

Computation of limit cycles and forced oscillations in discrete-time piecewise linear feedback systems through a complementarity approach

Luigi Iannelli, Francesco Vasca

Abstract—Limit cycles and forced oscillations in piecewise linear (PWL) feedback systems are difficult to be computed without a priori knowledge of the structure of the periodic solution. Even in that case the explicit computation of the solution is possible only assuming simple models. In this paper, by representing discrete-time PWL feedback systems as linear complementarity systems, we show that periodic oscillations can be computed by solving suitable static linear complementarity problems. An efficient algorithm for computing such solutions is adopted. Limit cycles in autonomous relay feedback systems and forced oscillations in pulse width modulated DC/DC power converters are easily found by solving the proposed complementarity problem, provided that the discretized complementarity system well approximates the original continuous-time system.

I. INTRODUCTION

We consider piecewise linear (PWL) feedback systems in the Luré form, i.e. representable as the feedback interconnection of a linear time invariant dynamical system Σ_d with a PWL static characteristic (φ, λ) , as shown in Fig. 1. Σ_d represents the linear system with a minimal state space realization. The static characteristic (φ, λ) is a PWL multi-valued mapping, which includes piecewise linear functions (e.g. saturation), set-valued functions (e.g. relay, quantizer) and unbounded characteristics (e.g. ideal diode and Zener diode characteristics). The analysis of limit cycles and periodic forced oscillations in this class of nonsmooth dynamical systems has attracted a wide interest in the literature, see among others [1], [2]. A strong motivation for such interest is the relevance of oscillations in the behavior of several practical systems representable in Luré form. Interesting classes of such systems are nonlinear circuits and power electronics converters [3], [4]. A typical approach for computing periodic oscillations in PWL feedback systems consists of solving a nonlinear system obtained by iteratively applying a Poincaré map to an a priori assumed form of the solution [1], [5], [6]. However closed form solutions can only be given for very special cases. Alternative widely used approaches are the Describing Function and the Tsypkin method. The former is very useful in order to obtain a rough estimation of period and amplitude of the oscillation, but may fail to predict limit cycles if high order harmonics are involved in the solution [7]. The Tsypkin method [8], [9] instead is rigorous but it becomes difficult to be applied when the periodic oscillation is not unimodal (two switchings per period, no sliding solutions) and, above all, when the static

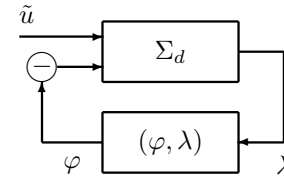


Fig. 1. Feedback interconnection of a linear time invariant system Σ_d with a piecewise linear characteristic (φ, λ) .

nonlinearity is a PWL multi-valued mapping different from a relay.

In this paper we propose the use of the complementarity formalism [10] for computing periodic oscillations of known period in systems in Luré form with a PWL feedback characteristic, both autonomous and forced by periodic inputs. The complementarity formalism has been shown to be useful for well-posedness and stability analysis of PWL feedback systems [11], [12], [13]. The problem of the computation of a periodic oscillation is here formulated as a static linear complementarity problem with possible multiple solutions [14]. Numerical results considering a feedback system with a bounded static characteristic (autonomous relay feedback system) and a feedback system with unbounded set-valued characteristics (power electronics circuit) show the effectiveness of the proposed approach.

II. COMPLEMENTARITY MODEL OF PIECEWISE LINEAR FEEDBACK SYSTEMS

A linear complementarity problem (LCP) can be defined as follows:

Problem 1 ($LCP(q, M)$): Given a real vector q and a real matrix M , find a real vector z such that

$$z \geq 0 \quad (1a)$$

$$q + Mz \geq 0 \quad (1b)$$

$$z^T (q + Mz) = 0, \quad (1c)$$

where the inequalities are considered componentwise.

