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Robust Invariant Set Theory Applied to Networked Buffer-Level Control

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***Abstract*— A manufacturer producing several items keeps them into safety stocks (buffers) in order to supply an external stochastic demand without interruptions. We consider the classical problem of determining stock levels guaranteeing that a stochastic bounded demand is always satisfied. A simple n-th integrator model with additive disturbances is employed. Invariant set theory for linear and switched linear systems is used to compute robust positive invariant sets and controlled robust invariant sets for two commonly used scheduling policies. This paper provides the explicit expression of the invariant sets for any arbitrary n.**

I. INTRODUCTION

We consider a manufacturer producing several items which are kept in safety stocks (buffers) in order to supply an external stochastic demand. Production is make-to-stock, i.e., a demand is satisfied by using items in stock while triggering a stock replenishment order. This model is generic and it corresponds to several implementations such as: *(i)* the items are distinct components assembled into a single product at a single facility, (in this case, buffers connect the production line of intermediate products with the assembly line of final products), *(ii)* the items are distinct products produced at a single facility and supplying distinct demands, *(iii)* the items represent the same product manufactured and distributed at different locations, *(iv)* the items represent the same product and distinct inventories are kept for high-priority and low-priority customers. In this manuscript we will not focus on a specific problem and use the general term "item typology" with the following interpretation: items of the same typology i are kept in the same buffer x_i and have and associated demand d_i. The demand d_i can be satisfied if $x_i \geq d_i$.

A classical control problem consists of scheduling the production of each item typology in order to guarantee that buffers have enough stocks to satisfy the demand (i.e. stockout never occurs). Over the last decades this problem has been studied from many different angles. Computing an optimal *dynamic scheduling policy* (i.e., a policy which takes into account the current state of inventory levels before deciding which item should be produced next) is not an easy task. It requires the choice of three unknowns: which item to produce, when to start the production and for how long. Starting from a preliminary study of Zheng and Zipkin [1], the author of [2] provides the explicit form of the optimal scheduling policy for two item typologies requiring identical production times. De Vericourt and coauthors in [3] generalized the results in [2] when the two typologies require different production times. The authors also explicitly noticed the difficulties of generalizing the results to more than two item typologies. Further extensions of the results in [3] can be found in [4] and references therein. The authors in [5]–[7] study the case of multiple item typologies by focusing on fixed scheduling sequence. Independently of the adopted scheduling policy, the stockouts are typically modeled either as lost demand, thus constraining buffer levels to assume positive values, or by backordering demand and allowing buffer levels to take negative values corresponding to the unmet demand. In both cases the stockout risk is *minimized* by introducing a buffer shortage cost. Moreover, the scheduling policies proposed in the aforementioned works are independent of initial buffer levels and could lead to stockouts even under nominal operation.

Differently from the aforementioned literature, in this paper we follow the approach presented in [8]–[10], impose hard constraints on buffer levels and study the conditions which guarantee that stockouts never occur. We use a discrete-time inventory model and assume that multiple product typologies can be produced during time instant k and $k+1$. This is possible by either having multiple production lines or switching the production of one line between different typologies during the sampling time. The paper is divided into two parts. In the first part, we use a simple n-th integrator model with additive disturbances: $x_i(k+1) = x_i(k) + u_i(k) + d_i(k)$ where x_i is the level of buffer i, u_i the production rate of the i-th typology and $-d_i$ is its demand for $i = 1, \dots, n$. We apply hybrid robust control invariant theory [11]–[14] and compute the largest sets of initial buffer levels $x_i(0)$ such that no-stockout ($x_i(k) \geq 0$ at all time instants k) is guaranteed for all admissible demands $d_i(k)$, $i = 1, \dots, n$. We study both the case of a generic admissible control law $u_i(k)$ (and thus compute "robust invariant sets") as well as specific production control laws $u_i(k) = f(x_1(k), \dots, x_n(k))$ (and thus compute "robust controlled invariant sets"). Our main contribution is to provide the analytic expression of the aforementioned invariant and controlled invariant sets, *for any arbitrary* n. Since the proposed control laws f are piecewise linear functions, we make of use hybrid system theory and tools in order to compute the corresponding robust invariant sets.

