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Controller synthesis for $\mathcal{L}_2$ behaviors using rational kernel representations

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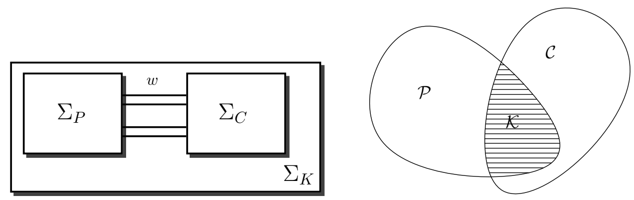
Abstract— **This paper considers the controller synthesis problem for the class of linear time-invariant $\mathcal{L}_2$ behaviors. We introduce classes of LTI $\mathcal{L}_2$ systems whose behavior can be represented as the kernel of a rational operator. Given a plant and a controlled system in this class, an algorithm is developed that produces a rational kernel representation of a controller that, when interconnected with the plant, realizes the controlled system. This result generalizes similar synthesis algorithms in the behavioral framework for infinitely smooth behaviors that allow representations as kernels of polynomial differential operators.**



(a) Full interconnection. (b) $\mathcal{P} \cap \mathcal{C} = \mathcal{K}$.

Fig. 1. Interconnection problems.

I. INTRODUCTION

The analysis of system interconnections is at the heart of many problems in modeling, simulation and control. Indeed, when focusing on control, the controller synthesis question amounts to finding a dynamical system (a controller) that, after interconnection with a given plant, results in a controlled system that is supposed to perform a certain task in a more desirable manner than the plant. Usually the control synthesis problem is formulated as a feedback optimization problem in which the plant and controller interact through a number of distinguished channels that have been divided in input- and output variables.

The behavioral theory of dynamical systems has been advocated as a conceptual framework in which especially interconnection structures of dynamical system can be studied in an input-output independent setting. There are many conceptual, pedagogic and practical reasons for doing so and we refer to [9], [10] for a detailed account on this matter.

One key problem concerning the interconnection of dynamical systems involves the question when a given dynamical system Σ_K can be implemented (or realized) as the interconnection of a dynamical system Σ_P, that is supposed to be given, and a second dynamical system Σ_C, that is supposed to be designed. With the interpretation that Σ_P and Σ_K denote the plant- and (desired) controlled system, this question is therefore equivalent to a synthesis question for the controller Σ_C.

Within the behavioral framework this question received a very complete and elegant answer for the class of linear time-invariant systems that admit representations in terms of polynomial difference or polynomial differential operators [5], [6]. A rather complete theory has been developed for such representations that covers, among other things, H_∞, LQ and H_2 optimal control.

It is the purpose of this paper to reconsider the controller synthesis question for specific classes of linear and time-invariant $\mathcal{L}_2$ systems that admit representations in terms of *rational functions*. In doing so, we depart from the setting proposed in [11] of considering infinitely smooth trajectories as solutions of "rational" differential equations. Instead, we view rational functions in $\mathcal{H}_\infty$ as multiplicative operators on $\mathcal{L}_2$ functions and define $\mathcal{L}_2$ systems through the kernel of such operator. In this way, rational functions naturally define dynamical systems in the frequency domain and offer distinct algebraic advantages over polynomial kernel representations.

The paper is organized as follows. Section II contains the formulation of the main problem that is discussed in this paper. In Section III some notational remarks about spaces and operators are introduced. Sections IV and V contain the introduction of $\mathcal{L}_2$ behaviors, the interconnection problem and a novel controller synthesis algorithm. An example using this synthesis algorithm is given in Section VI. In the last section of this paper, the results of this paper are discussed and some recommendations for further research on $\mathcal{L}_2$ systems are given.

II. PROBLEM FORMULATION

Following the behavioral formalism, a dynamical system [1] is described by a triple:

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B}), \tag{1}$$

where $\mathbb{T} \subseteq \mathbb{R}$ or $\mathbb{T} \subseteq \mathbb{C}$ is the time- or frequency-axis, $\mathbb{W}$ is the variable signal space, which typically contains inputs and outputs and will be taken to be a finite dimensional vector space throughout, and $\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the behavior, that is defined in more explicit terms in Section IV.

Using (1) it is possible to describe plants, controllers and desired controlled systems (as Σ_P, Σ_C and Σ_K respectively). Fig. 1 illustrates the interconnection Σ_K of two systems $\Sigma_P = (\mathbb{T}, \mathbb{W}, \mathcal{P})$ and $\Sigma_C = (\mathbb{T}, \mathbb{W}, \mathcal{C})$. It is defined as $\Sigma_K = (\mathbb{T}, \mathbb{W}, \mathcal{P} \cap \mathcal{K})$ and motivated by the idea that the

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behavior of the interconnection satisfies the laws of both Σ_P and Σ_C.

Fig. 1(a) gives an illustration of the problem treated in this paper, namely given a plant Σ_P and a desired controlled system Σ_K, construct, if it exists, a controller Σ_C that after interconnection with the plant results in the desired controlled system.

We address this problem for very specific classes of $\mathcal{L}_2$ systems. More specifically, we address the problems of existence, (non-) uniqueness of controllers, together with the problem to parametrize all controllers that establish a desired controlled system after interconnection.

As mentioned in the introduction, earlier research, using infinitely smooth trajectories, has been carried out for this problem [6], [9], [10]. This paper contributes to the controller synthesis question by considering various $\mathcal{L}_2$ systems, represented through rational operators.