In the sequel conditions (1) that define the $LCP(q, M)$ will be more compactly indicated by means of the complementarity condition

$$0 \leq w \perp z \geq 0, \quad (2)$$

with $w = q + Mz$. We now introduce the concept of a complementarity system.

Dept. of Engineering, University of Sannio, Benevento, Italy, email: {luigi.iannelli, vasca}@unisannio.it

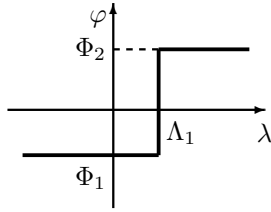


Fig. 2. A relay-like characteristic.

Definition 1: A discrete-time linear complementarity system (LCS) is the following linear system subject to complementarity constraints on z and w variables:

$$x_k = Ax_{k-1} + Bz_k + Eu_k \quad (3a)$$

$$w_k = Cx_{k-1} + Dz_k + Fu_k \quad (3b)$$

$$0 \leq w_k \perp z_k \geq 0, \quad (3c)$$

where $x \in \mathbb{R}^{N_x}$, $u \in \mathbb{R}^{N_u}$, $z \in \mathbb{R}^{N_z}$ and $w \in \mathbb{R}^{N_w}$, k is the discrete-time variable and A, B, C, D, E, F are real matrices of suitable dimensions.

It appears evident how, in order to have a well-defined model (3), the LCP($Cx_{k-1} + Fu_k, D$) given by (3b) and (3c) must be feasible for each discrete step k , since its solution z_k affects the dynamics. The PWL feedback system in Fig. 1 can be represented as a linear complementarity system (3). Assume that Σ_d is a discrete-time linear time invariant system:

$$x_k = A_d x_{k-1} + B_d(-\varphi_k) + E_d \tilde{u}_k \quad (4a)$$

$$\lambda_k = C_d x_{k-1} + D_d(-\varphi_k) + F_d \tilde{u}_k, \quad (4b)$$

with x, \tilde{u}, φ and λ real vectors of suitable dimensions (φ and λ of the same dimension) and (A_d, B_d, C_d) minimal.

The class of PWL characteristics (φ, λ) can be represented in the following complementarity formalism

$$\varphi = A_s \lambda + B_s z + g_s \quad (5a)$$

$$w = C_s \lambda + D_s z + h_s \quad (5b)$$

$$0 \leq w \perp z \geq 0, \quad (5c)$$

where the real matrices A_s, B_s, g_s, C_s, D_s and h_s have suitable dimensions. Such representation can be considered really general for describing set-valued PWL mappings [15], [16].

For instance the relay characteristic in Fig. 2, which has the two breaking points (Φ_1, Λ_1) and (Φ_2, Λ_2) , can be represented in the form (5) by choosing

$$\varphi = -(\Phi_2 - \Phi_1)z_1 + \Phi_2 \quad (6a)$$

$$w_1 = \lambda + z_2 - \Lambda_1 \quad (6b)$$

$$w_2 = -z_1 + 1 \quad (6c)$$

$$0 \leq w \perp z \geq 0. \quad (6d)$$

We can now build up the general system model (3) by substituting (5) in (4):

$$x_k = A_d x_{k-1} - B_d (A_s \lambda_k + B_s z_k + g_s) + E_d \tilde{u}_k \quad (7a)$$

$$\lambda_k = C_d x_{k-1} - D_d (A_s \lambda_k + B_s z_k + g_s) + F_d \tilde{u}_k \quad (7b)$$

$$w_k = C_s \lambda_k + D_s z_k + h_s \quad (7c)$$

$$0 \leq w_k \perp z_k \geq 0. \quad (7d)$$

By looking at (7b), if the matrix $D_d A_s$ has no eigenvalues in -1 , the matrix $\Theta \triangleq I + D_d A_s \in \mathbb{R}^{m \times m}$ is invertible and

$$\lambda_k = \Theta^{-1} (C_d x_{k-1} - D_d B_s z_k - D_d g_s + F_d \tilde{u}_k). \quad (8)$$