Due to page limitations, we prefer to present the interpretation of the main results rather then their lengthy proofs. We refer the interested reader to [15] for the proofs of all

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the results reported in this paper.

We believe that the application of receding horizon scheduling polices in manufacturing plants [16]–[18] can benefit from the proposed study. In fact, it is well known that when a receding horizon (or moving horizon) scheduling policy is used, the persistent feasibility of the closed-loop systems is hardly guaranteed [17]. The results of this paper allow to simply compute an invariant set which can be used as a terminal set constraint in the receding horizon scheduling policy. This is a key element for guaranteeing the persistent feasibility of the closed-loop system. The next section discusses into details the problem formulation and the paper outline.

II. Problem Formulation and Paper Outline

We consider the following discrete time model

$$x(k+1) \quad = \quad x(k) + u(k) + d(k) \tag{1}$$

with the uniform sampling time equal to $\Delta T = t_{k+1} - t_k$. System (1) represents a set of n buffers where n types of items are stored. The state is $x(k) = [x_1(k), \dots, x_n(k)] \in \mathbb{R}^n$ where $x_i(k)$ represents the level of the i-th buffer at time t_k, the positive input is $u(k) = [u_1(k), \dots, u_n(k)] \in \mathbb{R}^n$ where $u_i(k)$ is the production rate for the i-th item typology during the sampling interval $[t_k, t_{k+1})$. The negative vector $d(k) = [d_1(k), \dots, d_n(k)] \in \mathbb{R}^n$ represents the external demand and $d_i(k)$ is the demand of the i-th typology in the sampling interval $[t_k, t_{k+1})$.

In this work a scaled model will be used, i.e., $x_i(k)$ denotes the number of products of type i stored at time instant k divided by ΔT and $u(k)$ and $d(k)$ represent production and demand rates, respectively, rather than absolute quantities.

System (1) is subject to the following constraints for all $k \geq 0$

$$x(k) \in \mathbb{X}^n \triangleq \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq M_i \quad i = 1, \dots, n\} \tag{2a}$$

$$u(k) \in \mathbb{U}^n \triangleq \{u \in \mathbb{R}^n \mid \sum_{i=1}^{n} u_i \leq P^{max}, \quad 0 \leq u_i \leq p_i^{max} i = 1, \dots, n\} \tag{2b}$$

$$d(k) \in \mathbb{W}^n \triangleq \{d \in \mathbb{R}^n \mid \sum_{i=1}^{n} d_i \geq -D^{max}, \quad -d_i^{max} \leq d_i \leq 0 \quad i = 1, \dots, n\} \tag{2c}$$

$$p_i^{max} \geq d_i^{max} \quad i = 1, \dots, n. \tag{2d}$$

Constraint (2a) sets an upper-bound M_i to the maximum buffer capacity and it imposes that a stockout never occurs ($x_i \geq 0$), i.e. it guarantees that demand is always satisfied. Constraint (2b) sets an upper-bound p_i^{max} to the production rate u_i of the i-th typology. Moreover, at any instant k the overall production can not exceed the maximum value P^{max} (which represents the maximum production capacity of the plant or machine). Constraint (2c) sets an upper-bound d_i^{max} to the demand $d(k)$ of the i-th typology. Moreover, at any instant k the overall demand can not exceed the maximum value D^{max}. Constraint (2d) guarantees that the plant production capacity for each item is greater than the maximum demand. In this paper we study the nontrivial case $\sum_{i=1}^{n} p_i^{max} \geq P^{max} \geq D^{max} \geq 0$.

Remark 1: In model (1) production and demand of all typologies at time k are synchronous. In case of one production line, switches between different typologies will occur during the sampling interval $[t_k, t_{k+1})$; these are not captured in the proposed model. Model (1)-(2) also does not include delays and setup times. Nevertheless, we will show that nonintuitive and interesting results can be obtained by studying the robust feasibility of model (1)-(2).