III. Notation

A. Hardy spaces

Hardy spaces are denoted by $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$, where $p = 1, 2, \ldots, \infty$, and defined by:

$$\mathcal{H}_p^+ := \{f : \mathbb{C}^+ \to \mathbb{C}^{\mathtt{q}} \mid \|f\|_{\mathcal{H}_p^+} < \infty\},$$
$$\mathcal{H}_p^- := \{f : \mathbb{C}^- \to \mathbb{C}^{\mathtt{q}} \mid \|f\|_{\mathcal{H}_p^-} < \infty\},$$

where $\mathbb{C}^+ := \mathrm{Re}\{s\} > 0$ and $\mathbb{C}^- := \mathrm{Re}\{s\} < 0$, with $s = \sigma + j\omega$. So, functions in $\mathcal{H}_p^+$ are analytic[1] in $\mathbb{C}^+ \cup \{\infty\}$ and functions in $\mathcal{H}_p^-$ are analytic in $\mathbb{C}^- \cup \{-\infty\}$. The $\mathcal{H}_p^+$ spaces are the classical Hardy spaces [4].

The norms of functions in $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$ are defined as:

$$\|f\|_{\mathcal{H}_p^+} = \begin{cases} \lim\limits_{\sigma \downarrow 0} \left(\int\limits_{-\infty}^{\infty} |f(\sigma + j\omega)|^p \mathrm{d}\omega \right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \lim\limits_{\sigma \downarrow 0} \sup\limits_{\omega \in \mathbb{R}} |f(\sigma + j\omega)|, & p = \infty, \end{cases}$$

and

$$\|f\|_{\mathcal{H}_p^-} = \begin{cases} \lim\limits_{\sigma \uparrow 0} \left(\int\limits_{-\infty}^{\infty} |f(\sigma + j\omega)|^p \mathrm{d}\omega \right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \lim\limits_{\sigma \uparrow 0} \sup\limits_{\omega \in \mathbb{R}} |f(\sigma + j\omega)|, & p = \infty. \end{cases}$$

It is remarked that the tangential limits $\sigma \to 0$ in the above expressions exist, which makes the Hardy spaces well defined normed spaces, cf. [4].

B. Rational functions and Units

The prefixes $\mathcal{R}$ and $\mathcal{U}$ denote, respectively, rational functions and units in the Hardy spaces $\mathcal{H}_p^+$ and $\mathcal{H}_p^-$ as in

$$\mathcal{RH}_p^+ := \{f \in \mathcal{H}_p^+ \mid f \text{ is rational}\},$$
$$\mathcal{RH}_p^- := \{f \in \mathcal{H}_p^- \mid f \text{ is rational}\},$$

[1] A function is analytic if it is complex differentiable.

and

$$\mathcal{UH}_\infty^+ := \{U \in \mathcal{RH}_\infty^+ \mid U^{-1} \in \mathcal{RH}_\infty^+\},$$
$$\mathcal{UH}_\infty^- := \{U \in \mathcal{RH}_\infty^- \mid U^{-1} \in \mathcal{RH}_\infty^-\}.$$

Note that units are necessarily square rational matrices.

C. Laplace transformation

The Laplace transform $\mathcal{L} : L_2(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) \to \mathcal{L}_2(\mathbb{C}, \mathbb{C}^{\mathtt{q}})$ defines an isometry between the L_2 Hilbert space and the inner product space $\mathcal{L}_2$:

$$\mathcal{L}_2 := \mathcal{H}_2^+ \oplus \mathcal{H}_2^- = \{f : \mathbb{C} \to \mathbb{C}^{\mathtt{q}} \mid \|f\|_2 < \infty\},$$

which inherits the following norm:

$$\|f\|_2^2 = \int\limits_{-\infty}^{\infty} f(j\omega)^H f(j\omega) \mathrm{d}\omega,$$

and the inner product on complex valued functions:

$$\langle f, g \rangle = \int\limits_{-\infty}^{\infty} f(j\omega)^H g(j\omega) \mathrm{d}\omega.$$

Any element $w \in \mathcal{L}_2$ can be uniquely decomposed as $w = w_+ + w_-$, where

$$w_+ := \Pi_+ w, \text{ with } \Pi_+ : \mathcal{L}_2 \to \mathcal{H}_2^+,$$
$$w_- := \Pi_- w, \text{ with } \Pi_- : \mathcal{L}_2 \to \mathcal{H}_2^-.$$

Here, Π_+ and Π_- denote the canonical projections from $\mathcal{L}_2$ onto $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$, respectively.

D. Mappings in $\mathcal{RH}_\infty^+$ and $\mathcal{RH}_\infty^-$

Elements of $\mathcal{RH}_\infty^+$ and $\mathcal{RH}_\infty^-$ (also known as stable- and anti-stable functions of $\mathcal{RH}_\infty^{\text{stable}}$ and $\mathcal{RH}_\infty^{\text{anti-stable}}$) define operators in the following manner. Let $\Theta \in \mathcal{RH}_\infty^+$ and define $\Theta : \mathcal{L}_2 \to \mathcal{L}_2$ by:

$$(\Theta w)(s) := \Theta(s) w(s), \quad \text{where } w \in \mathcal{L}_2,$$

which is the usual "multiplication" or Laurent operator in the frequency domain [4]. Similarly, let $\Psi \in \mathcal{RH}_\infty^-$ and define $\Psi : \mathcal{L}_2 \to \mathcal{L}_2$ by:

$$(\Psi w)(s) := \Psi(s) w(s), \quad \text{where } w \in \mathcal{L}_2.$$

When restricted to the domains $\mathcal{H}_2^+$ or $\mathcal{H}_2^-$, these operators define functions as:

Lemma 3.1: Let $\Theta \in \mathcal{RH}_\infty^+$, with possible domains $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$. Then

$$\Theta : \mathcal{L}_2 \to \mathcal{L}_2, \quad \Theta : \mathcal{H}_2^+ \to \mathcal{H}_2^+, \quad \Theta : \mathcal{H}_2^- \to \mathcal{L}_2.$$

Similarly, let $\Psi \in \mathcal{RH}_\infty^-$, with possible domains $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$. Then

$$\Psi : \mathcal{L}_2 \to \mathcal{L}_2, \quad \Psi : \mathcal{H}_2^+ \to \mathcal{L}_2, \quad \Psi : \mathcal{H}_2^- \to \mathcal{H}_2^-.$$

The proof of this lemma and more details about Hardy spaces can be found in [4].