Now system (7) can be written as (3) with

$$A := A_d - B_d A_s \Theta^{-1} C_d \quad (9a)$$

$$B := B_d A_s \Theta^{-1} D_d B_s - B_d B_s \quad (9b)$$

$$C := C_s \Theta^{-1} C_d \quad (9c)$$

$$D := D_s - C_s \Theta^{-1} D_d B_s \quad (9d)$$

$$E := (E_d - B_d A_s \Theta^{-1} F_d \quad g) \quad (9e)$$

$$F := (C_s \Theta^{-1} F_d \quad h), \quad (9f)$$

where we have included in the vector u also the ones needed to represent the constant term coming from g_s and h_s , see (5) and (6), i.e.

$$g := B_d (A_s \Theta^{-1} D_d - I) g_s \quad (10a)$$

$$h := h_s - C_s \Theta^{-1} D_d g_s. \quad (10b)$$

Note that being Θ singular, it means that the feedback structure has an algebraic loop not solvable and we get an ill-posed problem. Note that in the case $D_d > 0$ and $A_s \geq 0$, the matrix $\Theta = (I + D_d A_s)$ is invertible [17]. The same can be proved if $D_d \geq 0$ and $A_s \geq 0$ and diagonal, which is the case for the matrix A_s in complementarity models of nondecreasing PWL characteristics (φ, λ) , [13].

Thus we have proven the equivalence of the class of systems considered in equations (4)-(5) to the LCS (3). In the following we will assume that, given an initial condition x_0 and an input sequence $\{u_k\}$, the system (4)-(5) will have a unique solution denoted by the sequence $\{x_k\}$. Such assumption means that the LCS (3) has a unique solution $\{x_k\}$, even though the LCP($Cx_{k-1} + Fu_k, D$) might not admit a unique solution z_k (indeed, the sequence $\{Bz_k\}$ will be unique).

III. COMPUTATION OF PERIODIC OSCILLATIONS

Consider the system (3) forced by a periodic external signal $\{u_k\}$ of period N . Assume the system admits a periodic forced oscillation, i.e. $x_{k+N} = x_k \forall k$. The state evolution gives

$$x_N = A^N x_0 + \sum_{i=1}^N A^{N-i} (Bz_i + Eu_i) = x_0. \quad (11)$$

By solving with respect to x_0 and defining $\Pi_N \triangleq (I - A^N)^{-1}$:

$$x_0 = \Pi_N \sum_{i=1}^N A^{N-i} (Bz_i + Eu_i). \quad (12)$$

Note that u_i , $i = 1, \dots, N$ is a known input whereas z_i , $i = 1, \dots, N$ are unknowns. The matrix Π_N satisfies the following properties:

$$A\Pi_N = \Pi_N A \quad (13a)$$

$$\Pi_N A^N = \Pi_N - I. \quad (13b)$$

By writing (3b) for $k = 1, \dots, N$

$$w_1 = Cx_0 + Dz_1 + Fu_1 \quad (14a)$$

$$w_2 = CAx_0 + CBz_1 + Dz_2 + CEu_1 + Fu_2 \quad (14b)$$

\vdots

$$w_N = CA^{N-1}x_0 + \sum_{i=1}^{N-1} CA^{N-1-i}Bz_i + Dz_N \\ + \sum_{i=1}^{N-1} CA^{N-1-i}Eu_i + Fu_N. \quad (14c)$$

By substituting (12) in (14)

$$w_k = CA^{k-1}\Pi_N \left(\sum_{i=1}^N A^{N-i}(Bz_i + Eu_i) \right) \\ + \sum_{i=1}^{k-1} CA^{k-1-i}Bz_i + Dz_k + \sum_{i=1}^{k-1} CA^{k-1-i}Eu_i + Fu_k \quad (15)$$

for $k = 1, \dots, N$.

