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From passivity under sampling to a new discrete-time passivity concept

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***Abstract*— The concept of "average passivity" is introduced making use of the Differential Difference Representation (DDR) of nonlinear discrete-time dynamics. It gives a first insight towards the introduction of a passivity notion which is equivalento to the continuous-time criterium when applied under sampling.**

I. INTRODUCTION

Robust control strategies based on passivity properties or more in general dissipativity concepts [17], [6] are widely investigated from theory to practice in terms of Lyapunov design or H_∞ control as many other efficient approaches for capturing and respecting the physical structure of the process (see for example [3], [15], [2] and the references therein). Specialized studies were developed in discrete time [7], [4], [10], [13], [14] where additional difficulties occur due to generic nonlinearity in the control variable of the dissipation inequalities. The notion of passivity itself deserves a deeper analysis in discrete time. In particular, in a sampled data context, the study is further complicated since the sampled equivalent model of a passive continuous-time plant does not satisfy a standard discrete-time dissipation inequality. Such a pathology reflects into the fact that eventhough some dissipation inequality is preserved see [9], [16], standard discrete-time passivity of the sampled model is lost. How do evolve dissipation inequalities under sampling needs specific attention as it directly affects the digital redesign.

In a linear context, it has been shown in [5], that passivity under zero-order-holding sampling device is maintained with respect to a "modified" output matrix. Such a result is presently generalized to nonlinear input-affine dynamics with respect to a "modified" output matrix. Then, such a "modified output mapping" is interpreted as the average of the "true output mapping" over the sampling time so providing an interesting physical meaning. This is made possible by considering the equivalent representation of sampled dynamics as two coupled differential/difference equations. Arguing so, it becomes possible to define passivity concepts for nonlinear discrete-time dynamics without direct input-output link. Average dissipativity concepts are so introduced for nonlinear discrete-time dynamics in their differential difference representations - DDR- [11]. The case of linear time invariant dynamics is studied as an example. The paper is organized as follows. Section II recalls dissipativity concepts and criteria for continuous-time input-affine systems Σ_c. On these bases, the studied problem is described. Assuming Σ_c dissipative with supply rate $< u, y >$ and storage function V, is its sampled equivalent dynamics dissipative?, with respect to what output mapping ? what supply rate? what storage function?. In section III, after recalling the differential/difference representation of discrete-time dynamics specialized for dynamics under sampling we describe the "modified" output mapping with respect to which passivity under sampling is preserved at the the sampling instants. The case of linear systems studied in [5] is recovered. In section IV, a novel average dissipativity notion is introduced for nonlinear discrete-time dynamics in their DDR. Sufficient conditions are described through KYP-type properties. The case of linear time invariant systems is discussed as an example so providing in the discrete-time linear context too novel concepts of average dissipativity, average passivity or average positive realness. The main contribution in Section V says that under sampling continuous-time passivity is transformed into average passivity over each sampling time. For, we interpret the "modified " output as the average with respect to the input signal of the "real" output mapping. Dissipation inequalities under sampling are described. These definitions are applied to the elementary RC circuit.

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II. PROBLEM SETTLEMENT AND SOME RECALLS

The continuous-time case - In this paper, we consider single input-affine dynamics Σ_c over $X = R^n$

$$\dot{x} \quad = \quad f(x) + u(t)g(x) \tag{1}$$

with output mapping $y = h(x)$. The set of admissible inputs consists of all U-valued piecewise continuous functions defined on R, f and g are smooth (i.e. C^∞) vector fields and h is a smooth mapping. Without loss of generality we assume $f(0) = 0$ and $h(0) = 0$.

We review some basic concepts related to the notions of passivity and dissipativity (see [17], [6], [3], [15], [2] for further details). Let w be a real-valued function on $U \times Y$, called the *supply rate*, we assume that for any $u \in \mathcal{U}$ and for any $x_0 \in X$, the output $y(t)$ of Σ_c is such that

$$\int_0^t |w(y(s), u(s))|ds < \infty \quad \text{for all } t \geq 0. \tag{2}$$

Definition 2.1: Σ_c with supply rate w is said to be *dissipative* if there exists a C^0 nonnegative function $V : X \to R$, called the *storage function* which satisfies $V(0) = 0$ such that for all $u \in \mathcal{U}$, $x_0 \in X$, $t \geq 0$

$$V(x(t)) - V(x_0) \leq \int_0^t w(y(s), u(s))ds \tag{3}$$

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with $x(t) = \Phi(t, x_0, u)$.

