Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Cascade Structural Model Approximation of Identified State Space Models

Bo Wahlberg and Henrik Sandberg

Abstract— **General black-box system identification techniques such as subspace system identification and FIR/ARX least squares system identification are commonly used to identify multi-input multi-output models from experimental data. However, in many applications there are a priori given structural information. Here the focus is on linear dynamical systems with a cascade structure, and with one input signal and two output signals. Models of such systems are important in e.g. cascade control applications. It is possible to incorporate such a structure in a prediction error method, which, however, is based on rather advanced numerical non-convex optimization techniques to calculate the corresponding structured model estimate. We will instead study how to use model approximation techniques to approximate a general black-box estimate with a structured model. This will avoid the use of numerical optimization and works well with e.g. subspace system identification, which is a standard method in process industry where cascade systems are very common. The problems of cascade structural model approximation and model reduction are rather non-standard, and we will study several new methods. The basic idea is to first find a higher order but structured model approximation using standard H_∞ model matching techniques, and then in a second step use so-called structured balanced model reduction to find lower order structured approximation. Structured balanced model reduction is a rather new approach, with powerful model order selection tools and error bound results. The results of the corresponding two step model approximation approach seem promising, as illustrated by a simple numerical example.**

I. Introduction

System identification deals with estimation and validation of models of dynamical systems from experimental data. Most system identification methods concern, however, single-input single-output (SISO) systems. Many of these results can be generalized to multi-input multi-output (MIMO) systems. In particular, subspace system identification methods have shown very useful when dealing with MIMO systems. This is a black-box technique to identify state-space models and it is difficult to take a priori information of the underlying system into account to specify the model structure. The current work has been motivated by a discussion on use of subspace identification in process industry presented in [17]. In this application area it is common to first identify unstructured sub-models, which then in a second step are merged into a high order model. The complexity of this combined model has then to be reduced in order to apply e.g. model predictive control algorithms. In industry, simple standard model reduction techniques, such as balanced model reduction, are used, which do not take the structure into account.

The objective of this contribution is to study identification and approximation of systems with a cascade or series structure as illustrated in Fig. 1.

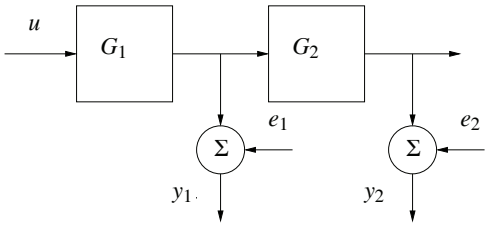

Fig. 1. Cascade system.

The corresponding input/output relations are

$$\begin{aligned} y_1(t) &= G_1(q)u(t) + e_1(t) \\ y_2(t) &= G_2(q)G_1(q)u(t) + e_2(t). \end{aligned}$$

The input signal is denoted by $u(t)$ and the two output signals are $y_1(t)$ and $y_2(t)$, respectively. The transfer functions are $G_1(q)$ and $G_2(q)$, and are assumed to be stable. Here q is the shift operator. The signals $e_1(t)$ and $e_2(t)$ denote the measurement noise processes. We assume that the dimensions of the input and the two output signals are all one (the scalar case).

There are several questions that have to be answered and important user choices to be made when applying system identification methods to a data set obtained from a cascade system of the form $\{u(t), y_1(t), y_2(t)\}$, $t = 1 \ldots N$. Any single-input-multi-output (SIMO) system identification method, such as subspace system identification, can be applied, but it is not straightforward to impose the cascade model structure.

It is possible to apply a Prediction Error Method (PEM) or the Maximum Likelihood (ML) method, [3], to a constrained model structure that only allows for models of cascade form. Because of the product $G_1(q)G_2(q)$ simple linear in the parameters model structures such as FIR or ARX models are not directly applicable. However, structured PEM and ML are asymptotically statistically optimal methods to solve the structured cascade system identification problem. In [16], [15], the statistical properties of identified cascade models are analyzed in detail and an initial discussion on how to take the uncertainty of the estimates into account to use model reduction to find structured models are given. Our aim is to develop cascade structural model approximation methods for identified state space models.

The outline of this paper is as follows. An introduction to system identification using model approximation/reductions

This work was partially supported by The Swedish Foundation for Strategic Research, The Swedish Research Council and the Linnaeus Center ACCESS at KTH.

The authors are with the Automatic Control Lab and ACCESS, School of Electrical Engineering, KTH, SE-100 44 Stockholm, Sweden. E-mail: {bo,hsan}@ee.kth.se

with a focus on cascade systems is given in Section II. Section III reviews structured model reduction of cascade systems. The idea start with a high order but structured approximation of the identified model, and then in a second step approximate this with a lower order one while keeping the structure. The proposed method is closely related to balanced model reductions. Section IV deals with using H_∞ approximation model matching techniques in order to find cascade structured, but possible high order, approximation of unstructured estimates. Two methods for cascade structural model approximation are outlined together with corresponding error bounds and model order selection tools are also outlined in this section. These methods are evaluated in Section V on two simple numerical examples. Finally, Section VI concludes the paper.

