Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Synchronization in networks of nonlinear oscillators with coupling delays

Wim Michiels and Henk Nijmeijer

Abstract— **We consider the synchronization problem of an arbitrary number of coupled nonlinear oscillators with delays in the interconnections. The emphasis is on coupled Lorenz systems. The network topology is described by a directed graph. Unlike the conventional approach of deriving directly sufficient synchronization conditions, our approach starts from a linearized stability analysis in a (gain, delay) parameter space of a synchronized equilibrium and extracts insights from an analysis of its bifurcations and from the corresponding emerging behavior. Instrumental to this analysis a factorization of the characteristic equation is employed that not only facilitates the analysis and reduces computational cost, but also allows to determine the precise role of the individual agents and the topology of the network in the (in)stability mechanisms. The study reveals fundamental limitations to synchronization and it explains under which conditions on the topology of the network and on the characteristics of the coupling, the systems are expected to synchronize. Furthermore, the main result shows that for sufficiently large coupling gains, coupled Lorenz systems exhibit a generic behavior that does not depend on the number of systems and the topology of the network, as long as some basic assumptions are satisfied, including the strong connectivity of the graph. The results are illustrated with networks of Lorenz systems with several topologies.**

I. Introduction

We consider p identical nonlinear oscillators described by

$$\begin{array}{l} \dot{x}_i(t) = f(x_i) + B u_i(t), \\ y_i(t) = C^T x_i(t), \qquad i = 1, \dots p, \end{array} \tag{1}$$

where $x_i \in \mathbb{R}^n,\ i = 1, \dots, p,\ B, C \in \mathbb{R}^{n \times 1}$ and $f:\ \mathbb{R}^n \mapsto \mathbb{R}^n$ is twice continuously differentiable. We further assume that for $u_i = 0$ the system (1) has at least one unstable equilibrium of focus type, which we denote by x^* in what follows. In several parts of the paper the nonlinear oscillators are specified as Lorenz systems,

$$\left\{ \begin{array}{rcl} \dot{x}_{i,1}(t) &=& \sigma(x_{i,2}(t) - x_{i,1}(t)) + u_i(t) \\ \dot{x}_{i,2}(t) &=& r x_{i,1}(t) - x_{i,2}(t) - x_{i,1}(t) x_{i,3}(t) \\ \dot{x}_{i,3}(t) &=& -b x_{i,3}(t) + x_{i,1}(t) x_{i,2}(t), \\ y_i(t) &=& x_{i,1}(t), \quad i = 1, \dots p, \end{array} \right. \tag{2}$$

with the parameter values given by

$$\sigma = 10,\ r = 28,\ b = 8/3. \tag{3}$$

Note that for $u_i = 0$ each Lorenz system has three equilibria given by

$$(0,0,0), \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1 \right), \tag{4}$$

the latter two corresponding to unstable foci. Furthermore, with the parameter values (3) it exhibits a chaotic attractor [9].

In order to describe the coupling between the oscillators we define a directed graph

$$\mathcal{G}(\mathcal{V}, \mathcal{E}, G), \tag{5}$$

characterized by the node set $\mathcal{V} = \{1, \dots, p\}$, a set of edges $\mathcal{E}$ where $(i,l) \in \mathcal{E}$ if and only if $\alpha_{i,l} \neq 0$, and a weighted adjacency matrix G with zero diagonal entries and non-diagonal entries equal to $\alpha_{i,l} \geq 0$. Next, we couple the systems (1) by means of the 'control' law

$$u_i(t) = k \left(\sum_{(i,l)\in\mathcal{E}} \alpha_{i,l} \left(y_l(t-\tau) - y_i(t) \right) \right), \quad i = 1, \dots, p, \tag{6}$$

where $k > 0$ represents the 'controller' gain and τ the transmission delay. It is important to point out that we do not assume that G is symmetric.

The aim of the paper is to study the effect of the coupling (6) with k and τ as parameters on the synchronization of the systems (1), and to reveal synchronization mechanisms and conditions. Instrumental to this study a complete characterization of the local stability / instability regions of the synchronized equilibrium $(x^*, \dots, x^*)$ of (1) and (6) in the (k, τ) parameter space is made. Note that achieving stability can be seen as an extension of the use of Pyragas type feedback [8] to stabilize an unstable equilibrium as in [4]. It can also be interpreted as a situation where so-called oscillator death is achieved [1]. Beyond the stability analysis of (synchronized) equilibria, our goal is to gain insights in and reveal explanations for the occurrence of more complex synchronized behavior, by investigating properties of the solutions on the onset of instability and by investigating the generality of the obtained results w.r.t. the network topology and the number of coupled systems.

A motivation and overview of synchronization problems and results can be found in [10], [11].