The last inequality is called the *dissipation inequality*. Dissipative systems with supply rate given by the inner product

$$w(y,u) =< u, y >= y^T u \quad (4)$$

where the super script ^{T} denotes transpose are *passive systems* satisfying

$$V(x(t)) - V(x_0) \leq \int_0^t y^T(s)u(s)ds. \quad (5)$$

Definition 2.2: Σ_c with supply rate w is said to be *lossless* if for all $u \in \mathcal{U}$, $x_0 \in X$, $t \geq 0$

$$V(x(t)) - V(x_0) = \int_0^t w(y(s), u(s))ds. \quad (6)$$

Definition 2.3: Σ_c is *positive real* if for all $u \in \mathcal{U}$, $t \geq 0$

$$0 \leq \int_0^t y^T(s)u(s)ds \quad (7)$$

whenever $x_0 = 0$.

Passivity can be characterized in terms of the Kalman-Yakubovitch-Popov property [3]: there exists a C^1-nonnegative function $V : X \to R$ with $V(0) = 0$ s.t. $\forall x \in X$

$$\mathrm{L}_f V(x) \leq 0;\ \mathrm{L}_g V(x) = h^T(x). \quad (8)$$

Proposition 2.1: If Σ_c satisfies the KYP property, then it is passive with storage function V. Conversely, a passive system having a C^1 storage function has the KYP property. We note that the KYP property can be interpreted as the infinitesimal version of the dissipation inequality (6).

The discrete-time case - Considering now a discrete-time dynamics Σ_d in the form of a map

$$x_k \to x_{k+1} \quad = \quad F(x_k, u_k) \quad (9)$$

the definition below is usual.

Definition 2.4: Σ_d with output mapping $y = h(x)$ is passive if there exists a C^0 nonnegative function $V : X \to R$, which satisfies $V(0) = 0$, such that for all $u_k \in \mathcal{U}_d$, all $x_k \in X$

$$V(x_{k+1}) - V(x_k) \leq y_k^T u_k \quad (10)$$

or equivalently

$$V(x_N) - V(x_0) \leq \sum_{i=0}^{N} y_i^T u_i \quad (11)$$

for all $N \geq 0$, $u_i \in \mathcal{U}_d$; i.e the stored energy along any trajectory from x_0 to x_N does not exceed the supply. Accordingly, one defines *discrete-time losslessness* replacing the inequality (10) by an equality and *discrete-time positive realness* when $y_k^T u_k \geq 0$.

Remark. From (11), a passive system with a positive definite storage function is Lyapunov stable. Reciprocally, V is not increasing along trajectories such that $y_k = 0$. Recalling that such a constraint defines the zero dynamics, one deduces that a passive system with a positive storage function V has a Lyapunov stable zero dynamics. KYP-type properties in discrete-time have been described in [7], [4], [10].

Some notations - In the sequel, $e^f = 1 + \sum_{i\geq 1} \frac{\mathrm{L}_f^i}{i!}$, indicates the operator Lie series associated with a smooth vector field f on X regarded as the Lie derivative $\mathrm{L}_f = \sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i}$, 1 indicates the identity operator and I_n the identity function on R^n; for any smooth real valued function h, the following result holds $e^f h(x) = h(e^f x)$.

III. PASSIVITY UNDER SAMPLING

Assuming some dissipation inequality as in (3) and the control piecewise constant, we investigate the possible preservation under sampling of such inequality. More precisely, assuming Σ_c dissipative with supply rate (4) and computing its sampled equivalent model, for $u(t)$ constant over time intervals of length $\delta \in]0, T^*]$, a finite interval ($u_k \in \mathcal{U}_d$), the nonlinear difference equation

$$x_k \to x_{k+1} = e^{\delta(f+u_k g)} x_k = F^\delta(x_k, u_k) \quad (12)$$

with output mapping $y = h(x)$ describes Σ^δ, the *sampled equivalent model* to (1) in the form of a map; i.e. the state evolutions (resp. output evolutions) of (12) and (1) coincide at the sampling instants $t = k\delta$ ($k \geq 0$), under constant input $u(t) = u_k$ for $t \in [k\delta, (k+1)\delta[$ for the same initial condition x_0.