This work focuses on model (1)-(2) and studies the buffer conditions and the production laws which guarantee that demand is always satisfied (no-stockout). We make use of robust invariant set theory [11]–[13] for linear and switched linear systems. We first provide a brief review on invariant set theory in Section III-A. Then, we assume

$$\begin{array}{l} p_i^{max} = p_j^{max} = P^{max} = 1 \quad i, j = 1, \dots, n \\ d_i^{max} = d_j^{max} = D^{max} = 1 \quad i, j = 1, \dots, n \\ M_i = M_j = M \quad i, j = 1, \dots, n \end{array} \tag{3}$$

For system (1) under constraints (2) and assumption (3),

1) in Section IV we compute the largest set of initial buffer levels $x(0)$ such that for all admissible demands there exists a control law satisfying production constraints (2b) and buffer level constraints (2a). This set is the maximal control robust invariant set $\mathcal{C}_\infty$ (defined in Section III),
2) in Section V and VI we compute the largest set of initial buffer levels $x(0)$ such that for two specific production control laws $u = f(x)$, production constraints (2b) and buffer level constraints (2a) are satisfied for all admissible demands. This set is the maximal robust positive invariant set $\mathcal{O}_\infty$ for the closed loop system (defined in Section III).

The first production law is the *Replenish Lowest Buffer* (RLB) law: at each time instant k the system produces at the maximum rate the product typology corresponding to the lowest buffer level. The second production law is the *Distribute Production Capacity* (DPC) law: at each time instant k the control policy starts producing the item typology with the smallest buffer level. The buffer will be filled up to the maximum demand value or up to the maximum buffer level reachable with the residual production capacity. If there is a residual production capacity it will be used to replenish the second lowest buffer, and so on, until the maximum production capacity P^{Max} is reached or all buffers have been replenished at their maximum demand values. In RLB policies only one product typology can be produced at a time. In DPC policies more product typologies can be produced at the same time.

While computing the robust invariant sets described above is an interesting numerical exercise, we remark that the main contribution of this work is to provide the analytical expression for the invariant sets *for arbitrary* n.

In Section VII the results are extended to system (1)–(2) without the simplifying assumption (3). Illustrative examples are presented in Section VIII.

Remark 2: The first problem presented in this paper (Section IV) has been investigated in [8]–[10] for a larger class of multi-inventory dynamical systems. The results presented in [8]–[10] are more general than the one presented in Section IV. However, by focusing on the simpler system class (1), our approach allows to explicitly compute the expression of the maximal control robust invariant set $\mathcal{C}_\infty$ for arbitrary n.

Remark 3: In push systems the buffer holds products waiting to be manufactured. Therefore the problem becomes the dual of the one presented in this section. The main feasibility condition consists in having each buffer level below a fixed constant and avoiding that the queue of products waiting to be manufactured grows to infinity [19]–[21]. The results presented in this work can be extended to push systems.

III. Background on Robust Invariant Sets

This section has been extracted from [11]–[13] and provides the basic definitions and results for robust invariant sets for constrained linear and piecewise linear systems. A comprehensive survey of papers on set invariance theory can be found in [22].

Definition 1 (P-collection): A set $\mathcal{C} \subseteq \mathbb{R}^n$ is called the *P-collection* (in $\mathbb{R}^n$) if it is collection of a finite number of n-dimensional polytopes, i.e.

$$\mathcal{C} = \{\mathcal{C}_i\}_{i=1}^{N_C}, \tag{4}$$

where $\mathcal{C}_i := \{x \in \mathbb{R}^n \mid C_i^x x \leq C_i^c\}$, $\dim(\mathcal{C}_i) = n$, $i = 1, \dots, N_C$, with $N_C < \infty$.

In the following we consider PWA systems subject to an additive disturbance $w(k)$:

$$x(k+1) = f_a(x(k), w(k)) = \tilde{A}_r x(k) + \tilde{g}_r + w(k), \tag{5a}$$
$$\text{if } \; x(k) \in \tilde{\mathcal{P}}_r, \; r \in \{1, 2, \dots, R\}, \tag{5b}$$
$$x(k) \in \mathbb{X}, \quad w(k) \in \mathbb{W}, \quad \forall\, k \geq 0. \tag{5c}$$

where the active dynamic r is defined by the polyhedron $\tilde{\mathcal{P}}_r$ and R represent the number of different dynamics. The sets $\mathbb{X}$ and $\mathbb{W}$ are compact and polytopic and contain the origin in their interior. We will denote the set of states over which the PWA system (5) is defined as $\tilde{\mathcal{S}}_{\text{PWA}} = \bigcup_{r\in\mathcal{R}} \tilde{\mathcal{P}}_r$, where $\tilde{\mathcal{S}}_{\text{PWA}}$ is a P-Collection.