W. Michiels is with the Department of Computer Science, Katholieke Universiteit Leuven, Belgium, Wim.Michiels@cs.kuleuven.be

H. Nijmeijer is with the Faculty of Mechanical Engineering, Eindhoven University of Technology, The Netherlands, h.nijmeijer@tue.nl

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

In [5], [12] and the references therein constructive conditions for the synchronization of systems with delays in the coupling are presented using Lyapunov type arguments. In [12] a symmetric network of four systems is considered. Synchronization in general networks is addressed in [2] and the references therein, without taking into account delay effects. The latter effects are however crucial in this paper.

Throughout the paper we make the following assumptions:

Assumption 1.1: The graph $\mathcal{G}$ is strongly connected.

Assumption 1.2: The adjacency matrix G satisfies

$$\sum_{l=1}^{p} \alpha_{i,l} = 1, \quad i = 1,\ldots,p.$$

The following results are direct corollaries.

Corollary 1.3: G has a simple eigenvalue equal to one, with corresponding eigenvector $[1 \ \cdots \ 1]^T$.

Corollary 1.4: All eigenvalues of G have modulus smaller or equal than one.

In what follow we denote the eigenvalues of G as $\lambda_i(G),\ 1 \leq i \leq p$, where we take the convention

$$\lambda_1(G) = 1.$$

We use the notation $\sigma(\cdot)$ for the spectrum and denote with $\Re(\lambda)$ and $\Im(\lambda)$ the real and imaginary part of a complex number λ.

II. A COORDINATE TRANSFORMATION

We discuss a coordinate transformation, which isolates the dynamics on the synchronization manifold. The remaining "error" dynamics then determines whether synchronized behavior is stable.

Define the matrix $\tilde{G} \in \mathbb{R}^{(p-1)\times(p-1)}$:

$$\tilde{G} = \begin{bmatrix} 0 & \alpha_{2,3} & \alpha_{2,4} & \cdots & \alpha_{2,p} \\ \alpha_{3,2} & 0 & \alpha_{3,4} & & \alpha_{3,p} \\ & & & \ddots & \\ \alpha_{p,2} & \alpha_{p,3} & \cdots & \alpha_{p,p-1} & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,p} \end{bmatrix}. \tag{7}$$

It satisfies:

Property 2.1: $\sigma(\tilde{G}) = \sigma(G) \setminus \{1\}$

By means of the new variables

$$\left\{ \begin{array}{rcl} e_2(t) & = & x_2(t) - x_1(t) \\ & \vdots & \\ e_p(t) & = & x_p(t) - x_1(t) \end{array} \right. ,$$

and using (7) we can bring (1)-(6) in the form (8), shown at the top the next page. From this equation it can be seen that a synchronized solution, characterized by $e_2 \equiv 0, \ldots, e_p \equiv 0$, can only exist in three cases:

1) the delay is equal to zero;
2) the overall motion is τ-periodic;
3) $\sum_{l=1}^{p} \alpha_{i,l} = \sum_{l=1}^{p} \alpha_{k,l},\ \forall i,k \in \{1,\ldots,p\}$. Since G can always be scaled (and the factor absorbed in the gain k) this corresponds to Assumption 1.2.

Because we are primarily interested in explaining synchronized chaotic behavior in the presence of delays in the coupling, we can take Assumption 1.2 without loosing generality and the equations (8) simplify to:

$$\begin{aligned} \dot{x}_1 &= f(x_1(t)) + kBC^T \cdot \\ &(x_1(t-\tau) - x_1(t)) + kBC^T \sum_{l=1}^{p} \alpha_{1,l}\ e_l(t-\tau), \end{aligned} \tag{9}$$

$$\begin{bmatrix} \dot{e}_2 \\ \vdots \\ \dot{e}_p \end{bmatrix} = \begin{bmatrix} f(x_1+e_2) - f(x_1) - kBC^T e_2 \\ \vdots \\ f(x_1+e_p) - f(x_1) - kBC^T e_p \end{bmatrix} k \ + \tilde{G} \otimes BC^T \begin{bmatrix} e_2(t-\tau) \\ \vdots \\ e_p(t-\tau) \end{bmatrix}. \tag{10}$$

The solutions on the synchronization manifold are characterized by

$$\dot{x}_1(t) = f(x_1(t)) + kBC^T (x_1(t-\tau) - x_1(t)). \tag{11}$$

If all the solutions of (9) and (10) converge to a bounded forward invariant set, then the synchronization between the agents is achieved locally if the linearization of (10),

$$\begin{bmatrix} \dot{e}_2(t) \\ \vdots \\ \dot{e}_p(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial f}{\partial x}(x_1(t)) - kBC^T\right) e_2(t) \\ \vdots \\ \left(\frac{\partial f}{\partial x}(x_t(t)) - kBC^T\right) e_p(t) \end{bmatrix} +k\ \tilde{G} \otimes BC^T \begin{bmatrix} e_2(t-\tau) \\ \vdots \\ e_p(t-\tau) \end{bmatrix}, \tag{12}$$

is uniformly asymptotically stable. In order to simplify the analysis, we let R and I be defined as