A. The DDR of sampled dynamics

In place of a map (9), an alternative state space representation of discrete-time dynamics as two coupled differential/difference equations modeling the free and controlled dynamics respectively has been proposed in [11]. The usefulness of such a representation has been discussed in several papers putting in light how to characterize structural and control properties in discrete time in a format comparable to their differential geometric formulations usual in a continuous-time setting. These analogies are further employed when sampled dynamics are investigated. In this case, the DDR associated with a sampled dynamics described in the form of a map exists and is uniquely defined due to the invertibility of the drift term: for sufficiently small δ ensuring series convergence, (12) is drift invertible $F^\delta(x, 0) = e^{\delta f}x; (F^\delta)^{-1}(x, 0) = e^{-\delta f}x = F^{-\delta}(x, 0)$ and so is $F^\delta(x, u)$ (with inverse $F^{-\delta}(x, u)$), for $u \in \mathcal{U}_d$, a neighborhood of 0. It follows that, equivalently to (12), the sampled dynamics Σ^δ can be described by two coupled equations [12]; i.e. for all $\tau \in]0, \delta]$

$$x^+ \quad = \quad e^{\tau f} x; \quad x^+(0) = x^+ \quad (13)$$

$$\frac{dx^+(\tau u)}{d(\tau u)} \quad = \quad G^\tau(x^+(\tau u), \tau u) \quad (14)$$

setting by definition

$$G^\tau(., \tau u) := \frac{dF^\tau(., \tau u)}{d\tau u}|_{F^{-\tau}(.,\tau u)} \quad (15)$$

$$= \quad \frac{1}{\tau}\int_0^\tau e^{-s ad_{f+ug}} g ds = G_1^\tau + \sum_{i\geq 1} \frac{(\tau u)^i}{i!} G_{i+1}^\tau.$$

Given $x_k \in X$ and $u_k \in \mathcal{U}_d$, integrating (14) between 0 and δu_k for the initial state condition $x^+(0)$ specified by (13), $x^+(0) = x^+ = e^{\delta f} x_k$, one recovers (12) at time $t = (k+1)\delta$

$$x_{k+1} := x^+(\delta u_k) = x^+(0) + \int_0^{\delta u_k} G^\delta(x^+(v), v)dv \quad (16)$$

and identically for any smooth output mapping

$$\begin{aligned} h(x_{k+1}) &:= h(x^+(\delta u_k)) & (17) \\ &= h(x^+(0)) + \int_0^{\delta u_k} \mathrm{L}_{G^\delta(.,v)} h(x^+(v))dv. \end{aligned}$$

Combinatoric relations between these two representations of the sampled equivalent Σ^δ to Σ_c are in [12].

B. Usual passivity under sampling

From (5) written at time $t = (k+1)\delta$ for $x_0 = x_k$ and $u(\tau) = u_k; \tau \in [k\delta, (k+1)\delta[$, it follows that if Σ_c passive with storage function V, then for all $\delta \in]0, T^*]$, its sampled equivalent dynamics Σ^δ satisfies the dissipation inequality

$$V(x_{k+1}) - V(x_k) \leq (\int_0^\delta y^T(s)ds)u_k \quad (18)$$

for all $u_k \in \mathcal{U}$, $x_k \in X$, $k \geq 0$.
It is possible to interpret (18) as the discrete-time dissipation inequality of the form (10) for some "modified output mapping". For, set $H^\delta(x_k, u_k)$

$$H^\delta(x_k, u_k) := \int_0^\delta y^T(s)ds = \int_0^\delta e^{s(f+u_k g)} h(x_k)ds \quad (19)$$

$$= \int_0^\delta e^{sf} h(x_k)ds + u_k \int_0^\delta \Big(\int_0^s \mathrm{L}_{G^s(.,\tau u_k)} h(x^+(\tau u_k))d\tau \Big)ds.$$

Theorem 3.1: Assuming Σ_c with output mapping $y = h(x)$ passive (resp. lossless or positive real) with storage function V, then for all $\delta \in]0, T^*]$, $u_k \in \mathcal{U}_d$, its sampled equivalent dynamics Σ^δ with output mapping $H^\delta(., u_k)$ defined in (19) is discrete-time passive (resp. lossless or positive real) with the same storage function V.