We will also consider PWA systems subject to the input $u(k)$ and the disturbance $w(k)$:

$$x(k+1) = f(x(k), u(k), w(k)) = \\ = A_r x(k) + B_r u(k) + g_r + w(k), \tag{6a}$$
$$\text{if } \; [x(k)^T \; u(k)^T]^T \in \mathcal{P}_r, \quad r \in \mathcal{R}, \tag{6b}$$
$$x(k) \in \mathbb{X}, \quad u(k) \in \mathbb{U}, \quad w(k) \in \mathbb{W}, \quad \forall\, k \geq 0. \tag{6c}$$

where the sets $\mathbb{X}$, $\mathbb{U}$ and $\mathbb{W}$ are compact and polytopic and contain the origin in their interior.

With slight abuse of notation, we will use $f(x(k), u(k), \mathbb{W})$ to denote the set of states $x(k+1)$ which is reachable for any $w(k) \in \mathbb{W}$, given $[x(k)^T \;\; u(k)^T] \in \mathcal{P}_r$.

A. Definitions

We will first introduce some basic concepts before defining the invariant sets that we wish to compute. For any integer k, $\mathbf{w}_k$ denotes the sequence $\{w(0), w(1), \dots, w(k-1)\}$, i.e., $\mathbf{w}_k \in \mathbb{W}^k$, and $\phi(k; x(0), \mathbf{w}_k)$ denotes the solution of $x(k+1) = f_a(x(k), w(k))$ at time k if the initial state is $x(0)$ and the disturbance sequence is $\mathbf{w}_k$. For the autonomous PWA system (5), we will denote the k-step reachable set for initial states x contained in the set $\mathcal{S}$ as

$$\text{Reach}(k; \mathcal{S}, \mathbb{W}) \triangleq \{\phi(k; x(0), \mathbf{w}_k) \in \mathbb{R}^n \mid \\ x(0) \in \mathcal{S}, \mathbf{w}_k \in \mathbb{W}^k\}.$$

Two different types of sets are being considered: *invariant sets* and *control invariant sets*. We will first discuss invariant sets. The invariant sets are computed for autonomous systems. These types of sets are useful to answer questions such as: "For a *given* feedback controller $u = k(x)$, find the set of states whose trajectory will never violate the system constraints". The following definitions are derived from [13], [22]–[25].

Definition 2 (Robust Positive Invariant Set): A set $\mathcal{O}$ is said to be a robust positive invariant set for the autonomous PWA system in (5) if $\text{Reach}(1; \mathcal{O}, \mathbb{W}) \subseteq \mathcal{O}$.

Definition 3 (Maximal Robust Positive Invariant Set $\mathcal{O}_\infty$): The set $\mathcal{O}_\infty$ is the maximal robust invariant set of the autonomous PWA system (5) if $0 \in \mathcal{O}_\infty$, $\mathcal{O}_\infty$ is robust invariant and $\mathcal{O}_\infty$ contains all robust invariant sets that contain the origin.

Control invariant sets are defined for systems subject to external inputs. These types of sets are useful to answer questions such as: "Find the set of states for which *there exists* a controller such that the system constraints are never violated". The following definitions are derived from [13], [22]–[25].

Definition 4 (Robust Control Invariant Set): A set $\mathcal{C} \subseteq \mathbb{X}$ is said to be a robust control invariant set for the PWA system in (6) if for every $x(k) \in \mathcal{C}$ there exists a $u(k) \in \mathbb{U}$ such that $f(x(k), u(k), \mathbb{W}) \subseteq \mathcal{C}$.