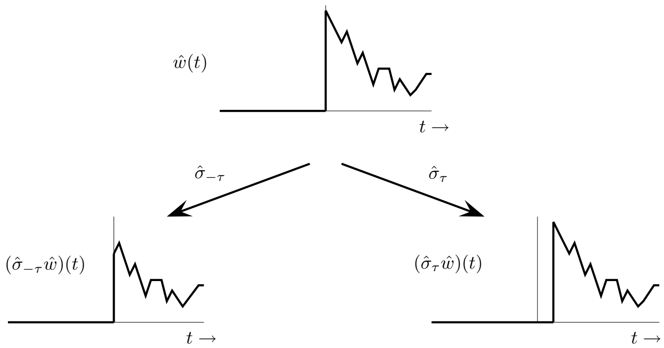

Fig. 2. τ-shift of $\hat{w} = \mathcal{L}^{-1}\{w\}$ with $w \in \mathcal{H}_2^+$.

E. τ-shift operators

We define the τ-shift operator $\hat{\sigma}_\tau$ on a signal $\hat{w} : \mathbb{R} \rightarrow \mathbb{R}$ as:

$$(\hat{\sigma}_\tau \hat{w})(t) = \hat{w}(t - \tau),$$

where $\hat{w}$ is the inverse Laplace transform of w, which is an element of $\mathcal{L}_2$, $\mathcal{H}_2^+$ or $\mathcal{H}_2^-$ (so, $\hat{w} = \mathcal{L}^{-1}\{w\}$).
A τ-shift is called a *left-shift* if $\tau < 0$ (which means that the signal shifts left with respect to the time axis) and is named a *right-shift* if $\tau > 0$ (so, the signal shifts right with respect to the time axis).

For any $\tau \in \mathbb{R}$, we introduce the shift operators $\sigma_\tau : \mathcal{L}_2 \rightarrow \mathcal{L}_2$, $\sigma_\tau : \mathcal{H}_2^+ \rightarrow \mathcal{H}_2^+$ and $\sigma_\tau : \mathcal{H}_2^- \rightarrow \mathcal{H}_2^-$ by defining:

$$(\sigma_\tau w)(s) = \quad \mathrm{e}^{-s\tau} w(s),$$

$$(\sigma_\tau w)(s) = \begin{cases} \mathrm{e}^{-s\tau} w(s), & [\tau > 0] \\ \mathrm{e}^{-s\tau}\left(w(s) - \int\limits_0^{-\tau} \hat{w}(t)\mathrm{e}^{-st}\mathrm{d}t\right), & [\tau < 0] \end{cases}$$

$$(\sigma_\tau w)(s) = \begin{cases} \mathrm{e}^{-s\tau}\left(w(s) - \int\limits_{-\tau}^{0} \hat{w}(t)\mathrm{e}^{-st}\mathrm{d}t\right), & [\tau > 0] \\ \mathrm{e}^{-s\tau} w(s), & [\tau < 0] \end{cases}$$

respectively. Obviously, σ_0 is the identity map. Note that $\sigma_\tau : \mathcal{L}_2 \rightarrow \mathcal{L}_2$ defines an isometry (for all $\tau \in \mathbb{R}$) and that $\sigma_\tau : \mathcal{H}_2^+ \rightarrow \mathcal{H}_2^+$ and $\sigma_\tau : \mathcal{H}_2^- \rightarrow \mathcal{H}_2^-$ define isometries only if $\tau \geq 0$ and $\tau \leq 0$, respectively. When interpreted in the time domain, a left- and right-shift for a signal $w \in \mathcal{H}_2^+$ are illustrated in Fig. 2.

Definition 3.1: A subset $\mathcal{P}$ of $\mathcal{L}_2$ (or $\mathcal{H}_2^+$ or $\mathcal{H}_2^-$) is said to be *left-shift invariant* if $\sigma_\tau \mathcal{P} \subseteq \mathcal{P}$ for all $\tau < 0$.
The set $\mathcal{P}$ is said to be *right-shift invariant* if $\sigma_\tau \mathcal{P} \subseteq \mathcal{P}$ for all $\tau > 0$.

IV. Rational representations of behaviors

In the previous section, stable- and anti-stable rational operators have been introduced on Hilbert spaces. In this section we will associate behaviors as linear shift invariant subsets of $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ defined through the null spaces of these operators. Throughout this section, the variables w are elements of $\mathcal{L}_2$, $\mathcal{H}_2^+$ or $\mathcal{H}_2^-$.

First, behaviors associated with mappings P from the space of rational stable Hardy functions are discussed.
For any $P \in \mathcal{RH}_\infty^+$, the following three dynamical systems are defined:

$$\begin{aligned} \Sigma_P &:= (\mathbb{C}, \mathbb{C}^{\mathtt{q}}, \mathcal{P}(P)), \\ \Sigma_{P,+} &:= (\mathbb{C}, \mathbb{C}^{\mathtt{q}}, \mathcal{P}_+(P)), \\ \Sigma_{P,-} &:= (\mathbb{C}, \mathbb{C}^{\mathtt{q}}, \mathcal{P}_-(P)), \end{aligned} \tag{2a}$$

where

$$\begin{aligned} \mathcal{P}(P) &:= \{w \in \mathcal{L}_2 \mid Pw = 0\} = \ker P \subset \mathcal{L}_2, \\ \mathcal{P}_+(P) &:= \{w \in \mathcal{H}_2^+ \mid Pw = 0\} = \ker P \subset \mathcal{H}_2^+, \\ \mathcal{P}_-(P) &:= \{w \in \mathcal{H}_2^- \mid Pw \in \mathcal{H}_2^+\} = \ker \Pi_- P \subset \mathcal{H}_2^-. \end{aligned} \tag{2b}$$

Here, Π_- is the canonical projection that is introduced before. For these sets we have the following properties:

Lemma 4.1: For $P \in \mathcal{RH}_\infty^+$, the behaviors $\mathcal{P}(P)$, $\mathcal{P}_+(P)$ and $\mathcal{P}_-(P)$ are linear and right-shift invariant subsets of $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$, respectively. A system Σ with either of these behaviors is called an $\mathcal{L}_2$ right-shift invariant system.