Proof: It must be shown that the discrete-time dissipation inequality holds for all $u_k \in \mathcal{U}_d$, $x_k \in X$, $k \geq 0$

$$V(x_{k+1}) - V(x_k) \leq < u_k, H^\delta(x_k, u_k) > \quad (20)$$

For, it is sufficient to rewrite $\int_0^\delta y(s)ds$ in (18) as $\int_0^\delta h(x^+(su_k))ds$ with

$$x^+(su_k) = e^{s(f+u_k g)} x_k = e^{sf} x_k + \int_0^s G^s(x^+(\tau u_k), \tau u_k)d\tau.$$

■

Remark. The result can be extended to system Σ_c with with direct input-output link by considering $\bar{y} = \bar{h}(x, u)$ in place of $h(x)$. in such a case (19) generalizes as

$$\bar{H}^\delta(x_k, u_k) := \int_0^\delta e^{s(f+u_k g)} \bar{h}(x_k, u_k)ds =$$

$$\int_0^\delta e^{sf} \bar{h}(x_k, u_k)ds + u_k \int_0^\delta (\int_0^s \mathrm{L}_{G^s(.,\tau u_k)} h(x^+(\tau u_k), u_k)d\tau)ds.$$

C. The linear case as an example

Let (A, B, C) be the minimal realization of a linear time invariant - LTI - continuous-time system on R^n

$$\dot{x} = Ax + Bu; \qquad y = Cx$$

assumed passive with quadratic $V = \frac{1}{2}x^T P x$ ($P \geq 0$ a symmetric positive matrix); i.e. $\dot{V} \leq y^T u$.
Under zero-order holding device, the sampled equivalent $(A^\delta := e^{\delta A}, B^\delta := \int_0^\delta e^{\tau A} B d\tau, C)$ to (A, B, C) does not maintain passivity; i.e. the equivalent discrete-time dissipation inequality $\bar{V}(x_{k+1}) - \bar{V}(x_k) \leq y_k^T u_k$ does not hold true for $\bar{V} = V$; i.e. passivity is lost under sampling. A major obstruction is the lack of direct input-output link. To overcome this problem, it has been proposed in [5] to modify the output mapping of the sampled equivalent model so that to get preservation of the dissipativity inequality as well as preservation of the continuous-time energy at the sampling instants. More precisely, the LTI sampled system $(A^\delta, B^\delta, C^\delta, D^\delta)$, with output mapping

$$y^\delta(x_k, u_k) = C^\delta x_k + D^\delta u_k \quad (21)$$

with $C^\delta = C\int_0^\delta e^{\tau A} d\tau$ and $D^\delta = C\int_0^\delta B^\tau d\tau$ satisfies at the sampling instants the dissipation inequality

$$V(x_{k+1}) - V(x_k) \leq < u_k, y^\delta(x_k, u_k) >$$

so recovering the linear version of (19). The same holds true when considering linear systems with direct input-output link.
In the sequel we show that the right member of (20) can be interpreted as the supply rate associated with a certain average output mapping directly deduced from the adopted DDR. For, *average dissipativity* is introduced.