Definition 5 (Maximal Robust Control Invariant Set $\mathcal{C}_\infty$): The set $\mathcal{C}_\infty$ is said to be the maximal robust control invariant set for the PWA system in (6) if it is robust control invariant and contains all robust control invariant sets contained in $\mathbb{X}$.

In the interest of space we will not describe the procedure for computing the maximal robust control invariant subset (when system (6) is considered) or the maximal positive invariant subset (if system (5) is considered). We refer the reader to [13], [24], [26] for details.

IV. Computation of the Maximal Robust Control Invariant Set

The objective of this section is to compute the maximal robust control invariant set for system (1) subject to constraints (2)-(3) for arbitrary n. Given the set $L^n = \{1, 2, \dots, n\}$ we denote by $\text{group}(f, n)$ the set composed of all the subsets $L^n_{f,i}$ of L^n of dimension f: $\text{group}(f, n) = \left\{L^n_{f,1}, \dots, L^n_{f,n_f}\right\}$ with $n_f = \begin{pmatrix} n \\ f \end{pmatrix}$. As an example group$(2,3) = \{\{1,2\},\{1,3\},\{2,3\}\}$ and $L^3_{2,1} = \{1,2\}$. The following theorem provides the explicit form of the maximal robust control invariant set for arbitrary *n*. In the interest of space we only state the main theorem. The proof can be found in [15].

Theorem 1: The maximal robust control invariant set $\mathcal{C}_\infty$ for system (1) subject to constraints (2)-(3) is

$$\begin{aligned}\mathcal{C}_\infty = \quad & \{x \in \mathbb{R}^n \,|x_i \leq M,\ i = 1,\ldots,n, \\ & \textstyle\sum_{j\in L^n_{f,i}} x_j \geq f-1, L^n_{f,i} \in \text{group}(f,n), \\ & i = 1,\ldots,n_f,\ f = 1,\ldots,n\}\end{aligned} \quad (7)$$

.

The explicit form (7) of the maximal robust control invariant set provides an interesting insight which is shown next through an example. Let $n = 3$, the invariant set for system (1) subject to constraints (2)-(3) is so composed: *(i)* every buffer level is greater or equal to 0 and smaller or equal to M, *(ii)* the sum of any two buffer levels is greater or equal to 1, *(iii)* the sum of all three buffer levels is greater or equal to 2. The above set of constraints guarantee the existence of a control law $u(k) = f(x(k))$ which does not lead to stockout for any feasible demand profile. More examples can be found in Section VIII.

V. Computation of Maximal Robust Positive Invariant Set under the RLB Feedback Policy

Consider system (1) subject to (2)-(3) and define the RLB control law for $i = 1,\ldots,n$, as follows :

$$u_i(k) = \begin{cases} 1 & \text{if } (x_i(k) \leq x_j(k)\ \forall j \neq i),\ (x_i(k) \leq M-1) \\ 0 & \text{otherwise.}\end{cases} \quad (8)$$

The RLB control law works as follows: at each time instant k the system produces at the maximum rate the product typology corresponding to the lowest buffer level i, unless $x(k)$ is greater than $M-1$.

Remark 4: The control law (8) is multi valued along some switching hyperplanes. This does not cause any problem from a theoretical and practical point of view. The results of this section hold for any single-valued implementation of the control law extracted from (8).

The objective is to compute the maximal robust positive invariant set for system (1) subject to constraints (2)-(3) under the control law (8) for arbitrary n. The following theorem provides a closed form solution. The proof can be found in [15].

Theorem 2: The maximal robust positive invariant set $\mathcal{O}_\infty$ for system (1) with $n \geq 2$ subject to constraints (2)-(3) under the control law (8) is

$$\begin{aligned}\mathcal{O}_\infty = \quad & \{x \in \mathbb{R}^n \mid x_i \leq M,\ i = 1,\ldots,n, \\ & \textstyle\sum_{j\in L^n_{f,i}} x_j \geq \alpha_f,\ L^n_{f,i} \in \text{group}(f,n), \\ & i = 1,\ldots,n_f,\ f = 1,\ldots,n\}\end{aligned} \quad (9)$$