II. System Identification and Model Approximation

The problem of first estimating an unstructured SIMO model and then in a second step find a structured cascade model is closely related to model approximation. A simple approach is to first estimate $G_1(q)$ from $u(t)$ and $y_1(t)$ and then the series transfer function $G_3(q) = G_2(q)G_1(q)$ from $u(t)$ and $y_2(t)$. Denote the corresponding estimates by $\hat{G}_1(q)$ and $\hat{G}_3(q)$, respectively. Another approach is to apply a state space system identification method, e.g. a subspace approach, to obtain an unstructured state-space model estimate

$$\begin{aligned} x(t+1) &= \hat{A}x(t)+\hat{B}u(t) \\ y_1(t) &= \hat{C}^1x(t) \\ y_2(t) &= \hat{C}^3x(t) \end{aligned} \tag{1}$$

and take

$$\hat{G}_1(q) = \hat{C}^1(qI-\hat{A})^{-1}\hat{B}, \tag{2}$$

$$\hat{G}_3(q) = \hat{C}^3(qI-\hat{A})^{-1}\hat{B}, \tag{3}$$

which both will be of the same order as the total state space model. We will assume that the true system and the corresponding estimate are both stable.

One way to find the cascade transfer function $G_1(q)$ and $G_2(q)$ is to minimize the cost function

$$\bar{V}(G_1,G_2) = \frac{1}{\lambda_1}\int_{-\pi}^{\pi}|G_1(e^{i\omega})-\hat{G}_1(e^{i\omega})|^2\Phi_u(\omega)d\omega + \frac{1}{\lambda_2}\int_{-\pi}^{\pi}|G_2(e^{i\omega})G_1(e^{i\omega})-\hat{G}_3(e^{i\omega})|^2\Phi_u(\omega)d\omega,$$

where $\Phi_u(\omega)$ is the input spectral density. This approach is called the asymptotic ML approach in [14], and is closely related to PEM methods. One of the first ideas of using model reduction in system identification is [18]. Model reduction in system identification has more recently been further studied by Tjärnström and co-workers in a series of papers, [10], [9], [11]. The general result is that it is best to first estimate a high order model, which gives a good description of the true system, and then in a second step approximate this model with a reduced order one. It is, however, important to take the statistical properties of the model estimate into account when doing the model reduction.

There seem to be few results on how to find reduced order models with specific structures without using nonconvex numerical optimization methods. However, recently there have been some techniques developed, see, for example [12], [13], [6], [7]. We will here use a method that is similar to the one developed in [7], since it comes with a priori error bounds.

For cascade systems, it is rather straightforward to find a model of the first transfer function $G_1(q)$ by for example using a balanced model approximation of $\hat{G}_1(q)$ defined by (2). The more difficult problem is how to find $G_2(q)$ from $\hat{G}_3(q)$ in (3), (and $G_1(q)$). In principle one has to solve

$$\min_{G_2}\|G_1G_2-\hat{G}_3\|,$$

for fixed G_1 with some suitable norm. A naive approach would be to take

$$\hat{G}_2(q) = \frac{\hat{G}_3(q)}{G_1(q)}$$

where $G_1(q)$ is the approximation of $\hat{G}_1(q)$ obtained in the first step, and then apply model reduction on $\hat{G}_2(q)$ to find $G_2(q)$. A problem with this idea is that $\hat{G}_2(q)$ may be unstable and/or anti-causal, which means that a standard model reduction method would fail. A way to overcome this is to use the approximative stable inverse

$$\frac{1}{G_1(q)} \approx \frac{K}{1+KG_1(q)}$$

where the gain $K>0$ is designed to assure stability of the right hand side. This is equal to designing a stable P-regulator with gain K for feedback control of $G_1(q)$ and then performing model reduction of

$$\hat{G}_2(q) = \frac{\hat{G}_3(q)}{G_1(q)+1/K} \tag{4}$$

to find a reduced order model $G_2(q)$. See [4] for more ideas of using feedback inversion in system identification.

The idea of using feedback to find G_2 is closely related to applying cascade control to the estimated state space model (1). That is find a high gain controller for the inner loop, e.g.

$$u(t) = -K(y_1(t)-r_2(t))$$

such that the closed loop system is stable. Here $r_2(t)$ is the reference signal to the second system. The interpretation of the approximation is that the dynamics of closed loop inner system should be much faster than G_2. Then do model reduction of closed loop system

$$\begin{aligned} x(t+1) &= (\hat{A}-K\hat{B}\hat{C}^1)x(t)+K\hat{B}r_2(t) \\ y_2(t) &= \hat{C}^3x(t) \end{aligned}$$

to find a reduced order state-space model of G_2.