Definition 4.1: The classes of all linear and right-shift invariant systems in $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ that admit representations as the kernel of a rational element $P \in \mathcal{RH}_\infty^+$ are denoted by $\mathbb{M}, \mathbb{M}_+$ and $\mathbb{M}_-$.

Similarly, for any $\hat{P} \in \mathcal{RH}_\infty^-$, the following three dynamical systems are introduced as:

$$\begin{aligned} \Sigma_{\hat{P}} &:= (\mathbb{C}, \mathbb{C}^{\mathtt{q}}, \mathcal{P}(\hat{P})), \\ \Sigma_{\hat{P},+} &:= (\mathbb{C}, \mathbb{C}^{\mathtt{q}}, \mathcal{P}_+(\hat{P})), \\ \Sigma_{\hat{P},-} &:= (\mathbb{C}, \mathbb{C}^{\mathtt{q}}, \mathcal{P}_-(\hat{P})), \end{aligned} \tag{3a}$$

where the behaviors are given by:

$$\begin{aligned} \mathcal{P}(\hat{P}) &:= \{w \in \mathcal{L}_2 \mid \hat{P}w = 0\} = \ker \hat{P} \subset \mathcal{L}_2, \\ \mathcal{P}_+(\hat{P}) &:= \{w \in \mathcal{H}_2^+ \mid \hat{P}w \in \mathcal{H}_2^-\} = \ker \Pi_+ \hat{P} \subset \mathcal{H}_2^+, \\ \mathcal{P}_-(\hat{P}) &:= \{w \in \mathcal{H}_2^- \mid \hat{P}w = 0\} = \ker \hat{P} \subset \mathcal{H}_2^-, \end{aligned} \tag{3b}$$

As introduced before, $\Pi_+ : \mathcal{L}_2 \rightarrow \mathcal{H}_2^+$ is the canonical projection.

Lemma 4.2: For $\hat{P} \in \mathcal{RH}_\infty^-$ the behaviors $\mathcal{P}(\hat{P})$, $\mathcal{P}_+(\hat{P})$ and $\mathcal{P}_-(\hat{P})$ are linear and left-shift invariant subsets of $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$, respectively. A system Σ with either of these behaviors is called an $\mathcal{L}_2$ left-shift invariant system.

Definition 4.2: The classes of all linear and left-shift invariant systems in $\mathcal{L}_2$, $\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ that admit representations as the kernel of a rational element $\hat{P} \in \mathcal{RH}_\infty^-$ are denoted by $\mathbb{L}, \mathbb{L}_+$ and $\mathbb{L}_-$.

Now dynamical systems (1) can be described using $\mathcal{L}_2$ behaviors, some properties, using rational elements, are introduced:

Theorem 4.1: Let $P, K \in \mathcal{RH}_\infty^+$ and let $\mathcal{P}_{(\pm)} = \mathcal{P}_{(\pm)}(P)$ and $\mathcal{K}_{(\pm)} = \mathcal{K}_{(\pm)}(K)$ be as defined in (2). Then the following statements are equivalent:

i $\mathcal{K} \subset \mathcal{P}$,
ii $\mathcal{K}_+ \subset \mathcal{P}_+$,
iii $\mathcal{K}_- \subset \mathcal{P}_-$,
iv $\exists F \in \mathcal{RH}_\infty^+$ such that $P = FK$.

Moreover, $\mathcal{K} = \mathcal{P} \iff \mathcal{K}_+ = \mathcal{P}_+ \iff \mathcal{K}_- = \mathcal{P}_- \iff \exists U \in \mathcal{UH}_\infty^+$ such that $P = UK$.

The proof of this theorem can be found in the Appendix. Also anti-stable mappings can be used in the representations, which yields the following theorem:

Theorem 4.2: Let $\hat{P}, \hat{K} \in \mathcal{RH}_\infty^-$ and let $\mathcal{P}_{(\pm)} = \mathcal{P}(\hat{P})$ and $\mathcal{K}_{(\pm)} = \mathcal{K}(\hat{K})$ as in (3). Then the following statements are equivalent:

i $\mathcal{K} \subset \mathcal{P}$,
ii $\mathcal{K}_+ \subset \mathcal{P}_+$,
iii $\mathcal{K}_- \subset \mathcal{P}_-$,
iv $\exists \hat{F} \in \mathcal{RH}_\infty^-$ such that $\hat{P} = \hat{F}\hat{K}$.

Moreover, $\mathcal{K} = \mathcal{P} \iff \mathcal{K}_+ = \mathcal{P}_+ \iff \mathcal{K}_- = \mathcal{P}_- \iff \exists \hat{U} \in \mathcal{UH}_\infty^-$ such that $\hat{P} = \hat{U}\hat{K}$.

The proof of this theorem is similar to the one of Theorem 4.1 and therefore is not included in this paper.

V. Controller Synthesis

A. Full Interconnection problem

For each of the above classes of $\mathcal{L}_2$ systems, the synthesis problem defined in Section II can now be formally stated as follows:

Problem 5.1: Given two linear left-shift invariant systems Σ_P and Σ_K in the class $\mathbb{L}$ (or $\mathbb{L}_+$ or $\mathbb{L}_-$).

i Verify whether there exists $\Sigma_C \in \mathbb{L}$ ($\mathbb{L}_+$ or $\mathbb{L}_-$) such that $\mathcal{P} \cap \mathcal{C} = \mathcal{K}$. Any such system is said to *implement* $\mathcal{K}$ *for* $\mathcal{P}$ by full interconnection through w (Fig. 1(a)).
ii If such controller exists, find a representation $C_0 \in \mathcal{RH}_\infty^-$ of $\Sigma_C = (\mathbb{T}, \mathbb{W}, \mathcal{C})$ in the sense that $\mathcal{C} = \ker C_0$ (or $\mathcal{C} = \ker \Pi_+ C_0$ or $\mathcal{C} = \ker C_0$).
iii Characterize the set $\mathbb{C}_{\text{par}}$ of all $C \in \mathcal{RH}_\infty^-$ for which $\Sigma_C = (\mathbb{T}, \mathbb{W}, \ker C)$ implements $\mathcal{K}$ for $\mathcal{P}$.