IV. A NOVEL AVERAGE PASSIVITY IN DISCRETE-TIME

As above specialized to the sampled case, under some mild conditions (invertibility of the drift or invertibility of $f(x, \bar{u})$ for some $\bar{u} \in \mathcal{U}_d$), any nonlinear difference equation (12) can be represented as two coupled difference and differential equations - DDR - (see [11] for further details).

$$x^+ = F_0(x); \quad x^+(0) = x^+ \quad (22)$$

$$\frac{dx^+(u)}{du} = G(x^+(u), u) \quad (23)$$

so getting for all $u_k \in \mathcal{U}_d$

$$x_{k+1} := x^+(u_k) = x^+(0) + \int_0^{u_k} G(x^+(v), v)dv \quad (24)$$

and for any mapping $h : X \to R$

$$y_{k+1} = h(x^+(u_k)) := h(x^+(0)) + \int_0^{u_k} \mathrm{L}_{G(.,v)} h(x^+(v))dv. \quad (25)$$

In this framework, a discrete-time system is described by (22-23) and the output mapping $y^+(u) = h(x^+(u))$. It has to be stressed that such a representation induces u-dependency of the output mapping through u-dependency of the state dynamics at the basis of the definitions introduced below.

Definition 4.1: Given Σ_d with output mapping $y(u) = h(x^+(u))$ then, for any fixed $v_d \in \mathcal{U}_d$, $y_{av}(v_d)$ denotes the *average output mapping* over $]0, v_d]$ of $y(u)$ defined as

$$y_{av}(v_d) := \frac{1}{v_d}\int_0^{v_d} h(x^+(v))dv. \qquad (26)$$

It is now possible to enounce standard dissipativity concepts making reference to such an an average output mapping so introducing the "average dissipativity " notions.

Definition 4.2: Σ_d is said to be *average passive* if there exists a C^0 nonnegative function $V : X \to R$, called the *storage function*, such that for all $u_k \in \mathcal{U}_d$, $x_k \in X$, $k \geq 0$

$$V(x_{k+1}) - V(x_k) \leq y_{av}^T(u_k)u_k. \qquad (27)$$

Definition 4.3: Σ_d is said to be *average lossless* if for all $u_k \in \mathcal{U}_d$, $x_k \in X$, $k \geq 0$, the equality holds true

$$V(x_{k+1}) - V(x_k) = y_{av}^T(u_k)u_k. \qquad (28)$$

Definition 4.4: Σ_d is said to be *average positive real* if for all $u_k \in \mathcal{U}_d$, $x_k \in X$, $k \geq 0$

$$0 \leq \int_0^{u_k} y(v)dv = y_{av}^T(u_k)u_k \qquad (29)$$

whenever $x_0 = 0$.

The following Lemma is a direct consequence of the adopted state-space structure.

Lemma 4.1: Given Σ_d, (27) rewrites as

$$V(F_0(x_k)) - V(x_k) + \int_0^{u_k} \mathrm{L}_{G(.,v)}V(x^+(v))dv \leq \int_0^{u_k} h(x^+(v))dv. \qquad (30)$$

According to this, some sufficient KYP-type properties can be given to describe discrete-time average passivity.

Proposition 4.1: Given Σ_d, if there exists a C^1 nonnegative function $V : X \to R$ with $V(0) = 0$ such that for all $x \in X$

$$V(F_0(x)) - V(x) \leq 0 \qquad (31)$$

$$\mathrm{L}_{G_1}V(x) = h^T(x);\ \mathrm{L}_{G_i}V(x) = 0; \quad i \geq 2 \quad (32)$$

then Σ_d is average passive for any $u_k \in \mathcal{U}_d$ with storage V.

Proof: The result is a direct consequence of (30). (31) corresponds to setting $u_k = 0$ while (32) can be interpreted as the infinitesimal version of $\int_0^{u_k} \mathrm{L}_{G(.,v)}V(x^+(v))dv \leq \int_0^{u_k} h(x^+(v))dv$ rewritten as $u_k \int_0^1 \mathrm{L}_{G(.,su_k)}V(x^+(su_k))ds \leq u_k \int_0^1 h(x^+(su_k))ds$. ■

Regarding losslessness, the conditions are also necessary.

Proposition 4.2: Given Σ_d, if there exists a C^1nonnegative function $V : X \to R$ with $V(0) = 0$ such that for all $x \in X$, (32) hold true and

$$V(F_0(x)) - V(x) = 0 \qquad (33)$$

then Σ_d is average lossless for any $u_k \in \mathcal{U}_d$, with storage function V. Conversely, average losslessness of Σ_d with storage V implies (33-32).

These notions can be extended to systems with direct input-output link.