By collecting all the terms z_i and u_i and by using properties (13) one can write

$$w_1 = C\Pi_N A^{N-1}Bz_1 + Dz_1 + C\Pi_N A^{N-2}Bz_2 + \dots \\ + C\Pi_N A^{N-1}Eu_1 + Fu_1 + C\Pi_N A^{N-2}Eu_2 + \dots \quad (16a)$$

$$w_2 = (C(\Pi_N - I)B + CB)z_1 \\ + (C\Pi_N A^{N-1}B)z_2 + Dz_2 + \dots + \\ + (C(\Pi_N - I)E + CE)u_1 \\ + (C\Pi_N A^{N-1}E)u_2 + Fu_2 + \dots \quad (16b)$$

and so on. The equations above together with the complementarity conditions (3c) can be rewritten as the following LCP(q_N, M_N):

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} = M_N \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} + q_N \quad (17a)$$

$$0 \leq w_k \perp z_k \geq 0, \quad k = 1, \dots, N \quad (17b)$$

where

$$M_N = \begin{pmatrix} D & 0 & \dots & 0 \\ 0 & D & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D \end{pmatrix} \\ + \begin{pmatrix} C\Pi_N A^{N-1}B & C\Pi_N A^{N-2}B & \dots & C\Pi_N B \\ C\Pi_N B & C\Pi_N A^{N-1}B & \dots & C\Pi_N AB \\ \vdots & \vdots & \ddots & \vdots \\ C\Pi_N A^{N-2}B & C\Pi_N A^{N-3}B & \dots & C\Pi_N A^{N-1}B \end{pmatrix} \quad (18)$$

and q_N is reported in (19). Note that M_N is a block circulant matrix. The solution of LCP(q_N, M_N) will be a real vector of dimension $N_z \cdot N$. If M_N is a P-matrix then the solution is unique [10] and the system (3) has a unique periodic oscillation of period N . Moreover, the periodic oscillation can be obtained by solving LCP (17) and by using equations (11)-(12). If M_N is not a P-matrix the LCP (17) could have no solution or multiple solutions. Any solution of the LCP will correspond to a periodic oscillation of the system (3) and viceversa. In order to obtain the possible multiple solutions of (17) we will use the algorithm proposed in [14].

Note that we can extend such considerations also to the case of autonomous systems, since $\{u_k\}$ constant is periodic of any period, in particular the period N of autonomous oscillations.

A. Detection of multiple LCP solutions

Consider an autonomous relay feedback system, i.e. the system in Fig. 1 with $\tilde{u} = 0$ and (φ, λ) representable as in (6) with $\Lambda_1 = 0$. It is well known that depending on Σ_d , such type of system can exhibit limit cycles. On the other hand, $x_k = 0 \forall k$ is also a solution for the system (3) so as it can be simply verified by considering $z_1 = \Phi_2/(\Phi_2 - \Phi_1)$, $z_2 = 0$, $w_1 = 0$, $w_2 = -\Phi_1/(\Phi_2 - \Phi_1)$ that from (6) implies $\varphi = 0$ and $\lambda = 0$, see (4).

The fact that autonomous systems with PWL characteristics passing through the origin have the trivial solution $x_k = 0 \forall k$ and can also exhibit limit cycles, demonstrate that it is of primary importance to find the different solutions of the LCP (17). To this aim we use the algorithm proposed in [14]. Say \bar{z} a solution of the LCP (17). Now it is possible to construct a new LCP that has all solutions of (17) except \bar{z} , i.e. to exclude \bar{z} from the solutions of (17). Let Γ be the set of indexes such that the corresponding components of the vector \bar{z} are positive, i.e. $\Gamma = \{k \in \{1, \dots, N_z \cdot N\} : \bar{z}_k > 0\}$. It can be simply shown that for any $z \geq 0$ it is $z = \bar{z}$ if and only if the following inequalities are satisfied:

$$z_k \geq \bar{z}_k, \quad \forall k \in \Gamma \quad (20a)$$

$$\sum_{k \in \Gamma} z_k \leq \sum_{k \in \Gamma} \bar{z}_k \quad (20b)$$

$$\sum_{k \notin \Gamma} z_k \leq 0 \quad (20c)$$

$$q_N = \begin{pmatrix} C\Pi_N A^{N-1}E + F & C\Pi_N A^{N-2}E & \cdots & C\Pi_N E \\ C\Pi_N E & C\Pi_N A^{N-1}E + F & \cdots & C\Pi_N AE \\ \vdots & \vdots & \ddots & \vdots \\ C\Pi_N A^{N-2}E & C\Pi_N A^{N-3}E & \cdots & C\Pi_N A^{N-1}E + F \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \quad (19)$$

where the last inequality can be dropped if all components of \bar{z} are strictly positive, i.e. if the size of Γ is $N_z \cdot N$. Otherwise the number of inequalities (20) is given by the size of Γ plus two ((20b) and (20c) are scalar inequalities). The inequalities (20) can be rewritten in the matrix form

$$Pz + r \geq 0 \quad (21)$$

for suitable P and r where r contains the values \bar{z}_k . Since $Pz + r \geq 0$ if and only if $z = \bar{z}$, it will be $\min(Pz + r) < 0$ for any $z \neq \bar{z}$. By assuming that the different solutions of the LCP (17) are isolated one can say that there exists a $\gamma > 0$ such that $\min(Pz + r) \leq -\gamma$ for all solutions of the LCP that are different from \bar{z} . The new solution of the LCP (17), different from \bar{z} , can be obtained by taking the first $N_z \cdot N$ components of the solution of the LCP(q_{mod}, M_{mod}) where

$$q_{mod} = \begin{pmatrix} q_N \\ r \\ -\gamma \end{pmatrix} \quad (22a)$$

$$M_{mod} = \begin{pmatrix} M_N & 0 & 0 \\ P & I & 0 \\ 0 & e^T & -1 \end{pmatrix} \quad (22b)$$

where e^T is a vector of ones of suitable dimension. The procedure presented above can be iterated in order to obtain all solutions of the LCP(q_N, M_N) [14].

IV. APPLICATION SCENARIOS

In this section we show how to compute a limit cycle in a autonomous relay feedback system and a forced periodic oscillation in a PWM power electronics DC/DC buck converter. Such systems can be represented in the form of Fig.1 with Σ_d being a continuous-time system. Therefore in order to apply the proposed procedure, the first step consists of obtaining the discrete-time system (3). Assume that Σ_d is a continuous-time linear time invariant system

$$\dot{x} = \tilde{A}_d x + \tilde{B}_d(-\varphi) + \tilde{E}_d \tilde{u} \quad (23a)$$

$$\lambda = \tilde{C}_d x + \tilde{D}_d(-\varphi) + \tilde{F}_d \tilde{u}. \quad (23b)$$

In order to obtain the discrete-time model (3) one can discretize (23) obtaining (4) and then follow the procedure presented in Section II. Equivalently one can also compute the closed loop continuous-time complementarity model and then discretize. Substituting (5) in (23) one obtains

$$\dot{x} = A_c x + B_c z + E_c u \quad (24a)$$

$$w = C_c x + D_c z + F_c u \quad (24b)$$

$$0 \leq w \perp z \geq 0, \quad (24c)$$

where the matrices are given by (9) with A_d, B_d, C_d, D_d, E_d and F_d replaced by the corresponding matrices with the tilde, respectively. By discretizing (24) by using the backward Euler method with sampling period T_s it is possible to get the following discrete-time linear complementarity system:

$$x_k = x_{k-1} + T_s A_c x_k + T_s B_c z_k + T_s E_c u_k \quad (25a)$$

$$w_k = C_c x_k + D_c z_k + F_c u_k \quad (25b)$$

$$0 \leq w_k \perp z_k \geq 0 \quad (25c)$$

and thus one obtains (3) with

$$A := (I - T_s A_c)^{-1} \quad (26a)$$

$$B := (I - T_s A_c)^{-1} T_s B_c \quad (26b)$$

$$C := C_c (I - T_s A_c)^{-1} \quad (26c)$$

$$D := D_c + C_c (I - T_s A_c)^{-1} T_s B_c \quad (26d)$$

$$E := (I - T_s A_c)^{-1} T_s E_c \quad (26e)$$

$$F := F_c + C_c (I - T_s A_c)^{-1} T_s E_c. \quad (26f)$$

A similar result can be obtained by discretizing the model (24) with the trapezoidal instead of the Euler rule.

Let us consider the case in which the continuous-time complementarity system (24) has a periodic oscillation of period T (in the case of a forced oscillation, $u(t)$ is also periodic of period T). By assuming consistency of the discretization, i.e. the discrete-time system approximates the continuous-time system, the discrete-time complementarity system (3) will have a periodic oscillation of period N , where N is related to the discretization step T_s and the period T . Moreover, in this case the continuous-time instants at which conditions (24c) change, i.e. when one or more components of w or z become zero, do not need to be known a priori and do not need to be sampling time instants. In other words the shape of the periodic oscillation is not fixed a priori. Such arguments are often valid from a practical point of view, so as it will be shown by our numerical results. However from a more theoretical point of view one should prove consistency of the discretization which is a non trivial task [4], [18], [19].

A. Limit cycles in autonomous relay feedback systems

Consider the continuous-time system (23) with the following matrices:

$$\tilde{A}_d = \begin{pmatrix} -3 & 1 & 0 \\ -3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad \tilde{B}_d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \tilde{E}_d = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (27a)$$

$$\tilde{C}_d = (1 \ 0 \ 0), \quad \tilde{D}_d = 0, \quad \tilde{F}_d = 0, \quad (27b)$$

with $\tilde{u} = 0$ and the characteristic (φ, λ) being a signum, i.e. the relay characteristic depicted in Fig. 2 with $\Phi_1 = -1$,

$\Lambda_1 = 0$ and $\Phi_2 = 1$. It is simple to verify by using the Describing Function approach or the Tsytkin method that such continuous-time system exhibits a limit cycle. Construct the model (24) by using the complementarity representation (6) for the relay, discretize the system with $T_s = 0.07\text{s}$ and choose $N = 46$ samples per period.

For each k the relay characteristic can be written in the form (6) which corresponds to the model (5) with $A_s = 0$, $B_s = (-2 \ 0)$, $g_s = 1$, $C_s = (1 \ 0)^T$, $h_s = (0 \ 1)^T$ and

$$D_s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $z_k \in \mathbb{R}^2$ and we are considering $N = 46$ discrete-time instants, the vector z in the LCP(q_N, M_N) given by (17) will have dimension $N_z \cdot N = 92$. From (6) it follows that $(\bar{\varphi}_k, \bar{\lambda}_k) = (0, 0)$ corresponds to $\bar{z}_{1k} = 1/2$ and $\bar{z}_{2k} = 0$. Then, from (4) being $u_k = (\tilde{u}_k \ 1)^T$ and $\tilde{u}_k = 0$ for all k , it follows that $\bar{z}_k = (\bar{z}_{1k} \ \bar{z}_{2k})^T = (1/2 \ 0)^T$, $\bar{w}_k = (\bar{w}_{1k} \ \bar{w}_{2k})^T = (0 \ 1/2)^T$, and $\bar{x}_k = 0$ for all k satisfy (3). Therefore $z_k = \bar{z}_k$ for all k will be also a solution of LCP (17). Since we are interested in the solution of (17) that corresponds to the nontrivial solution of (3), i.e. the one corresponding to the limit cycle, we need to eliminate \bar{z}_k for all k from the possible solutions of (17) by implementing (22). In this case it is simple to verify that the size of the set Γ is N (half of the $N_z \cdot N = 2N$ components of \bar{z} are strictly positive), and