The ideas presented above are quite ad hoc and do not use the full power of the state-of-the-art in model approximation. In the next section we will develop more efficient methods for the cascade model approximation problem.

III. Structured Model Reduction of Cascade Systems

In this section, we apply results from [7] to a system in cascade form, which is a special case of an interconnected linear system. The results in [7] are given in continuous time, but the results can be extended to discrete time as is shown here.

A high-order model $\hat{G}_1(q)$ of $G_1(q)$ can be obtained directly using standard identification techniques, as described in Section II. Assume also that a high-order approximation $\hat{G}_2(q)$ of $G_2(q)$ has been obtained in some way, for example from (4). More advanced methods for computing $\hat{G}_2(q)$ are given in Section IV using H_∞ model matching.

A realization of the cascade system $\begin{bmatrix} \hat{G}_1 \\ \hat{G}_2\hat{G}_1 \end{bmatrix}$ is given by

$$\begin{aligned} \begin{bmatrix} \hat{x}_1(t+1) \\ \hat{x}_2(t+1) \end{bmatrix} &= \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{B}_2\hat{C}_1 & \hat{A}_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} u(t) \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \begin{bmatrix} \hat{C}_1 & 0 \\ 0 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}, \end{aligned} \tag{5}$$

where $x_1(t) \in \mathbb{R}^{n_1}, x_2(t) \in \mathbb{R}^{n_2}$ and,

$$\hat{G}_1(q) = \hat{C}_1(qI - \hat{A}_1)^{-1}\hat{B}_1, \quad \hat{G}_2(q) = \hat{C}_2(qI - \hat{A}_2)^{-1}\hat{B}_2.$$

The order of $\hat{G}_1(q)$ is smaller or equal to n_1, and the order of $\hat{G}_2(q)$ is smaller or equal to n_2. If standard model reduction methods, such as balanced truncation, is applied to (5), the particular structure of the realization is generally lost. This means that we cannot extract reduced order models $G_1(q)$ and $G_2(q)$ from such a reduction. The problem is that normal reduction usually mixes the states $\hat{x}_1(t)$ and $\hat{x}_2(t)$. The idea in structured model reduction [12], [13], [7] is to find a suitable block-diagonal coordinate transformation, i.e.,

$$\begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix}$$

for the realization (5). After such a transformation the block structure remains intact and the states are truncated from each subsystem. That is, we apply canonical projections $S_L^1, S_R^1, S_L^2, S_R^2$,

$$\begin{aligned} S_L^1 = (S_R^1)^T &= \begin{bmatrix} I_{r_1} & 0_{r_1 \times (n_1 - r_1)} \end{bmatrix} \in \mathbb{R}^{r_1 \times n_1}, \\ S_L^2 = (S_R^2)^T &= \begin{bmatrix} I_{r_2} & 0_{r_2 \times (n_2 - r_2)} \end{bmatrix} \in \mathbb{R}^{r_2 \times n_2}, \end{aligned}$$

where r_1 and r_2 are the orders of the reduced models. If the transformation and projection is applied to (5), the realization of the reduced model $\begin{bmatrix} G_1 \\ G_2G_1 \end{bmatrix}$ is now given by

$$\begin{aligned} \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix} &= \begin{bmatrix} S_L^1 T_1 \hat{A}_1 T_1^{-1} S_R^1 & 0 \\ (S_R^2 T_2 \hat{B}_2)(\hat{C}_1 T_1^{-1} S_R^1) & S_L^2 T_2 \hat{A}_2 T_2^{-1} S_R^2 \end{bmatrix} \\ \begin{bmatrix} B_1 \\ 0 \end{bmatrix} &= \begin{bmatrix} S_L^1 T_1 \hat{B}_1 \\ 0 \end{bmatrix}, \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} = \begin{bmatrix} \hat{C}_1 T_1^{-1} S_R^1 & 0 \\ 0 & \hat{C}_2 T_2^{-1} S_R^2 \end{bmatrix}. \end{aligned} \tag{6}$$

From this realization it is straightforward to identify realizations of the reduced subsystems,

$$\begin{aligned} G_1(q) &= C_1(qI - A_1)^{-1}B_1, \\ G_2(q) &= C_2(qI - A_2)^{-1}B_2. \end{aligned} \tag{7}$$

The question remains how to choose the transformations T_1, T_2 and the approximation orders r_1, r_2 so that the approximation error

$$\left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_2\hat{G}_1 - G_2G_1 \end{array} \right\|_\infty$$

is small. An attractive method is balanced truncation with (generalized) structured Gramians [7] ("structured balanced truncation"), since then a priori error bounds on the H_∞ error is obtained. The details are given next.

Remark 1: An alternative to structured balanced truncation is to apply model reduction separately to the models $\hat{G}_1(q)$ and $\hat{G}_2(q)$. A problem with this is that one does not take into account that all inputs to $\hat{G}_2(q)$ are filtered through $\hat{G}_1(q)$. This generally alters the relative importance of the states in $\hat{G}_2(q)$. For this reason it is often important to include a frequency-dependent weight in the approximation criterion [1], as is done in structured balanced truncation.