$$\begin{aligned} R &= \{i \in \{2,\ldots,p\} :\ \Im(\lambda_i(G)) = 0\}, \\ I &= \{i \in \{2,\ldots,p\} :\ \Im(\lambda_i(G)) > 0\} \end{aligned}$$

and we let T_r be a matrix satisfying

$$T_r^{-1} \tilde{G} T_r = D,$$

where D is a block triangular matrix whose diagonal blocks are given by

$$\{\lambda_i(G) :\ i \in R\} \cup \left\{ \begin{bmatrix} \Re(\lambda_i(G)) & \Im(\lambda_i(G)) \\ -\Im(\lambda_i(G) & \Re(\lambda_i(G))) \end{bmatrix} :\ i \in I \right\},$$

$$\left\{\begin{array}{l} \dot{x}_1(t) = f(x_1(t)) + kBC^T\left(\sum_l \alpha_{1,l}\right)(x_1(t-\tau)-x_1(t)) + kBC^T\sum_l \alpha_{1,l}\ e_l(t-\tau), \\ \left[\begin{array}{c}\dot{e}_2(t)\\ \vdots \\ \dot{e}_p(t)\end{array}\right] = \left[\begin{array}{c} f(x_1+e_2)-f(x_1)\\ \vdots \\ f(x_1+e_p)-f(x_1)\end{array}\right] - k\left(\left[\begin{array}{ccc}\sum_l \alpha_{2,l} & & \\ & \ddots & \\ & & \sum_l \alpha_{p,l}\end{array}\right]\otimes BC^T\right)\left[\begin{array}{c} e_2(t)\\ \vdots \\ e_p(t)\end{array}\right] \\ \quad +k\ \tilde{G}\otimes BC^T\left[\begin{array}{c} e_2(t-\tau)\\ \vdots \\ e_p(t-\tau)\end{array}\right] + \left(k\left[\begin{array}{c}\sum_l \alpha_{1,l}-\sum_l \alpha_{2,l}\\ \sum_l \alpha_{1,l}-\sum_l \alpha_{3,l}\\ \vdots \\ \sum_l \alpha_{1,l}-\sum_l \alpha_{p,l}\end{array}\right]\otimes BC^T\right)(x_1(t)-x_1(t-\tau))\end{array}\right. \tag{8}$$

The matrices T_r and D alway exist by the idendity $\sigma(D) = \sigma(\tilde{G})$ and Property 2.1. Note that this transfomation corresponds to a full triangularization if all eigenvalues of G are real. If G has nonreal eigenvalues then a full triangularization is not performed because it would result in a non-real matrix.

If we apply the state transformation induced by the matrix $(T_r \otimes I)$ to (12), then it follows that its zero solution is uniformly asymptotically stable if the following systems are uniformly asymptotically stable:

$$\begin{array}{r}\dot{\xi}_i = \left(\frac{\partial f}{\partial x}(x_1(t)) - kBC^T\right)\xi_i(t) + \\ k\lambda_i(G)BC^T\xi_i(t-\tau),\end{array} \tag{13}$$

for all $i \in I$ and

$$\left[\begin{array}{c}\dot{\xi}_i\\ \dot{\eta}_i\end{array}\right] = I\otimes\left(\frac{\partial f}{\partial x}(x_1(t)) - kBC^T\right)\left[\begin{array}{c}\xi_i\\ \eta_i\end{array}\right] + k \\ \left[\begin{array}{cc}\Re(\lambda_i(G)) & \Im(\lambda_i(G))\\ -\Im(\lambda_i(G)) & \Re(\lambda_i G))\end{array}\right]\otimes BC^T\left[\begin{array}{c}\xi_i(t-\tau)\\ \eta_i(t-\tau)\end{array}\right], \tag{14}$$

for all $i \in J$.

Remark 2.2: The analysis of networks using the master stability function [7], [3] is based on a similar decomposition of the error dynamics. For $\tau = 0$, this function maps $z \in \mathbb{C},\ \Re(z) \leq 0$ to the largest Lyapunov exponent of

$$\dot{\xi}_i = \frac{\partial f}{\partial x}(x_1(t))\xi_i(t) + zBC^T\xi_i(t).$$

An important difference w.r.t. the undelayed case considered in the literature is that the solutions on the synchronization manifold, governed by (11), depend on k and τ.

III. STABILITY OF SYNCHRONIZED EQUILIBRIA

When we linearize the system (1)-(6) around the synchronized equilibrium[1] $(x^*, \cdots x^*)$, we obtain

$$\left[\begin{array}{c}\dot{\nu}_1(t)\\ \vdots \\ \dot{\nu}_p(t)\end{array}\right] = I\otimes(A - kBC^T)\left[\begin{array}{c}\nu_1(t)\\ \vdots \\ \nu_p(t)\end{array}\right] + \\ kG\otimes BC^T\left[\begin{array}{c}\nu_1(t-\tau)\\ \vdots \\ \nu_p(t-\tau)\end{array}\right], \tag{15}$$

where $A = \frac{\partial f}{\partial x}(x^*)$.