$$r = (-1/2 \ \cdots \ -1/2 \ N/2 \ 0)^T$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & \cdots & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 & \cdots & -1 & 0 & -1 \end{pmatrix}$$

with P having $N + 2$ rows and $2N$ columns. By defining $z_{mod} = (z^T \ y^T \ v)^T$ the LCP(q_{mod}, M_{mod}) can be rewritten as (see (1) and (22))

$$z_{mod} \geq 0 \quad (30a)$$

$$q_N + M_N z \geq 0 \quad (30b)$$

$$r + Pz + y \geq 0 \quad (30c)$$

$$-\gamma + \sum_{i=1}^{N+2} y_i - v \geq 0 \quad (30d)$$

$$z_{mod}^T (q_{mod} + M_{mod} z_{mod}) = 0 \quad (30e)$$

where $z \in \mathbb{R}^{N_z \cdot N}$, $y \in \mathbb{R}^{N+2}$ and $v \in \mathbb{R}$. By choosing $\gamma = -1 \cdot 10^{-3} \cdot 2N$ the LCP (22) is solved by using the PATH tool [20]. In Fig. 3 it is shown the solution obtained through the LCP together with the numerical results of a time stepping simulation of the system. It should be stressed that the ‘‘switching’’ time instants are not fixed a priori and are automatically determined by the solution of the LCP. Moreover the solution of the LCP (30), thanks to

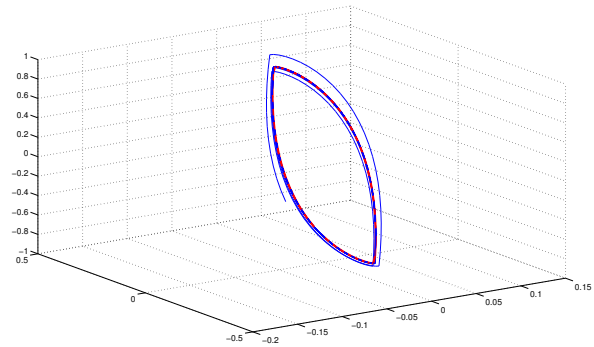


Fig. 3. Limit cycle of the relay feedback example computed by using the complementarity procedure (thick line) and a time-stepping simulation (thin line).

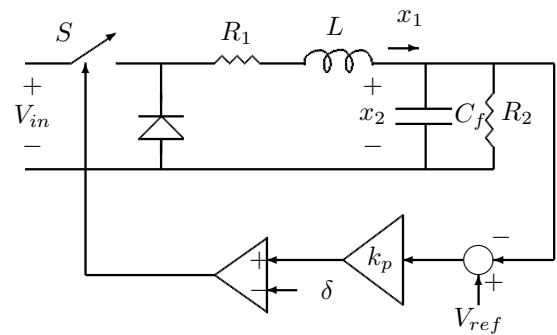


Fig. 4. Buck converter with PWM voltage-mode control.

the efficient PATH algorithm, is obtained much faster than a time-stepping simulation which must be very long in order to get a trajectory close enough to the steady state solution (on our hardware/software platform we had, respectively, a couple of minutes against about ten minutes).

B. Forced oscillations in power converters

A typical class of systems that exhibit forced oscillations consists of power electronics converters with Pulse Width Modulation. A DC/DC buck converter with output feedback control and PWM modulation is depicted in Fig. 4. The behavior of the controlled buck converter in discontinuous conduction mode [21] can be represented, in the usual operating conditions, by means of the controlled circuit reported in Fig. 5. The input voltage \tilde{V} is imposed by the feedback control law that is designed in order to regulate the output voltage x_2 to the desired value V_{ref} . In particular

$$\tilde{V} = V_{in} \text{step}(k_p (V_{ref} - x_2) - \delta) \quad (31a)$$

$$\delta(t) = \frac{\Delta}{NT_s} \text{mod}(t, NT_s), \quad (31b)$$

with the step function being described by the characteristic in Fig. 2 with $\Lambda_1 = 0$, $\Phi_1 = 0$, $\Phi_2 = 1$. It is not difficult to show that such system belongs to the class of PWL feedback system considered in this paper and, due to the external periodic signal δ (it is the PWM carrier signal or, equivalently, the dither signal [22]), a forced oscillation is