The structured (generalized) controllability Gramian and the structured observability Gramians for (5) are symmetric positive definite matrices

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \tag{8}$$

respectively, that satisfy the controllability Lyapunov inequality

$$\begin{aligned} \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{B}_2\hat{C}_1 & \hat{A}_2 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{B}_2\hat{C}_1 & \hat{A}_2 \end{bmatrix}^T - \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \\ + \begin{bmatrix} \hat{B}_1(\hat{B}_1)^T & 0 \\ 0 & 0 \end{bmatrix} < 0 \end{aligned} \tag{9}$$

and the observability Lyapunov inequality

$$\begin{aligned} \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{B}_2\hat{C}_1 & \hat{A}_2 \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} \hat{A}_1 & 0 \\ \hat{B}_2\hat{C}_1 & \hat{A}_2 \end{bmatrix} - \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \\ + \begin{bmatrix} (\hat{C}_1)^T\hat{C}_1 & 0 \\ 0 & (\hat{C}_2)^T\hat{C}_2 \end{bmatrix} < 0. \end{aligned} \tag{10}$$

The Gramians are generalized since they satisfy linear matrix inequalities (LMIs) instead of equalities. The inequalities are needed to guarantee a block-diagonal structure. The LMIs can be effectively solved using software like SeDuMi [8]. To obtain unique generalized Gramians, we solve the LMIs while minimizing the traces of the structured Gramians (8). Finding structured Gramians and reduced models for the cascade system is a well posed problem because of the following propositions.

Proposition 1: The matrices $\hat{A}_1$ and $\hat{A}_2$ are stable if, and only if, there are symmetric positive definite matrices P_1, Q_1, P_2, Q_2 that satisfy the Lyapunov inequalities (9)–(10).

Proof: Application of the Schur lemma. ■

Proposition 2: Assume there are symmetric positive definite matrices P_1, Q_1, P_2, Q_2 to the cascade system (5). Define the *structured singular values* by

$$\begin{aligned} \sigma_{1,i} &= \lambda_i^{1/2}(P_1Q_1) > 0, \quad i = 1, \ldots, n_1, \\ \sigma_{2,i} &= \lambda_i^{1/2}(P_2Q_2) > 0, \quad i = 1, \ldots, n_2. \end{aligned}$$

Then there are invertible matrices T_1, T_2 such that the structured Gramians are balanced, i.e.,

$$\begin{aligned} \Sigma_1 &= T_1^{-T}Q_1T_1^{-1} = T_1^TP_1T_1 = \operatorname{diag}\{\sigma_{1,1}, \ldots, \sigma_{1,n_1}\} \\ \Sigma_2 &= T_2^{-T}Q_2T_2^{-1} = T_2^TP_2T_2 = \operatorname{diag}\{\sigma_{2,1}, \ldots, \sigma_{2,n_2}\}. \end{aligned}$$

If these transformations T_1, T_2 are used to construct the reduced models $G_1(q)$ and $G_2(q)$ as in (6)–(7), then the models satisfy the upper error bound

$$\left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_2\hat{G}_1 - G_2G_1 \end{array} \right\|_\infty \leq 2 \sum_{i=r_1+1}^{n_1} \sigma_{1,i} + 2 \sum_{i=r_2+1}^{n_2} \sigma_{2,i}, \tag{11}$$

and $G_1(q)$ and $G_2(q)$ are guaranteed to be stable.

Proof: This is a discrete-time version of Proposition 2 in [7]. ■

Remark 2: The matrices T_1 and T_2 that balance the structured Gramians can be computed using the same techniques as in regular balancing, see [19].

Remark 3: The upper error bound (11) is useful since it helps to select the model orders r_1 and r_2.

IV. Model Approximation of Cascade System in H_∞-norm

The problem we would like to solve in this section is

$$\inf_{G_1\in R_{r_1}H_\infty, G_2\in R_{r_2}H_\infty} \left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_3 - G_2G_1 \end{array} \right\|_\infty. \tag{12}$$

Here R_rH_∞ denotes rational transfer functions in H_∞ of McMillan degree less or equal to r. The problem is to find stable transfer functions $G_1(q)$ and $G_2(q)$ of McMillan degree less or equal to r_1 and r_2, respectively, such that the given approximation criterion is small. (To simplify notation, we omit the argument q for the transfer functions in the rest of the section.) The approximation problem (12) is hard, but can be relaxed in various ways, as we shall see next. Note that the approximation criterion is not the same as in (11) since $\hat{G}_3$ does not (yet) have the form $\hat{G}_2\hat{G}_1$.