A similar problem formulation applies for the model classes $\mathbb{M}$, $\mathbb{M}_+$ and $\mathbb{M}_-$.

Our synthesis algorithm is inspired by the polynomial analog that has been treated in [5], [6] and leads to explicit rational representations of behaviors $\mathcal{C}$ that implement $\mathcal{K}$ for $\mathcal{P}$.

Theorem 5.1: Given the systems $\Sigma_P = (\mathbb{T}, \mathbb{W}, \mathcal{P})$ and $\Sigma_K = (\mathbb{T}, \mathbb{W}, \mathcal{K})$ in the class $\mathbb{L}_{(\pm)}$ (or $\mathbb{M}_{(\pm)}$).

i There exists a controller $\Sigma_C = (\mathbb{T}, \mathbb{W}, \mathcal{C}) \in \mathbb{L}_{(\pm)}$ (or $\mathbb{M}_{(\pm)}$) that implements $\mathcal{K}$ for $\mathcal{P}$ by full interconnection if and only if $\mathcal{K} \subset \mathcal{P}$.
ii Whenever one of the equivalent conditions of item i holds, the set $\mathbb{C}_{\text{par}}$ of all possible kernel representations of controllers that implement $\mathcal{K}$ for $\mathcal{P}$ by full interconnection is given in Step 5 of Algorithm 1 below.

The proof of Theorem 5.1 is inspired by the polynomial analog in [5] and [6] and is given in the next subsection.

B. Algorithm

The following algorithm results in the explicit construction of all controllers Σ_C that solve Problem 5.1 for the class $\mathbb{L}$ of $\mathcal{L}_2$ systems. A similar algorithm applies for the solution of Problem 5.1 for the model classes $\mathbb{L}_+$, $\mathbb{L}_-$ and $\mathbb{M}_{(\pm)}$.

Algorithm 1: *Given* $P, K \in \mathcal{RH}_\infty^-$ *that define the systems* Σ_P *and* Σ_K *as in* (3).
Aim: *Find all* $C \in \mathcal{RH}_\infty^-$ *that define the system* $\Sigma_C = (\mathbb{T}, \mathbb{W}, \mathcal{C}) \in \mathbb{L}$ *with* $\mathcal{C} = \ker C$*, such that* $\mathcal{C}$ *implements* $\mathcal{K}$ *for* $\mathcal{P}$ *in the sense that* $\mathcal{P} \cap \mathcal{C} = \mathcal{K}$ *by full interconnection.*
Step 1: Verify whether $\mathcal{K} \subset \mathcal{P}$. Equivalently, verify whether there exists a mapping $F \in \mathcal{RH}_\infty^-$ such that $P = FK$. If not, the algorithm ends and no controller exists that implements $\mathcal{K}$ for $\mathcal{P}$.
Step 2: Determine a unit $U \in \mathcal{UH}_\infty^-$ which brings F into column reduced form: $\overline{F} = FU = [F_1 \ , \ 0]$, where $F_1 \in \mathcal{RH}_\infty^-$ is square and of full rank.
Step 3: Extend the matrix $\overline{F}$ with $\overline{W} = [0 \ , \ I]$ such that

$$\overline{\Lambda} = \begin{bmatrix} \overline{F} \\ \overline{W} \end{bmatrix} = \begin{bmatrix} F_1 & 0 \\ 0 & I \end{bmatrix},$$

belongs to $\mathcal{UH}_\infty^-$. Factorize $\overline{W} = WU$ with $W = \overline{W}U^{-1}$.
Step 4: Set $\Sigma_C = (\mathbb{T}, \mathbb{W}, \mathcal{C})$ where $\mathcal{C} = \ker C_0$ and $C_0 = WK \in \mathcal{RH}_\infty^-$. The controller Σ_C then belongs to $\mathbb{L}$ and implements $\mathcal{K}$ for $\mathcal{P}$.
Step 5: Set

$$\mathbb{C}_{\text{par}} = \{Q_1 P + Q_2 WK \mid Q_1, Q_2 \in \mathcal{RH}_\infty^-, Q_2 \text{ full rank}\}.$$

Then $\mathbb{C}_{\text{par}}$ is a parametrization of all controllers $\Sigma_C = (\mathbb{T}, \mathbb{W}, \mathcal{C})$ that implement $\mathcal{K}$ for $\mathcal{P}$ by ranging over all kernel representations $\mathcal{C} = \ker C$ with $C \in \mathbb{C}_{\text{par}}$.

Proof: Proof of Theorem 5.1:

i (⇒): This is trivial.
(⇐): If $\mathcal{K} \subset \mathcal{P}$, then there exists a F as in Theorem 4.1 or 4.2. In the controlled behavior $\mathcal{K}$, the restrictions of the plant as well as the restrictions applied by the controller have to be satisfied:

$$\mathcal{K} = \ker \left[\begin{smallmatrix} P \\ C \end{smallmatrix}\right] = \ker K = \ker(\Lambda K), \text{ where } \Lambda \in \mathcal{UH}_\infty^-,$$

where $\Lambda = \text{col}(F, W)$ with W unknown. The extended matrix $\overline{\Lambda}$ in step 3 is a multiplication of a unit U with Λ, so Λ has to be a unit. Therefore:

$$\mathcal{K} = \ker \left[\begin{smallmatrix} P \\ C \end{smallmatrix}\right] = \ker \left(\left[\begin{smallmatrix} F \\ W \end{smallmatrix}\right] K \right),$$

so, $C = WK$ which results in $\mathcal{C} = \ker C = \ker\{WK\}$.
ii One can apply a multiplication with a unit $Q \in \mathcal{UH}_\infty^-$:

$$\mathcal{K} = \ker \Big(\underbrace{\left[\begin{smallmatrix} I & 0 \\ Q_1 & Q_2 \end{smallmatrix}\right]}_{Q} \left[\begin{smallmatrix} P \\ C \end{smallmatrix}\right] \Big) = \ker \left(\left[\begin{smallmatrix} P \\ Q_1 P + Q_2 WK \end{smallmatrix}\right] \right),$$

where $Q_1, Q_2 \in \mathcal{RH}_\infty^-$ and Q_2 is full rank. Then all possible rational functions C can be parametrized by $\mathbb{C}_{\text{par}}$ as in step 5.