where α_k for $k = 3,\ldots,n$ is defined recursively as follows

$$\alpha_k = (\alpha_{k-1}+1)k/(k-1),\ \alpha_2 = 2,\ \alpha_1 = 0. \quad (10)$$

Theorem 2 shows that when the buffer model (1)-(3) is controlled by the RLB Feedback Policy in (8) the maximal robust positive invariant set can be computed explicitly for arbitrary n. The explicit form (9)-(10) provides also an interesting insight illustrated next through an example. Let $n = 3$, then stockout will never occur with the RLB policy if the initial buffer levels satisfy the following conditions: *(i)* every buffer level is greater than 0, *(ii)* the sum of any two buffer levels is greater than 2, *(iii)* the sum of all three levels greater than $9/2$. We remark that the system variables are normalized to the maximum production rate. By comparing $\mathcal{O}_\infty$ in (9)-(10) and $\mathcal{C}_\infty$ in (7) for $n = 3$ it is clear that there is room for improvement on the RLB policy, i.e., there might exist a control law which excludes stockout with lower initial buffer levels. Next we show that the DPC policy is one of those. More examples can be found in Section VIII.

VI. Computation of maximal robust positive invariant set under the DPC Feedback Policy

Consider system (1) subject to (2)-(3). At time k let $m(k) = [m_1,\ldots,m_n]$ be the index vector of the state components in increasing order, i.e., $m_i \in [1,n]$ for $i = 1,\ldots,n$, $m_i \neq m_j$ for all $i \neq j$, and $x_{m_i} \geq x_{m_j}$ for all $j < i$. Define the following DPC control law for $i = 1,\ldots,n$

$$u_{m_i(k)}(k) = \begin{cases}\min\{(1 - x_{m_i(k)}(k)), & \\ \qquad (1 - \sum_{j=1}^{i-1} u_{m_j(k)})\} & \text{if } x_{m_i(k)}(k) < 1 \\ 0 & \text{otherwise}\end{cases} \quad (11)$$

The DPC control policy produces the typologies whose buffer levels are less than the corresponding demand value; it starts from the smallest and it continues increasing order until the machine has production capacity. The buffer will be filled up to the maximum demand value or with the residual production capacity.

The following theorem shows that the DCP policy (11) is the "best" policy in the sense that $\mathcal{O}_\infty = \mathcal{C}_\infty$.

Theorem 3: The maximal robust positive invariant set for system (1) subject to constraints (3) under the control law (11) is equal to $\mathcal{C}_\infty$ in (7), i.e.

$$\begin{aligned}\mathcal{O}_\infty = \{ \quad & x \in \mathbb{R}^n | x_i \leq M,\ i = 1,\ldots,n, \\ & \textstyle\sum_{j\in L^n_{f,i}} x_j \geq f-1, L^n_{f,i} \in \text{group}(f,n), \\ & i = 1,\ldots,n_f,\ f = 1,\ldots,n\}.\end{aligned} \quad (12)$$

The proof of Theorem 3 can be found in [15].

VII. Extended Modeling

In this section the simplifying assumption (3) is removed and the results in Sections IV, V and VI are extended to system (1)–(2). For the sake of simplicity and without any loss of generality we assume $D^{max} = P^{max} = 1$.

Computation of $\mathcal{C}_\infty$: The following theorem provides the explicit form of the maximal robust control invariant set for system (1) subject to constraints (2) for arbitrary n.

Theorem 4: The maximal robust control invariant set $\mathcal{C}_\infty$ for system (1) subject to constraints (2) is

$$\begin{aligned}& \mathcal{C}_\infty = \{x \in \mathbb{R}^n | x_i \leq M_i,\ x_i \geq 0,\ i = 1,\ldots,n, \\ & \textstyle\sum_{j\in L^n_{f,i}} x_j \geq\ \sum_{j\in L^n_{f,i}} d_j^{max} - 1, L^n_{f,i} \in \text{group}(f,n), \\ & i = 1,\ldots,n_f,\quad f = 2,\ldots,n\}\end{aligned} \quad (13)$$

where group(f, n) is defined in Section IV.

The proof of Theorem 4 follows the lines of the proof of Theorem 1. We refer to [15] for detailed proof.