A. A fundamental lower error bound

From optimal Hankel norm approximation [19] it is known that

$$\inf_{G_1\in R_{r_1}H_\infty} \|\hat{G}_1 - G_1\|_\infty \geq \sigma_{r_1+1}(\hat{G}_1)$$
$$\inf_{G_3\in R_{r_1+r_2}H_\infty} \|\hat{G}_3 - G_3\|_\infty \geq \sigma_{r_1+r_2+1}(\hat{G}_3),$$

where $\sigma_i(G)$ is the i-th largest Hankel singular value of G. These can be used to derive a lower bound on the problem (12).

Proposition 3: The optimization problem (12) has the lower bound

$$\max\{\sigma_{r_1+1}(\hat{G}_1), \sigma_{r_1+r_2+1}(\hat{G}_3)\} \leq \inf_{G_1\in R_{r_1}H_\infty, G_2\in R_{r_2}H_\infty} \left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_3 - G_2G_1 \end{array} \right\|_\infty.$$

Proof: Define $G_3 = G_2G_1$. Then $G_3 \in R_{r_1+r_2}H_\infty$, and

$$\inf_{\substack{G_1\in R_{r_1}H_\infty \\ G_3\in R_{r_1+r_2}H_\infty}} \left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_3 - G_3 \end{array} \right\|_\infty \leq \inf_{\substack{G_1\in R_{r_1}H_\infty \\ G_2\in R_{r_2}H_\infty}} \left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_3 - G_2G_1 \end{array} \right\|_\infty.$$

Furthermore, we have

$$\left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_3 - G_3 \end{array} \right\|_\infty \geq \|\hat{G}_1 - G_1\|_\infty \geq \sigma_{r_1+1}(\hat{G}_1).$$

A similar inequality holds for $\sigma_{r_1+r_2+1}(\hat{G}_3)$, and the result follows. ■

The next problem is to design methods that generate suboptimal solutions G_1, G_2 to (12), with upper error bounds. If the distance between the upper and lower bound is small, then we know the suboptimal solution is close to the optimal solution.

B. Suboptimal solution with upper bound, Method 1

The suboptimal method presented here uses the structured model reduction technique in Section III. First we construct an intermediate model of high order but with the desirable cascade structure, and then we reduce it to obtain G_1, G_2.

The method, denoted **Method 1**, has two steps:

Step 1: A high-order candidate $\hat{G}_2$ for G_2 is obtained by solving the standard H_∞-model matching problem [2]

$$\begin{aligned} \hat{G}_2 &:= \arg \inf_{Q\in RH_\infty} \|\hat{G}_3 - Q\hat{G}_1\|_\infty, \\ \alpha_1 &:= \inf_{Q\in RH_\infty} \|\hat{G}_3 - Q\hat{G}_1\|_\infty, \end{aligned} \tag{13}$$

using for example the command `hinfsyn` in MATLAB. Note that there is no order constraint on $\hat{G}_2$, and that if $\hat{G}_3/\hat{G}_1$ is stable and causal, then $\alpha_1 = 0$ and we get $\hat{G}_2 = \hat{G}_3/\hat{G}_1$. Typically, the order of $\hat{G}_2$ is $n_2 = n_1 + n_3$, where n_1 and n_3 are the orders of $\hat{G}_1$ and $\hat{G}_3$, respectively.

In this paper, we assumed a priori that the models $\hat{G}_1$ and $\hat{G}_3$ come from system identification of a stable cascade system. Hence, it should be possible to find a stable $\hat{G}_2$ such that $\hat{G}_3 \approx \hat{G}_2\hat{G}_1$. If α_1 is a relatively large number, this indicates that the a priori assumption made is wrong, or that the identified models $\hat{G}_1$ and $\hat{G}_3$ are too bad.

The cascade model $\begin{bmatrix} \hat{G}_1 \\ \hat{G}_2\hat{G}_1 \end{bmatrix}$ approximates the given model $\begin{bmatrix} \hat{G}_1 \\ \hat{G}_3 \end{bmatrix}$ with the error α_1, since

$$\left\| \begin{array}{c} \hat{G}_1 - \hat{G}_1 \\ \hat{G}_3 - \hat{G}_2\hat{G}_1 \end{array} \right\|_\infty = \alpha_1. \tag{14}$$

In the next step the order is reduced.

Step 2: Using the structured model reduction technique in Section III with the system $\begin{bmatrix} \hat{G}_1 \\ \hat{G}_2\hat{G}_1 \end{bmatrix}$ obtained in Step 1 as input, approximations G_1 and G_2 are obtained such that (11) holds, i.e.,

$$\left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_1\hat{G}_2 - G_1G_2 \end{array} \right\|_\infty \leq 2\sum_{i=r_1+1}^{n_1} \sigma_{1,i} + 2\sum_{i=r_2+1}^{n_2} \sigma_{2,i},$$

where $\sigma_{1,i}$ and $\sigma_{2,i}$ are the structured singular values of $\hat{G}_1$ and $\hat{G}_2$, respectively, and $\deg G_1 = r_1$, $\deg G_2 = r_2$, $\deg \hat{G}_1 = n_1$, and $\deg \hat{G}_2 = n_2$.