A. The characteristic equation

a) Factorization : The characteristic function of (15) is given by

$$f(\lambda;\ k,\tau) := \det F(\lambda;\ k,\tau),$$

where the characteristic matrix F is defined as

$$F(\lambda;\ k,\tau) = I\otimes(\lambda I - A + kBC^T) - G\otimes kBC^T e^{-\lambda\tau}. \tag{16}$$

If we factorize $G = T\Lambda T_c^{-1}$, where $\Lambda \in \mathbb{C}^{p\times p}$ is triangular and $T_c \in \mathbb{C}^{p\times p}$, the characteristic function becomes

$$\begin{array}{l} f(\lambda;\ k,\tau) \\ = \left|I\otimes(\lambda I - A + kBC_c^T) - T_c\Lambda T_c^{-1}\otimes kBC_c^T e^{-\lambda\tau}\right| \\ = \left|T_c^{-1}\otimes I\right|\ \left|I\otimes(\lambda I - A + kBC^T)\right. \\ \quad \left. -T_c\Lambda T_c^{-1}\otimes kBC^T e^{-\lambda\tau}\right|\ \left|T_c\otimes I)\right| \\ = \left|I\otimes\left(\lambda I - A + kBC^T - kBC^T\lambda_i(G)e^{-\lambda\tau}\right)\right| \\ = \Pi_{i=1}^p f_i(\lambda;\ k,\tau), \end{array} \tag{17}$$

where

$$\begin{array}{l} f_i(\lambda;\ k,\tau) := \det F_i(\lambda;\ k\tau), \\ F_i(\lambda;\ k,\tau) := \lambda I - A + kBC^T - kBC^T\lambda_i(G)e^{-\lambda\tau}, \end{array}$$

for $i = 1,\ldots,p$.

Remark 3.1: This factorization of the characteristic function can also be obtained from the factorization of (9) and (10) into (9) and (13)-(14) in Section II if one takes into account that $x_1(t) \equiv x^*$ and further factorizes the characteristic

[1] Although the coupled system may have other equilibria (this is for instance the case for Lorenz systems), we restrict in this analysis to the synchronized equilibria as we are in the first place interested in studying synchronization phenomena.

function of (14). It follows from this observation that the zeros of

$$f_1(\lambda;\ k,\tau) = \det(\lambda I - A + kBC^T - kBC^T e^{-\lambda\tau})$$

describe the dynamics of the linearization of the "nominal" system (11), while the zeros of $f_2(\lambda;\ k,\tau)$,..., $f_p(\lambda;\ k,\tau)$ describe the behavior of the synchronization error dynamics.

b) Nullspaces and behavior on the onset of instability: We consider the null spaces of (16) corresponding to a characteristic root. For reasons of simplicity we restrict ourselves to the generic case where all the eigenvalues of G are simple. Let E_i be the eigenvector of G corresponding to the eigenvalue $\lambda_i(G),\ i = 1,\ldots,p$. By Corollary 1.3, we have $E_1 = [1\ 1\ \cdots\ 1]^T$.

If for some $l \in \{1,\ldots,p\}$, the equation

$$f_l(\lambda;\ k,\tau) = 0$$

has a zero at $\lambda = \hat{\lambda}$ with multiplicity m such that

$$F_l(\hat{\lambda};\ k,\tau)\ V = VJ, \quad V \in \mathrm{C}^{n\times m},\ J \in \mathrm{C}^{m\times m}, \tag{18}$$

where J a Jordan matrix with zero diagonal elements, i.e. a block diagonal matrix whose diagonal blocks have the form

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix},$$

then it can be shown by inspection that

$$F(\hat{\lambda};\ k,\tau)\ (E_l \otimes V) = (E_l \otimes V)J. \tag{19}$$

This implies that if we decompose $V = \mathrm{diag}(V_1,\ldots,V_r)$ with $V_i \in \mathbb{R}^{\gamma_i\times\gamma_i}$ a Jordan block for $1 \le i \le r$, then the corresponding solution of (15) can be written in the form

$$\begin{bmatrix} \nu_1(t) \\ \vdots \\ \nu_p(t) \end{bmatrix} = E_l \otimes VR(t)e^{\hat{\lambda}t} = \begin{bmatrix} e_{l,1}VR(t) \\ \vdots \\ e_{l,p}VR(t) \end{bmatrix} e^{\hat{\lambda}t}, \tag{20}$$

where $E_l = [e_{l,1}\cdots e_{l,p}]^T$ and

$$R(t) = \left[\begin{array}{cccc|c} c_{1,1} & c_{1,2}t & \cdots & c_{1,\gamma_1}t^{\gamma_1-1} & \cdots \end{array}\right. \\ \left.\begin{array}{c|cccc} & c_{r,1} & c_{r,2}t & \cdots & c_{r,\gamma_r}t^{\gamma_r-1} \end{array}\right]^T,$$

with the constants $c_{i,j},\ 1 \le i \le r,\ 1 \le j \le \gamma_i$ depending on the initial conditions.