Computation of $\mathcal{O}_\infty$ under the RLB Feedback Policy: Consider system (1) subject to the constraints (2). The RLB control law (8) becomes

$$u_i(k) = \begin{cases} p_i^{max} & \text{if} \quad (x_i(k) \leq x_j(k)\ \forall j \neq i) \\ & \quad\ \textit{and}\ (x_i(k) \leq M_i - p_i^{max}) \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

with $i = 1, \ldots, n$. The maximal robust positive invariant set $\mathcal{O}_\infty$ for system (1) subject to constraints (2) under the control law (14) can be computed numerically for a fixed n. Numerical tests (reported in [15]) show that the set $\mathcal{O}_\infty$ might be non-convex and difficult to write in explicit form for arbitrary n.

Computation of $\mathcal{O}_\infty$ under the DPC Feedback Policy: Consider system (1) subject to the constraints (2). The DPC control law (11) becomes

$$u_{m_i(k)}(k) = \begin{cases} \min\{(d^{max}_{m_i(k)} - x_{m_i(k)}(k)), \\ \qquad (P^{max} - \sum_{j=1}^{i-1} u_{m_j(k)})\} \\ \qquad\qquad \text{if } (x_{m_i(k)}(k) < d^{max}_{m_i(k)}) \\ 0 \qquad\qquad \text{otherwise} \end{cases} \tag{15}$$

We obtain the following:

Theorem 5: The maximal robust positive invariant set for system (1) subject to constraints (2) under the control law (15) is equal to $\mathcal{C}_\infty$ in (13).

We refer to [15] for detailed proof.

Additional Extensions: The report [15] contains a discussion on extending the model (1) to consider setup-times and control policies function of the disturbances.

VIII. Numerical Examples

All the sets in this section have been computed with the Multi-Parametric Toolbox [27]. We consider two or three item typologies in order to be able to graphically visualize the results. In the examples (1) and (3) constraints (3) are used. Additional examples with different upper-bounds to the production and demand rates can be found in [15]. Different numerical values for M are chosen to better visualize the sets.

Example 1: Computation of $\mathcal{C}_\infty$ under Assumption (3)
We compute the maximal robust control invariant set $\mathcal{C}_\infty$ for system (1) subject to constraints (3) for n=2 and n=3. $\mathcal{C}_\infty$ for n=2 is

$$0 \leq x_1 \leq M,\ 0 \leq x_2 \leq M,\ x_1 + x_2 \geq 1. \tag{16}$$

$\mathcal{C}_\infty$ for n=3 is

$$\begin{aligned} &0 \leq x_1 \leq M, \quad 0 \leq x_2 \leq M, \quad 0 \leq x_3 \leq M \\ &x_1 + x_2 \geq 1, \quad x_1 + x_3 \geq 1, \quad x_2 + x_3 \geq 1 \\ &x_1 + x_2 + x_3 \geq 2. \end{aligned} \tag{17}$$

Example 2: Computation of $\mathcal{C}_\infty$ for $n = 3$
The maximal robust control invariant set $\mathcal{C}_\infty$ for system (1) subject to constraints (2) for $n = 3$, $p^{max} = [1; 1; 1]$ and $d^{max} = [0.8; 0.7; 0.4]$ is

$$\begin{aligned} &0 \leq x_1 \leq M, \quad 0 \leq x_2 \leq M, \quad 0 \leq x_3 \leq M \\ &x_1 + x_2 \geq 0.5, \quad x_1 + x_3 \geq 0.2, \quad x_2 + x_3 \geq 0.1 \\ &x_1 + x_2 + x_3 \geq 0.9. \end{aligned} \tag{18}$$

The maximal robust control invariant set $\mathcal{C}_\infty$ for system (1) subject to constraints (2) for $n = 3$ and $d^{max} = [0.5; 0.3; 1]$ is

$$\begin{aligned} &0 \leq x_1 \leq M, \quad 0 \leq x_2 \leq M, \quad 0 \leq x_3 \leq M \\ &x_1 + x_3 \geq 0.5, \quad x_2 + x_3 \geq 0.3 \\ &x_1 + x_2 + x_3 \geq 0.8. \end{aligned} \tag{19}$$