□

VI. Example: Quadratic cost

In this example, the plant behavior $\mathcal{P}$ of an unstable plant Σ_P is described by the state-space realization:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = x(t), \end{cases} \qquad x(0) = x_0, \tag{4}$$

where $x(t) \in \mathbb{R}^{\mathbf{x}}$, $u(t) \in \mathbb{R}^{\mathbf{u}}$, $A \in \mathbb{R}^{\mathbf{x}\times\mathbf{x}}$ and $B \in \mathbb{R}^{\mathbf{x}\times\mathbf{u}}$. The desired controlled behavior $\mathcal{K}$ consists of all pairs $(u, y) \in L_2(\mathbb{R}_+, \mathbb{R}^{\mathbf{u}\times\mathbf{x}})$ that minimize the cost function:

$$J(x_0, x(t), u(t)) = \tfrac{1}{2} \int_0^\infty x^T(t)Qx(t) + u^T(t)Ru(t)\mathrm{d}t$$

subject to the system equations (4) of the plant model Σ_P. Here, $0 \leq Q \in \mathbb{R}^{\mathbf{x}\times\mathbf{x}}$, $0 < R \in \mathbb{R}^{\mathbf{u}\times\mathbf{u}}$.

As discussed in [3], [8], this controlled behavior can be written as a dynamical system Σ_K with the state-space realization:

$$\begin{cases} \dot{x}(t) = (A - BR^{-1}B^TS)\ x(t), \quad x(0) = x_0, \\ u(t) = -R^{-1}B^TS\ x(t), \\ y(t) = x(t), \end{cases} \tag{5}$$

where S is a solution of an Algebraic Ricatti Equation. The numerical values used for those matrices are the following:

$$A = \left[\begin{smallmatrix} 4 & 1 \\ 0 & 4 \end{smallmatrix}\right], \quad B = \left[\begin{smallmatrix} 2 & 4 \\ 0 & 1 \end{smallmatrix}\right], \quad Q = \left[\begin{smallmatrix} 4 & 0 \\ 0 & 4 \end{smallmatrix}\right], \quad R = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right],$$
$$S = \left[\begin{smallmatrix} 4 & -15 \\ -15 & 76 \end{smallmatrix}\right], \quad \alpha, \beta \in \mathbb{R}^+, \quad \alpha \neq \beta.$$

The dynamical systems are specified by trajectories in the time domain, but we are interested in $\mathcal{L}_2$ behaviors using rational kernel representations as system representations. Because the controlled system is autonomous, the left-shift invariance property is required, which restricts us to use anti-stable mappings for P, K and also C. This results in:

$$P(s) = \left[\begin{smallmatrix} -I & (sI-A)^{-1}B \end{smallmatrix}\right] \in \mathcal{RH}_\infty^-,$$
$$K(s) = \left[\begin{smallmatrix} (sI-(A-BR^{-1}B^TS))(sI-\alpha I)^{-1} & 0 \\ R^{-1}B^TS(sI-\beta I)^{-1} & (sI-\beta I)^{-1} \end{smallmatrix}\right] \in \mathcal{RH}_\infty^-,$$

where $w(s) = [y(s) \;\; u(s)]^T$, $\alpha, \beta > 0$ and $\alpha \neq \beta$. Due to the requirement, the anti-stable "poles" α and β are introduced. Of course, no "pole-zero" cancellation should occur when α and β are chosen. Using those representations, the full interconnection algorithm can be applied to the problem:

Step 1: The first step in the full interconnection algorithm is to verify whether $\mathcal{K} \subset \mathcal{P}$, which should be the case. Equivalently, we need to verify whether there exists a $F(s) \in \mathcal{RH}_\infty^-$ such that $P(s) = F(s)K(s)$:

$$F(s) = \begin{bmatrix} \Gamma(s) & \Lambda(s) \end{bmatrix} \in \mathcal{RH}_\infty^-, \text{ where}$$
$$\Gamma(s) = \left[-I-(Is-A)^{-1}B(Is-\beta I)R^{-1}B^TS(sI-\beta I)^{-1}\right] \cdot (Is-\alpha I)(sI-(A-BR^{-1}B^TS)) \quad \text{and}$$
$$\Lambda(s) = (Is-A)^{-1}B(Is-\beta I)\,.$$

Step 2: The next step is to column reduce $F(s)$. This can be done using algorithms as in [2], which results in: $F(s) = \left[\begin{smallmatrix} 0 & -\frac{s-\alpha}{(s-4)^2} & 2\frac{s-\beta}{s-4} & 0 \\ 0 & -\frac{s-\alpha}{s-4} & 0 & 0 \end{smallmatrix}\right] \in \mathcal{RH}_\infty^-$, which can be column reduced to $\overline{F}(s) = F(s)\ U(s) = \begin{bmatrix} F_0(s) & 0 \end{bmatrix}$, where

$$\overline{F}(s) = \begin{bmatrix} -\frac{s-\alpha}{(s-4)^2} & 2\frac{s-\beta}{s-4} & 0 & 0 \\ -\frac{s-\alpha}{s-4} & 0 & 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty^-$$

and

$$U(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \frac{s-\beta}{s-\alpha} \\ 0 & 1 & \frac{1}{2}\frac{s-\alpha}{s-\beta} & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{UH}_\infty^-.$$