From (19) and (20) the following can be concluded:

- for a mode corresponding to a zero of $f_l(\lambda;\ k,\tau)$, the relation between the different components of the state of an individual subsystem is determined by the (generalized) null space of F_l, while the relation between the corresponding states of the different subsystems is *solely* determined by the eigenvector E_l corresponding to the l-th eigenvalue of the adjacency matrix G. This implies that all modes can classified in at most p types, based on the relations between the behavior of different subsystem. By the argument spelled out in Remark 3.1 the type induced by E_1 corresponds to the nominal behavior of one subsystem, while the types induced by $E_2,\ldots,E_p$, correspond to the synchronization error dynamics.
- the modes induced by the zeros of $f_1(\lambda;\ k,\tau)$ all correspond to synchronized behavior of the different subsystems because $E_1 = [1\ \cdots\ 1]^T$. By Corollary 1.3, the occurrence of these modes is *independent* of the topology of the graph (recall that we adopted the convention $\lambda_1(G) = 1$).

B. Computation of stability regions in the delay parameter

Let $Q(\lambda;\ k)$ and $P(\lambda;\ k)$ be coprime polynomials satisfying

$$\frac{P(\lambda;\ k)}{Q(\lambda;\ k)} = kC^T(\lambda I - A + kBC^T)^{-1}B.$$

For a fixed value of k the following two propositions allow to compute stability / instability regions of (15) in the delay parameter space:

Proposition 3.2: For every $i \in \{1,\ldots,p\}$ we have

$$f_i(0;\ k,0) = 0 \Leftrightarrow f_i(0;\ k,\tau) = 0, \forall\tau \ge 0.$$

Proposition 3.3: The equation

$$f_i(\lambda;\ k,\tau) = 0, \quad i \in \{1,\ldots,p\}, \tag{21}$$

has a root $j\omega,\ \omega > 0$, for some value of τ if and only if

$$h_i(\omega;\ k) = 0, \tag{22}$$

where

$$h_i(\omega;\ k) = |Q(j\omega;\ k)|^2 - |P(j\omega;\ k)|^2\ |\lambda_i(G)|^2. \tag{23}$$

Furthermore, for any ω satisfying (22) the set of corresponding delay values is given by[2]

$$\mathcal{T}_\omega^{(i)} = \left\{\frac{1}{\omega}\left[\angle\left(\frac{P(j\omega)\lambda_i(G)}{Q(j\omega)}\right) + 2\pi l\right],\ l = 0,1,\ldots\right\}. \tag{24}$$

If $h_i'(\omega;\ k) > 0\ (< 0)$, then increasing the delay leads to a root crossing the imaginary axis towards instability (stability).

Proof. The proposition is a generalization of Theorem 11.9 of [6]. □

[2] We adopt the convention $\angle(\cdot) \in [0, 2\pi)$.

C. Asymptotic behavior of coupled Lorenz systems for large gain values

We look at the stability regions in the (k, τ) parameter space for large values of the gain k, for the specific case where the subsystems are Lorenz systems (2), linearized around one of the nontrivial equilibria (4). For the standard parameters (3) this allows us to make assertions about stability regions, stability switches and emerging behavior, which do *not* depend on the network topology, as we shall see.

If we linearize (2) and (6) around the synchronized equilibrium

$$\begin{array}{l}(x^*, \ldots, x^*),\\ x^* = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1),\end{array} \quad (25)$$

then we obtain (15), where

$$\begin{array}{l} A = \left[\begin{array}{ccc} -\sigma & \sigma & 0 \\ r-(r-1) & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{array}\right], \\ B^T = C^T = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]. \end{array} \quad (26)$$

It is easy to show that the stability of (15) does not depend on which equilibrium x^* in (25) is considered and we will choose the one in the positive octant in what follows.

Lemma 3.4: For large values of k the zeros of the functions $f_i(\lambda;\ k, 0),\ i = 2, \ldots, p$, are in the open left half plane. Furthermore, the system (15) with $\tau = 0$ has exactly two characteristic roots in the closed right half plane, which are equal to the unstable eigenvalues of A.

Lemma 3.5: Assume that $|\lambda_i| < 1$. Then for large values of k the zeros of the function $f_i(\lambda;\ k, \tau)$ are in the open left half plane for all values of the delay parameter.