Example 3: Computation of $\mathcal{O}_\infty$ under the RLB Feedback Policy and Assumption (3)
System (1) subject to constraints (2) under the RLB control law (8) for $n = 3$ is rewritten as an autonomous PWA system defined over four polyhedral regions reported in figure 1(a) ($M = 5$). The maximal robust positive invariant set $\mathcal{O}_\infty$ is

$$\begin{aligned} &0 \leq x_1 \leq M, \quad 0 \leq x_2 \leq M, \quad 0 \leq x_3 \leq M \\ &x_1 + x_2 \geq 2, \quad x_1 + x_3 \geq 2, \quad x_2 + x_3 \geq 2 \\ &x_1 + x_2 + x_3 \geq 4.5. \end{aligned} \tag{20}$$

and is depicted in figure 1(b) ($M = 5$).

Example 4: Computation of $\mathcal{O}_\infty$ under the RLB Feedback Policy
The system (1) subject to constraints (2) under the RLB control law (8) for $n = 2$, $p^{max} = [0.9; 0.8]$ and $d^{max} = [0.5; 0.3]$ is rewritten as an autonomous PWA system over three polyhedral regions. The corresponding maximal robust positive invariant set $\mathcal{O}_\infty$ is depicted in figure 2(a) ($M = 3$).

The system (1) subject to constraints (2) under the RLB control law (8) for $n = 3$, $p^{max} = [0.9; 0.9; 0.9]$ and $d^{max} = [0.5; 0.3; 0.1]$ is rewritten as an autonomous PWA system defined over four polyhedral regions. The corresponding maximal robust positive invariant set $\mathcal{O}_\infty$ is depicted in figure 2(b) ($M = 3$).

IX. Conclusions

We have addressed the classical problem of buffer level control by means of robust invariant set theory for linear and switched linear systems. The analytic expression of robust positive invariant sets and controlled robust invariant sets for two common scheduling policies has been provided. The results of this paper allow to compute the invariant sets for an arbitrary number of buffers without resorting to recursive algorithms.

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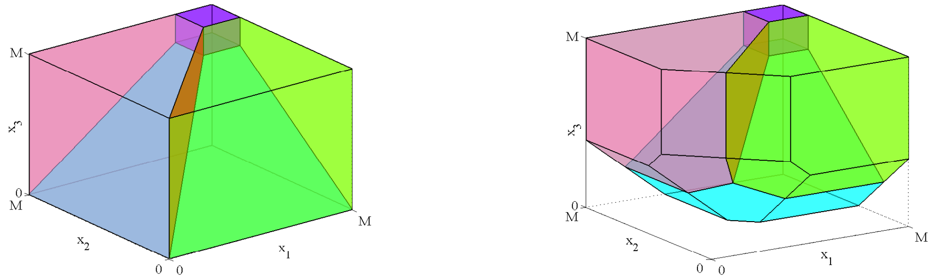
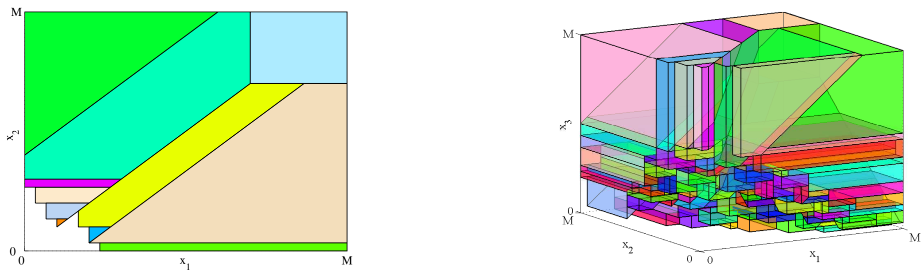


(a) S_{PWA} (b) $\mathcal{O}_\infty$

Fig. 1. S_{PWA} and $\mathcal{O}_\infty$ under the RLB Feedback Policy and the Assumption (3).



(a) n=2, p_1^{max}, d_1^{max} (b) n=3, p_2^{max}, d_2^{max}

Fig. 2. $\mathcal{O}_\infty$ under the RLB Feedback Policy.

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