Step 3-4: Then, as discussed in Step 4 of Algorithm 1, a possible controller behavior $\mathcal{C} = \ker C_0$ is expressed as:

$$C_0(s) = W(s)K(s) = \begin{bmatrix} \frac{s+6}{s-\alpha} & \frac{1}{s-\alpha} & 0 & 0 \\ \frac{1}{2}\frac{1}{s-\beta} & \frac{8}{s-\beta} & 0 & \frac{1}{s-\beta} \end{bmatrix} \in \mathcal{RH}_\infty^-.$$

As mentioned before, the behavioral framework does not require a separation of the variable w into inputs and outputs. This can be seen in the result above, because the controller restricts the outputs of the plant in the first row when a separation in the variable space is made. So, mathematically the interconnection with this controller results in the desired controlled behavior, but this representation is not directly implementable for a real system, because outputs of a plant can't always be used as inputs. For a practical reason, we consider the parametrization of all controllers that implement Σ_K in the next step.

Step 5: Another controller can be found using the matrices $Q_1(s)$ and $Q_2(s)$ as defined in Algorithm 1. When these matrices are chosen to be:

$$Q_1 = \begin{bmatrix} \frac{1}{2}\frac{s-4}{s-b} & -\frac{1}{2}\frac{1}{s-b} \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_\infty^- \text{ and } Q_2 = \begin{bmatrix} \frac{1}{2}\frac{s-a}{s-b} & -2 \\ 0 & 1 \end{bmatrix} \in \mathcal{RH}_\infty^-,$$

the resulting controller is equal to:

$$\begin{aligned} C(s) &= Q_1(s)P(s) + Q_2(s)C_1(s) \\ &= \begin{bmatrix} \frac{4}{s-b} & -\frac{15}{s-b} & \frac{1}{s-b} & 0 \\ \frac{1}{2}\frac{1}{s-b} & \frac{8}{s-b} & 0 & \frac{1}{s-b} \end{bmatrix} \in \mathcal{RH}_\infty^-, \end{aligned}$$

In fact, this controller allows a feedback implementation as it is equivalent to the general LQR, namely:

$$u(t) = -\begin{bmatrix} 4 & -15 \\ \frac{1}{2} & 8 \end{bmatrix} x(t).$$

Note: The values in the Ricatti solution S and the values in the estimated rational expressions are rounded to integers for simplification.

VII. Conclusions and recommendations

We considered the problem of controller synthesis for specific classes of $\mathcal{L}_2$ functions. Operators in the classes $\mathcal{RH}_\infty^+$ of stable rational functions and $\mathcal{RH}_\infty^-$ of anti-stable rational functions define linear right-shift invariant $\mathcal{L}_2$ behaviors and linear left-shift invariant $\mathcal{L}_2$ behaviors by considering their kernel spaces. Given two $\mathcal{L}_2$ systems Σ_P and Σ_K we solve the question to synthesize a third $\mathcal{L}_2$ system Σ_C that realizes

Σ_K in the sense that the full interconnection of Σ_P and Σ_C satisfies $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$. Necessary and sufficient condition for the existence of an $\mathcal{L}_2$ system Σ_C is the inclusion $\mathcal{K} \subset \mathcal{P}$. An explicit controller synthesis algorithm for the class of all controllers that implement an $\mathcal{L}_2$ controlled system for an $\mathcal{L}_2$ plant has been derived. An example is given to demonstrate the algorithm for the construction of a rational representation of $\mathcal{C}$.

This paper only introduced the case when the plant behavior $\mathcal{P}$ and controller behavior $\mathcal{C}$ are fully interconnected, which is not always the case. Therefore, some further research has to be done for the "partial interconnection" case using those classes of rational functions. Studies already started for infinite smooth continuous behaviors in [6]. In this case, disturbances like noise can be taken into account, which may yield in robust control problems.

APPENDIX
PROOF OF THEOREM 4.1

Proof:

(iv $\Rightarrow$ {i,ii,iii}):

- iv $\Rightarrow$ i:
  $\mathcal{K},\mathcal{P} \subset \mathcal{L}_2$, so $w \in \mathcal{L}_2$: If $P = FK$ and take a $w \in \mathcal{K}$. Then, $v = Kw = 0$, so also $Pw = FKw = Fv = 0$. This implies that $P(s)w(s) = 0$, so $w \in \mathcal{P}$, and $\mathcal{K} \subset \mathcal{P}$.
- iv $\Rightarrow$ ii:
  $\mathcal{K},\mathcal{P} \subset \mathcal{H}_2^+$, so $w \in \mathcal{H}_2^+$:
  This proof is identical to the case when $\mathcal{K},\mathcal{P} \subset \mathcal{L}_2$.
- iv $\Rightarrow$ iii:
  $\mathcal{K},\mathcal{P} \subset \mathcal{H}_2^-$, so $w \in \mathcal{H}_2^-$: Again, if $P = FK$ and $w \in \mathcal{K}$, one can say that $v = Kw \in \mathcal{H}_2^+$ and hence $Pw = FKw = Fv$. Now, $F \in \mathcal{RH}_\infty^+$, so $Fv \in \mathcal{H}_2^+$. So, $Pw \in \mathcal{H}_2^+$ which implies using (2) that $w \in \mathcal{P}$, which is equal to $\mathcal{K} \subset \mathcal{P}$.