The next result is a refinement of Proposition 3.3 for coupled Lorenz systems:

Lemma 3.6: Assume that $|\lambda_i| = 1$. The equation

$$f_i(\lambda;\ k, \tau) = 0 \quad (27)$$

has a root $j\omega,\ \omega > 0$, for some value of τ if and only if

$$-\frac{1}{2k} = \Re(T(\omega)) \quad (28)$$

where

$$T(\omega) = \frac{-\omega^2+(b+1)j\omega+br}{-j\omega^3-(b+1+\sigma)\omega^2+(br+b\sigma)j\omega+2\sigma br-2\sigma b}. \quad (29)$$

Furthermore, for any ω satisfying (28) the set of corresponding delay values is given by

$$\mathcal{T}_\omega^{(i)} = \left\{ \frac{1}{\omega}\left[\angle\left(\frac{\lambda_i(G)}{1+(kT(\omega))^{-1}}\right) + 2\pi l\right],\ l = 0, 1, \ldots \right\}. \quad (30)$$

Let us now apply the above results taking into account the parameter values (3). By Lemmas 3.4-3.5 the functions $f_i(\lambda,\ k, \tau)$ where $|\lambda_i(G)| < 1$ have their zeros in the left half plane for all values of τ if k is sufficiently large. So all stability switches are due to the functions $f_i(\lambda;\ k, \tau)$ such that $|\lambda_i| = 1$ and can be computed from Lemma 3.6. In Figure 1 the Nyquist plot of the function (29) is shown. For large values of k there are two distinct values of ω satisfying (28), which we refer to as $\omega_1(k) > \omega_2(k)$. We have

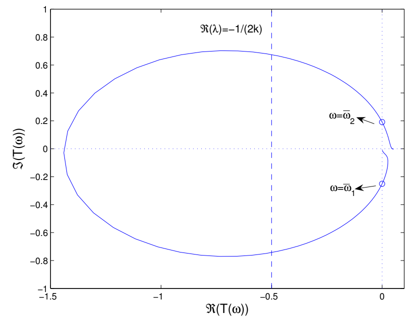

Fig. 1. The Nyquist plot of the function (29) for the parameter values (3).

$$\begin{array}{l}\bar{\omega}_1 := \lim_{k\to\infty} \omega_1(k) = 10.8148,\\ \bar{\omega}_2 := \lim_{k\to\infty} \omega_2(k) = 9.5879,\end{array} \quad (31)$$

$$T(\bar{\omega}_1) = -0.2503j, \quad T(\bar{\omega}_2) = 0.1914j. \quad (32)$$

From (30) and (31)-(32) it follows that for sufficiently large k the *first* stability switch is given by

$$\tau^*(k) := \frac{1}{\omega_2(k)} \angle\left(\frac{1}{1+\frac{1}{kT(\omega_2(k))}}\right) = \frac{\alpha(k)}{k},$$

where

$$\alpha(k) = \frac{k}{\omega_2(k)} \mathrm{atan}\left(\frac{1}{k\Im(T(\omega_2(k)))}\right).$$

Furthermore, by the crossing direction characterization of Proposition 3.3 this switch is towards stability (since $\omega_2 < \omega_1$) and results in an asymptotically stabile systems by Lemma 3.4. As this switch is due to a zero of $f_1(\lambda;\ k, \tau)$ it is independent of the network topology and the emanating solutions have the form

$$\left[\begin{array}{ccc} y_1(t) & \cdots & y_p(t)\end{array}\right] = \left[\begin{array}{ccc} V & \cdots & V \end{array}\right] e^{j\omega t},$$

where $F_1(j\omega;\ k, \tau)V = 0$, i.e. synchronization is preserved in the emanating solutions. Given that

$$\lim_{k\to\infty} \alpha(k) = (\bar{\omega}_2 \Im(T(\bar{\omega}_2)))^{-1} = 5.4481$$

the obtained results can be summarized as follows:

Theorem 3.7: Consider a network of coupled Lorenz systems with parameters (3).
There exists a number $\hat{k} > 0$ and a function

$$\tau^* : \ [\hat{k},\ \infty] \to \mathbb{R}_+,\ k \mapsto \tau^*(k), \quad (33)$$

satisfying the following properties:

1) there is a constant $\tilde{k} > \hat{k}$ such that for every $k > \tilde{k}$, the synchronized equilibrium has two characteristic roots in the open right half plane for $\tau \in [0,\ \tau^*]$, while it is asymptotically stable for $\tau \in (\tau^*,\ \tau^* + \epsilon)$ with ϵ sufficiently small;
2) at $\tau = \tau^*$ a synchronization preserving Hopf bifurcation occurs;
3) for all $k \in [\hat{k},\ \infty]$ we can factor

$$\tau^*(k) = \frac{\alpha(k)}{k},$$

where

$$\lim_{k\to\infty} \alpha(k) = 5.4481.$$

Furthermore, the number $\hat{k}$ and the function (33) are independent of the number of subsystems and of the network topology.