Returning to our original problem, (12), we have (using the triangle inequality) that

$$\begin{aligned} \max\{\sigma_{r_1+1}(\hat{G}_1), \sigma_{r_1+r_2+1}(\hat{G}_3)\} &\leq \left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_3 - G_2G_1 \end{array} \right\|_\infty \\ &\leq \alpha_1 + 2\sum_{i=r_1+1}^{n_1} \sigma_{1,i} + 2\sum_{i=r_2+1}^{n_2} \sigma_{2,i}. \end{aligned} \tag{15}$$

For the lower bound, we have used Proposition 3.

C. Suboptimal method with upper bound, Method 2

The second method is a slight variation of the first method in Section IV-B. The method, called **Method 2**, has three steps:

Step 1: The reduced-order model G_1 is obtained by approximating $\hat{G}_1$ directly. If regular balanced truncation [19] is used, we obtain the error bound $\|\hat{G}_1 - G_1\|_\infty \leq 2\sum_{i=r_1+1}^{n_1} \sigma_i(\hat{G}_1)$, where $\sigma_i(\hat{G}_1)$ are the Hankel singular values of $\hat{G}_1$.

Step 2: To find a high-order candidate $\hat{G}_2$ of G_2, we solve the standard H_∞-model matching problem [2]

$$\begin{aligned} \hat{G}_2 &:= \arg \inf_{Q\in RH_\infty} \|\hat{G}_3 - QG_1\|_\infty, \\ \alpha_2 &:= \inf_{Q\in RH_\infty} \|\hat{G}_3 - QG_1\|_\infty. \end{aligned} \tag{16}$$

Notice the difference to (13): Here $\hat{G}_2$ is in series with the approximation G_1, not with the original model $\hat{G}_1$. The order of the optimal $\hat{G}_2$ in (16) is typically $n_2 = r_1 + n_3$, which is smaller than the order of $\hat{G}_2$ in the (13).

Step 3: Solve the frequency-weighted model reduction problem $\inf_{G_2\in R_{r_2}H_\infty} \|(\hat{G}_2 - G_2)G_1\|_\infty$, where G_1 from Step 1 acts as frequency-dependent weight, and $\hat{G}_2$ comes from Step 2. This is *not* a standard model matching problem because of the degree constraint on G_2, and it is not known how to compute the optimal solution. However, there are many suboptimal methods available, for example Enns' classical method [1], or one of the many methods presented in the book [5]. A suboptimal solution with error bound and stability guarantee is also obtained if the structured model reduction method in Section III is applied. This is the method we use here. To do that, just delete the output channel y_1 from the cascaded realization. As input to the method we use the model $\hat{G}_2 G_1$ and none of the structured singular values $\sigma_{1,i}$ are truncated since G_1 shall remain intact. The bound (11) then becomes $\|\hat{G}_2 G_1 - G_2 G_1\|_\infty \leq 2\sum_{i=r_2}^{n_2} \sigma_{2,i}$, where $\sigma_{2,i}$ are the structured singular values corresponding to $\hat{G}_2$.

Returning to our original problem, (12), we have (using the triangle inequality) that

$$\begin{aligned} &\max\{\sigma_{r_1+1}(\hat{G}_1), \sigma_{r_1+r_2+1}(\hat{G}_3)\} \\ &\qquad \leq \left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_3 - G_2 G_1 \end{array} \right\|_\infty \\ &\qquad \leq 2\sum_{i=r_1+1}^{n_1} \sigma_i(\hat{G}_1) + \alpha_2 + 2\sum_{i=r_2+1}^{n_2} \sigma_{2,i}. \end{aligned} \tag{17}$$

V. Examples

For the two examples, we use the identified models

$$\begin{aligned} \hat{G}_1(q) &= \frac{q - 0.11}{q^3(q-0.1)(q-0.4)}, \\ \hat{G}_3(q) &= \hat{G}_1(q)\frac{q}{q^2 - 0.5q + 0.5} + \Delta(q). \end{aligned}$$

The first subsystem ($\hat{G}_1(q)$) is a low-pass filter, and the second subsystem has resonant poles. When $\Delta(q) = 0$, the system can be exactly realized in cascade form, using $G_1(q) = \hat{G}_1(q)$ and $G_2(q) = q/(q^2 - 0.5q + 0.5)$. Note also that there is almost a pole-zero cancellation in $\hat{G}_1(q)$. Hence, we expect the model reduction methods to detect this, and suggest that the corresponding state is truncated.