(iv $\Leftarrow$ {i,ii,iii}):

- iv $\Leftarrow$ i:
  $\mathcal{K},\mathcal{P} \subset \mathcal{L}_2$, so $w \in \mathcal{L}_2$: Using the definition of $\mathcal{K}$, one can write:

$$\begin{aligned}\mathcal{K} &= \{w \in \mathcal{L}_2 \mid \langle Kw, v\rangle_{\mathcal{L}_2} = 0 \ \ \forall v \in \mathcal{L}_2\}\\ &= \{w \in \mathcal{L}_2 \mid \langle w, K^* v\rangle_{\mathcal{L}_2} = 0 \ \ \forall v \in \mathcal{L}_2\} = (\operatorname{im} K^*)^\perp_,\end{aligned}$$

  where $K^* : \mathcal{L}_2 \to \mathcal{L}_2$ is the dual- or adjoint operator in $\mathcal{RH}_\infty^-$ defined by $K^*(s) = K^T(s^{-1})$. Something similar can be applied to the plant behavior. So, $\mathcal{K} \subset \mathcal{P}$ implies that $\mathcal{P}^\perp \subset \mathcal{K}^\perp$ and using the previous definition of $\mathcal{K}$, this results in

$$\overline{(\operatorname{im} P^*)} \subseteq \overline{(\operatorname{im} K^*)},$$

  where the bar denotes the closure in $\mathcal{L}_2$.
  For rational operators the latter implies that:

$$(\operatorname{im} P^*) \subset (\operatorname{im} K^*),$$

  because in that case the images are closed.
  Then we can say that for some e_i[2], $P^* e_i \in \operatorname{im} K^*$, so there exists a v_i such that:

$$P^* e_i = K^* v_i.$$

  This can be extended to a set of v_i's, such that: $P^* = K^* X$ with $X = (v_1, \ldots, v_p) \in \mathcal{RH}_2^- \subset \mathcal{RH}_\infty^-$. Then, we can rewrite this to $P = X^* K$, where F is equal to the dual operator $X^* \in \mathcal{RH}_\infty^+$.
- iv $\Leftarrow$ ii:
  $\mathcal{K},\mathcal{P} \subset \mathcal{H}_2^+$, so $w \in \mathcal{H}_2^+$:
  This proof is similar to the one in the previous item, except that now the $\mathcal{H}_2^+$ inner product is used. However, $\mathcal{H}_2^+$ inherited this inner product from $\mathcal{L}_2$.
- iv $\Leftarrow$ iii:
  $\mathcal{K},\mathcal{P} \subset \mathcal{H}_2^-$, so $w \in \mathcal{H}_2^-$: Now, $\mathcal{K}$ can be written as:

$$\begin{aligned}\mathcal{K} &= \{w \in \mathcal{H}_2^- \mid \langle \Pi_- Kw, v\rangle_{\mathcal{H}_2^-} = 0 \ \ \forall v \in \mathcal{H}_2^-\}\\ &= \{w \in \mathcal{H}_2^- \mid \langle w, K^*\Pi_-^* v\rangle_{\mathcal{H}_2^-} = 0 \ \ \forall v \in \mathcal{H}_2^-\}\\ &= \left(\operatorname{im} K^*\Pi_-^*\right)^\perp_,\end{aligned}$$

  where K^* and Π_-^* are adjoint operators. This can also be done for the plant behavior $\mathcal{P}$. As in item (iv $\Leftarrow$ i), $\mathcal{P}^\perp \subset \mathcal{K}^\perp$, so: $(\operatorname{im} P^*\Pi_-^*) \subset (\operatorname{im} K^*\Pi_-^*)$. Then there exists a $X \in \mathcal{RH}_2^-$ such that $P^*\Pi_-^* = K^*\Pi_-^* X$. So, one can say that $\Pi_- P = X^*\Pi_- K$, where $F = X^*$.

Equality condition:

Using the previous items, one can say that $\mathcal{P} = \mathcal{K}$ if and only if $P = U_1 K$ and $K = U_2 P$ with both U_1 and U_2 in $\mathcal{RH}_\infty^+$. Moreover, if U_1 and U_2 satisfy these conditions, then $P = U_1 U_2 K$ and $K = U_2 U_1 P$. If P and K are full rank, we find that $U_1 = U_2^{-1}$, which completes the proof. □

[2] $e_i = [0, \ldots, 1, \ldots, 0]$, with the 1 on the i^{th} position.

REFERENCES

[1] Madhu Nagraj Belur, *Control in a behavioral context*, PhD Thesis Rijksuniversiteit Groningen, 2003.

[2] Th.G.J. Beelen, G.J. van den Hurk and C. Praagman, *A new method for computing a column reduced polynomial matrix*, Systems & Control Letters, number 10, pages 217-224, 1988.

[3] D.W.G. Bijnen, *Model reduction on constrained optimally controlled systems*, Graduation paper, Eindhoven University of Technology, 2008.

[4] Bruce A. Francis, *A course in H_∞ Control Theory*, Lecture Notes in Control and Information Sciences, Spinger-Verlag, 1987.

[5] Jan Willem Polderman and Jan C. Willems, *Introduction to Mathematical Systems Theory: A Behavorial Approach*, Springer, 1998.

[6] H.L. Trentelman and R. Zavala You and C. Praagman, *Polynomial Embedding Algorithms for Controllers in a Behavioral Framework*, IEEE Transactions on Automatic Control, vol. 52, pages 1-6, 2007.

[7] Siep Weiland and Anton Stoorvogel, *Rational representations of behaviors: Interconnectability and stabilizability*, MCSS, vol. 10, pages 125-164, 1997.

[8] Siep Weiland, Jochem Wildenberg, Leyla Özkan and Jobert Ludlage, *A Lagrangian Method for Model Reduction of Controlled Systems*, Proceedings of the 17$^{\text{th}}$ IFAC World Congress, Seoul, 2008.

[9] Jan C. Willems, *On Interconnections, Control and Feedback*, IEEE Transactions on Automatic Control, vol. 42, pages 326-339, 1997.

[10] Jan C. Willems, *The Behavioral Approach to Open and Interconnected Systems*, IEEE Control Systems Magazine, vol. 27, pages 46-99, 2007.

[11] Jan C. Willems and Y. Yamamoto, *Behaviors defined by rational functions*, Linear Algebra and Its Applications, Volume 425, pages 226-241, 2007.