. time

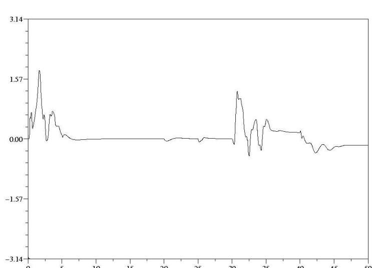

Fig. 3. Orientation tracking error $\tilde{g}_3$ vs. time

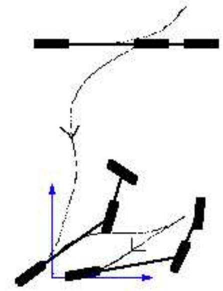

Fig. 4. Fixed reference, $t \in [0s, 5s)$

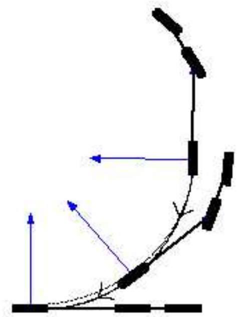

Fig. 5. Admissible arc of circle, $t \in [20s, 25s)$

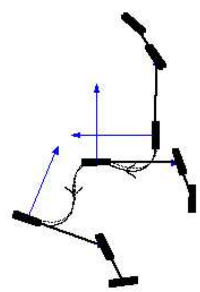

Fig. 6. Admissible trajectory with rapidly changing curvature, $t \in [25s, 30s)$

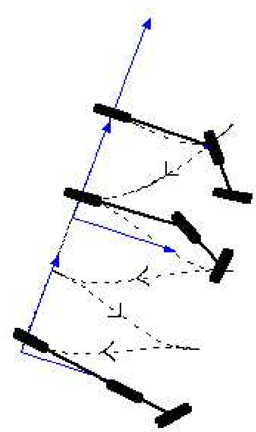

Fig. 7. Non-admissible lateral motion inducing maneuvers, $t \in [30s, 35s)$