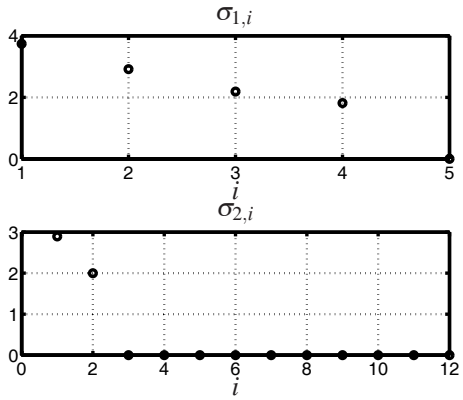

Fig. 2. The structured singular values from Method 1 in Example 1.

In these examples we assume that the system identification part of the problem has already been performed ($\hat{G}_1$ and $\hat{G}_3$ are given from the start). The objective of the examples is to check that the model reduction methods in Section IV give reasonable models of low order and to see how they react to errors in the identified models ($\Delta(q) \neq 0$).

Example 1: In the first example, choose $\Delta(q) = 0$. This is to check that the methods can detect when the model can be perfectly described by a model in cascade form.

The structured singular values for Method 1 are shown in Fig. 2. As can be seen, we have $n_1 = 5$ and $n_2 = 12$. The command `hinfsyn` in MATLAB that performs the model matching (13) terminates with $\alpha_1 = 5.91 \cdot 10^{-4}$. The structured singular values clearly indicate that we should choose $r_1 = 4$ and $r_2 = 2$. That $r_1 = 4$ means that the pole at 0.1 in $\hat{G}_1(q)$ that is close to a zero can be removed without causing a large input-output error. The norm of the original system and the approximation error are

$$\left\| \begin{array}{c} \hat{G}_1 \\ \hat{G}_3 \end{array} \right\|_\infty = 2.58, \quad \left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_3 - G_2 G_1 \end{array} \right\|_\infty = 1.32 \cdot 10^{-5},$$

and the upper error bound (15) becomes $4.22 \cdot 10^{-3}$. Hence, for practical purposes, the approximation error is essentially zero. The upper error bound is a factor 100 too conservative, but the structured singular values are still very useful to pick out the correct approximation orders r_1, r_2.

Method 2 also works well, and the singular values have the same qualitative behavior as the singular values in Fig. 2. Hence, we choose the same orders r_1 and r_2 for Method 2. Both the approximation error and the error bound (17) are slightly smaller for Method 2, $9.51 \cdot 10^{-6}$ and $1.11 \cdot 10^{-3}$, respectively. The computation time is also smaller for Method 2, 1.7 seconds, compared to 2.4 seconds for Method 1. This is because the LMIs in Method 2 are of lower dimension. Hence, for this particular example Method 2 is slightly better. But both methods perform well and recover the system in cascade form.

Example 2: In the second example, we choose the perturbation $\Delta(q) = 0.2/(q - 0.7)$. Such a perturbation could have been caused by noisy input-output data in the system identification. The structured singular values for Method 1 are shown in Fig. 3. As can be seen, now n_2 is increased

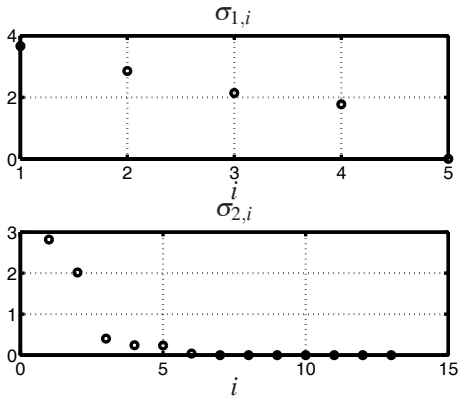

Fig. 3. The structured singular values from Method 1 in Example 2.

with one. The model matching problem (13) now results in $\alpha_1 = 0.348$, which is much larger than α_1 in Example 1. This is a clear indication that we will not be able to find a perfect match using a cascaded approximation. The reson is of course the perturbation $\Delta(q)$.

In Fig. 3, it is seen that there are large drops from $\sigma_{1,4}$ to $\sigma_{1,5}$, and from $\sigma_{2,2}$ to $\sigma_{2,3}$. This suggest that we should choose $r_1 = 4$ and $r_2 = 2$. But the structured singular values $\sigma_{2,3}, \ldots, \sigma_{2,6}$ are now clearly greater than zero, and one could also consider to choose $r_2 = 6$. Using $r_1 = 4$ and $r_2 = 2$, we obtain

$$\left\| \begin{array}{c} \hat{G}_1 \\ \hat{G}_3 \end{array} \right\|_\infty = 2.84, \quad \left\| \begin{array}{c} \hat{G}_1 - G_1 \\ \hat{G}_3 - G_2 G_1 \end{array} \right\|_\infty = 0.423,$$

and the upper error bound (15) is 2.20 which is about a factor 5 too conservative. Using $r_2 = 6$ instead, the approximation error becomes 0.347 and the upper bound 0.350. It turns out that no matter how r_1 and r_2 are chosen, the error is never smaller than 0.347. Hence, the constant α_1 is a good indicator on how good approximation one can get using Method 1.