IV. EXAMPLES

Ring topology, unidirectional coupling: We consider a ring topology with unidirectional coupling, described by the adjacency matrix

$$G = \begin{bmatrix} 0 & \cdots & & 0 & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{p\times p}, \quad (34)$$

which has the following properties:

$$\lambda_l(G) = e^{-j\frac{2\pi(l-1)}{p}}, \quad E_l = \begin{bmatrix} 1 \\ e^{-j\frac{2\pi(l-1)}{p}} \\ \vdots \\ e^{-j\frac{2\pi(p-1)(l-1)}{p}} \end{bmatrix}$$

for $l = 1, \ldots, p$. If (18) is satisfied for $\hat{\lambda} = j\omega,\ \omega > 0$, then the emanating solution (20) becomes

$$\begin{bmatrix} \nu_1(t) \\ \vdots \\ \nu_p(t) \end{bmatrix} = \begin{bmatrix} VR(t)\ e^{j\omega t} \\ VR(t)\ e^{j\omega t - \frac{2\pi(l-1)}{p}} \\ \vdots \\ VR(t)\ e^{j\omega t - \frac{2(p-1)\pi(l-1)}{p}} \end{bmatrix}. \quad (35)$$

It can be interpreted as a traveling wave solution, where the agents follow each with a phase shift of $360(l-1)/p$ degrees. Therefore, if $V \in \mathbb{C}^{p\times 1}$ and the characteristic root on the imaginary axis corresponds to a Hopf bifurcation of (1) and (6) for a critical value of some parameter, we refer to this bifurcation as a "Hopf $360(l-1)/p$" bifurcation.

With the individual agents taken as Lorenz systems (2) with parameters (3) and with $p = 4$ and $p = 12$ we have used Propositions 3.2-3.3 to compute stability regions in the delay parameter space of the synchronized equilibria (25). The results are displayed in Figure 2. The Hopf 0

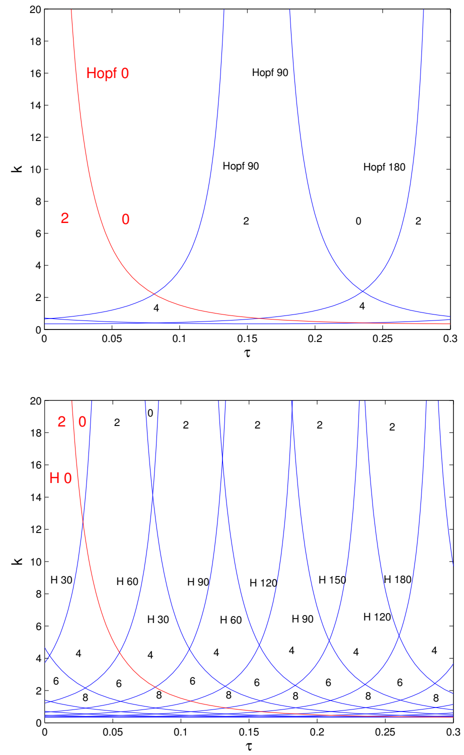

Fig. 2. Stability regions of the synchronized equilibrium (25) of Lorenz systems (2)-(3) coupled in a ring configuration described by (34), for $p = 4$ (above) and $p = 8$ (below). The numbers refer to the number of characteristic roots in the closed right half plane.

bifurcation curves are independent of the number of subsystems, because they are induced by the zeros of $f_1(\lambda;\ k, \tau)$. The first one corresponds to the function (33). By Theorem 3.7 the quantities indicated in red on the figure are independent of the number of agents and of the network topology.

Ring topology, bidirectional coupling: A ring topology with bidirectional coupling between the agents is described by the matrix

$$G = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 1 & & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{p\times p}, \quad (36)$$

satisfying

$$\lambda_l(G) = \cos\left(\frac{2\pi}{p}(l-1)\right),\ l = 1, \ldots, q,$$

where $q = (p+2)/2$ is p is even and $q = (p+1)/2$ if p is odd. All eigenvalues have multiplicity two, excepting $\lambda_1(G) = 1$ and, if p is even, $\lambda_{\frac{p+2}{2}}(G)$. The corresponding eigenvectors are

$$\begin{bmatrix} \cos\left(\frac{2\pi(l-1).(p-1)}{p}\right) & \cdots & \cos\left(\frac{2\pi(l-1).1}{p}\right) & 1 \end{bmatrix}^T$$

and

$$\left[\begin{array}{cccc} \sin\left(\frac{2\pi(l-1).(p-1)}{p}\right) & \cdots & \sin\left(\frac{2\pi(l-1).1}{p}\right) & 0 \end{array}\right]^T.$$

Note that if all subsystems are Lorenz systems described by (2)-(3) then for large values of k the stability switches are only associated with the eigenvalues ± 1 and corresponding eigenvectors $[1\ \pm 1\ 1\ \pm 1\ ]^T$ (see Lemma 3.5) and result in either synchronized motion or standing waves. This is due to the bidirectional coupling and in contrast to the case of unidirectional coupling addressed above, where traveling wave solutions naturally appear.

Finally, in all the examples synchronized chaotic behavior was observed for large k and with $\tau < \tau^*(k)$ sufficiently small. This is totally in agreement with the type of bifurcation at $\tau = \tau^*$.