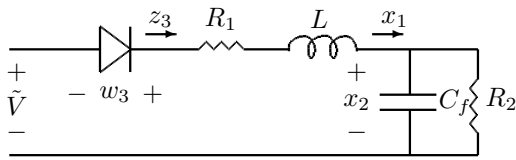


Fig. 5. Equivalent circuit of a controlled DC/DC buck converter.

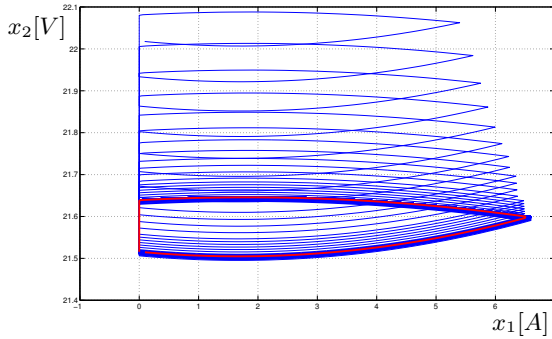


Fig. 6. Phase plane for a DC/DC buck converter in discontinuous conduction mode: LCP computed periodic solution (*thick line*); trajectory obtained through time-stepping simulation (*thin line*).

expected. Consider the dynamical equations describing the circuit together with the step characteristic (see (6)):

$$L\dot{x}_1 = -R_1x_1 - x_2 + z_3 + \tilde{V} \quad (32a)$$

$$C_f\dot{x}_2 = x_1 - x_2/R_2 \quad (32b)$$

$$\tilde{V} = V_{in}(1 - z_1) \quad (32c)$$

$$w_1 = -k_p x_2 + z_2 - \delta + k_p V_{ref} \quad (32d)$$

$$w_2 = -z_1 + 1 \quad (32e)$$

$$w_3 = x_1 \quad (32f)$$

$$0 \leq w \perp z \geq 0. \quad (32g)$$

Such equations, by choosing $u = (1 \quad -\delta + k_p V_{ref})^T$ can be recasted in the form (24). By considering the circuit parameters $R_1 = 0 \Omega$, $L = 20 \mu\text{H}$, $C_f = 222 \mu\text{F}$, $R_2 = 12.5 \Omega$, $V_{in} = 33 \text{V}$, $k_p = 0.1$, $\Delta = 1$, $N = 200$ and $T_s = 0.167 \mu\text{s}$ the solution of the complementarity problem (17) provides the steady state forced oscillation depicted in Fig. 6. Note that such behavior cannot be explicitly obtained by using the classical Poincaré maps without knowing a priori the structure of the cyclic switching sequence ($\tilde{V} = V_{in}$, diode ON; $\tilde{V} = 0$, diode ON; $\tilde{V} = 0$, diode OFF) and a numerical simulation of such system requires at least one hundred PWM periods before getting the steady state solution.

V. CONCLUSION

In this paper it has been shown how the complementarity framework can be used to compute periodic oscillations for linear time invariant systems connected in feedback through a PWL static mapping. Conditions for the existence of periodic oscillations in terms of solvability of a suitable static linear complementarity problem are obtained. The existence of limit cycles in autonomous PWL feedback systems has been

related to the existence of multiple solutions of the linear complementarity problem. The proposed procedure does not need to fix a priori the shape of the periodic oscillation and the LCP allows to compute also complex behaviors.

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