The singular values for Method 2 look qualitatively the same as in Fig. 3. Using $r_1 = 4$ and $r_2 = 2$, the approximation error and the error bound (17) is 0.425 and 2.22, respectively. This is slightly worse than for Method 1. Using $r_2 = 6$ instead, the approximation error becomes 0.347 and the upper bound 0.360, which again is slightly worse than for Method 1. The computation time is 2.0 seconds, compared to 2.1 seconds for Method 1.

In this example, neither method was able to compute a perfect approximation. This was not expected either, since $\Delta(q) \neq 0$. But both methods were able to compute two relatively good approximations in cascade form, and the singular values gave good insight about good approximation orders r_1, r_2. It should be remembered that both methods are suboptimal, and it is unknown how far away these solutions are from a truly optimal solution.

VI. Conclusion

The problem considered in this paper has been inspired by the discussion on the industrial applications of structural system identification given in [17]. A typical approach is to first estimate sub-models, which then are merged into a high order model. Next model reduction is used to find a lower order approximation suitable for e.g. model predictive control design. If a standard model reduction technique, such as balanced model reduction, is used the structure is, however, lost. The aim of this contribution has been to propose and study model approximation methods that retain the structure, and in particular for cascade systems. The idea is to first find a structured but higher order approximation. We have studied H_∞ methods, but it is as well possible to apply standard H_2 model matching techniques. The second step is then to use structured balanced truncation based on LMI techniques. The advantage is explicit error bounds, which can be used for model order selection.

Industrial applications often concern system models composed of cascade, feed-forward, feedback and multiplicative connections of linear dynamics and zero memory nonlinear elements. It would be interesting to develop model approximation techniques for such more general systems.

References

[1] D. Enns, "Model reduction with balanced realizations: an error bound and a frequency weighted generalization," in *Proceedings of the IEEE Conference on Decision and Control*, Las Vegas, Nevada, 1984.

[2] B. Francis, *A Course in H∞ Control Theory*. Springer, 1987.

[3] L. Ljung, *System Identification - Theory For the User, 2nd ed*. Upper Saddle River, New Jersey: Prentice Hall, 1999.

[4] O. Markusson, "Model and system inversion with applications in nonlinear system identification and control," Ph.D. dissertation, KTH, Jan. 2002.

[5] G. Obinata and B. D. O. Anderson, *Model Reduction for Control System Design*. London, UK: Springer-Verlag, 2001.

[6] H. Sandberg and R. M. Murray, "Frequency-weighted model reduction with applications to structured models," in *Proceedings of the 2007 American Control Conference*, New York City, New York, July 2007, pp. 941–946.

[7] ——, "Model reduction of interconnected linear systems using structured Gramians," in *Proccedings of the IFAC World Congess*, 2008, to appear. Available from `http://www.ee.kth.se/~hsan/ifac2008.pdf`.

[8] J. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11–12, pp. 625–653, 1999, available from `http://fewcal.kub.nl/sturm/software/sedumi.html`.

[9] F. Tjärnström, "Variance analysis of l2 model reduction when undermodeling - the output error cases," *Automatica*, vol. 39, no. 10, Oct. 2003.

[10] F. Tjärnström and L. Ljung, "L-2 model reduction and variance reduction," *Automatica*, vol. 38, no. 9, pp. 1517–1530, Sep. 2002.

[11] ——, "L2 model reduction and variance reduction," Department of Electrical Engineering, Linköping University, SE-581 83 Linköping, Sweden, Tech. Rep. LiTH-ISY-R-2801, Jun. 2007.

[12] A. Vandendorpe and P. Van Dooren, "On model reduction of interconnected systems," in *Proceedings International Symposium Math. Th. Netw. Syst.*, Belgium, 2004.

[13] ——, *Model Order Reduction: Theory, Research Aspects and Applications*, ser. Mathematics in Industry Series. Springer Verlag, 2007, ch. Model reduction of interconnected systems.

[14] B. Wahlberg, "Model reduction of high-order estimated models: The asymptotic ML approach," *International Journal of Control*, Jan. 1989.

[15] B. Wahlberg, H. Hjalmarsson, and J. Mårtensson, "Variance analysis for identification of cascade systems," in *Proceedings of the 47th IEEE Conference on Decision and Control*, 2008, to appear.

[16] ——, "Variance results for identification of cascade systems," *Automatica*, 2008, provisionally accepted.

[17] B. Wahlberg, M. Jansson, T. Matsko, and M. Molander, *Experiences from Subspace System Identification - Comments from Process Industry Users and Researchers*. Modeling, Estimation and Control, Lecture Notes in Information and Control Sciences, Springer, Sep. 2007, pp. 315–327.

[18] B. Wahlberg, "On model reduction in system identification," in *Proc. 1986 American Control Conference*, Jan. 1986.

[19] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, New Jersey: Prentice Hall, 1996.