V. Discussion and concluding remarks

We studied the synchronization of coupled nonlinear oscillators with delay in the coupling, (1) and (6), with the emphasis on coupled Lorenz systems. First, the state transformation to (8) led us to necessary conditions on the network topology for the existence of synchronized solutions.

Next we performed a stability analysis of synchronized equilibria in a (gain, delay) parameter space. Instrumental to this study we employed a factorization of the characteristic equation separating the nominal behavior and the synchronization error dynamics, and we revealed the precise role of the eigenvalues and the eigenvectors of the adjacency matrix of the graph on the behavior of the solutions. The latter allowed us to classify the modes of the system, as well as the Hopf bifurcation curves and the emerging behavior on the onset of instability. As a result of this analysis for the case of coupled Lorenz systems we proved that for sufficiently large gain values, there always exists a stability interval in the delay parameter space that does not contain the zero delay value. Furthermore, this behavior is *generic* because both the critical delay value, $\tau^*(k)$, and the type of corresponding bifurcation (a synchronization preserving Hopf bifurcation, in the sense that if the delay is reduced beyond the critical value the equilibrium becomes locally unstable *without loosing* the synchronization) do *not* depend on the network topology and the number of agents.

The presence of the synchronization preserving Hopf bifurcation at $\tau = \tau^*(k)$, the fact that for large values of k the functions $f_i(\lambda;\ k, \tau)$, $2 \leq i \leq p$, that describe the synchronization error around the equilibrium, have all zeros in the open left half plane for all $\tau \in [0,\ \tau^*(k) + \epsilon(k))$ with $\epsilon(k) > 0$, and the observed synchronized behavior in our experiments for *all* $\tau \in [0,\ \tau^*(k)]$ (chaotic for sufficiently small delay values), suggest that for every $\tau \in [0,\ \tau^*]$ all solutions are attracted to a bounded forward invariant set and that for $x_1(t)$ residing within this set the 'time-varying' systems (13)-(14) remain uniformly asymptotically stable. Lemma 3.5 further suggest that the stability of (13)-(14) is delay-independent if $|\lambda_i(G)| < 1$.

If the agents are not completely identical, then in general (perfectly) synchronized solutions do not exist (this can be seen from (8) where terms related to the deviations would appear in the right-hand side). Though the analysis in the paper has been performed step-by-step using a particular decomposition or factorization, holding for identical agents only, the final results for the coupled system (presence of a synchronized steady state solution, its stability regions and Hopf bifurcation curves in the (k, τ) plane, the structure of the eigenfunctions corresponding to the Hopf bifurcations) will be slightly perturbed only if the differences between the agents are sufficiently small. This means that Theorem 3.7 remains approximately valid in that for large k and particular values of τ there exists an almost synchronized equilibrium, which is stable but looses stability beyond $\tau \approx \tau^*$, while maintaining the solutions close to being synchronized. It could be an indication that for τ sufficiently small, the synchronization error dynamics exhibits to an attractor whose size can be made arbitrarily small by reducing the difference between the agents.

References

[1] F.M. Atay. Oscillator death in coupled functional differential equations near hopf bifurcation. *Journal of Differential Equations*, 221:190–209, 2006.

[2] I. Belykh, M. Hasler, M. Lauret, and H. Nijmeijer. Synchronization and graph topology. *International Journal of Bifurcation and Chaos*, 15(11):3423–3433, 2005.

[3] M. Chavez, D.-U. Hwang, A. Amann, H.G.E. Hentschel, and S. Boccaletti1. Synchronization is enhanced in weighted complex networks. *PRL*, 94(218701), 2005.

[4] P. Hövel and E. Schöll. Control of unstable steady states by time-delayed feedback methods. *Physical Review E*, 72:046203, 2005.

[5] H. Huijberts, H. Nijmeijer, and T. Oguchi. Anticipating synchronization of chaotic Lur'e systems. *Chaos*, 17(013117), 2007.

[6] W. Michiels and S.-I. Niculescu. *Stability and stabilization of time-delay systems. An eigenvalue based approach*. SIAM, 2007.

[7] L.M. Pecora and T.L. Carroll. Master stability function for synchronized coupled systems. *PRL*, 80(2109), 1998.

[8] K. Pyragas. Continuous control of chaos by self-controlling feedback. *Physics Letters A*, 170:421–428, 1992.

[9] S. Strogatz. *Nonlinear dynamics and chaos: with applications to biology, chemistry and engineering*. Perseus Books Publishing, 1994.

[10] S. Strogatz. *Sync: The emerging science of spontaneous order*. Hyperion: New York, 2003.

[11] C.W. Wu. *Synchronization in coupled chaotic circuits and systems*. World Scientific: Singapore, 2002.

[12] T. Yamamoto, T. Oguchi, and H. Nijmeijer. Synchronization of delay-coupled nonlinear oscillators with symmetric structure. In *Proceedings of the 6th annual Conference on Control Systems*, Nagoya, Japan, 2007.