A. Systems with direct input link

Given a discrete-time system described by (22-23), consider now the output mapping $\bar{h}(., u) : X \times \mathcal{U} \to X$, so defining $\bar{y}^+(u) = \bar{h}(x^+(u), u$. The dissipation inequality rewrites as

$$V(x^+(u_k)) - V(x_k) \leq \bar{y}_{av}^T(u_k)u_k$$

or equivalently

$$V(F_0(x_k)) - V(x_k) + \int_0^{u_k} \mathrm{L}_{G(.,v)}V(x^+(v))dv \leq \int_0^{u_k} \bar{h}^T(x^+(v), v)dv. \qquad (34)$$

Proposition 4.1 and Proposition 4.2 generalize as follows.

Proposition 4.3: Given Σ_d with output mapping $\bar{y} = \bar{h}(x, u) = \bar{h}_1 + \sum_{i\geq 1} \frac{u^i}{i!}\bar{h}_i$, if there exists a C^1nonnegative function V with $V(0) = 0$ such that for all $x \in X$

$$V(F_0(x)) - V(x) \leq 0 \qquad (35)$$

$$\mathrm{L}_{G_i}V(x) = \bar{h}_i^T(x); \quad i \geq 1 \qquad (36)$$

then Σ_d is average passive with storage V.

Proposition 4.4: Given Σ_d with output mapping $\bar{y} = \bar{h}(x, u) = \bar{h}_1 + \sum_{i\geq 1} \frac{u^i}{i!}\bar{h}_i$, if there exists a C^1nonnegative function $V : X \to R$, with $V(0) = 0$, such that for all $x \in X$ (33-36) hold true, then Σ_d, is average lossless with storage function V. Conversely, a discrete-time average lossless system with C^1 storage V satisfies (33-36).

This analysis shows that average dissipativity concepts do correspond to standard dissipativity concepts provided one makes reference to the average output mapping defined as in (26). This provides a key tool to deal with systems without direct input-output link by setting any standard dissipativity based control property on the DDR and its average output making reference to dissipativity average notions. As an example, the interconnection of two average passive systems is still average passive through connection with the average output mapping.

B. The linear case as an example

Given a LTI dynamics with direct input-output link (A_d, B_d, C_d, D_d) in its DDR form

$$x^+ = A_d x; \quad \frac{dx^+(u)}{du} = B_d; \quad \bar{y} = C_d x^+(u) + D_d u$$

with average output

$$\bar{y}_{av}^d(u_k) = C_d A_d x_k + \frac{1}{2}(C_d B_d + D_d)u_k$$

let us characterize *discrete-time average passivity.*

Theorem 4.1: Let (A_d, B_d, C_d, D_d) the minimal realization of a LTI system, the matrix inequality

$$\begin{pmatrix} A_d^T P A_d - P & A_d^T P B_d - A_d^T C_d^T \\ B_d^T P A_d - C_d A_d & -D_d - C_d B_d + B_d^T P B_d \end{pmatrix} \leq 0$$

has a solution for a symmetric matrix $P \geq 0$ iff (A_d, B_d, C_d, D_d) is average passive with supply rate $V(x) = \frac{1}{2}x^T P x$.

V. SAMPLED AVERAGE PASSIVITY

Let us now specialize these notions to the sampled case. Let Σ_c and for all $\delta \in]0, T^*]$, let Σ^δ its sampled equivalent dynamics defined in (13-14). The following result interprets the right hand sides of (18) or (20) as the supply rate associated with the average output mapping.

Proposition 5.1: Assuming Σ_c passive with storage function V, then for all $\delta \in]0, T^*]$, its sampled equivalent dynamics Σ^δ is *sampled average passive* with storage V; i.e.

$$V(x_{k+1}) - V(x_k) \leq < \delta u_k, y_{av}(\delta u_k > \tag{37}$$

for all $u_k \in \mathcal{U}$, $x_k \in X$, $k \geq 0$ with $y_{av}(\delta u_k) := \frac{1}{\delta u_k}\int_0^{\delta u_k} h(x^+(v))dv$.

Proof: A simple calculus shows that

$$\begin{aligned} \int_0^\delta y^T(s)ds u_k &= \int_0^\delta (e^{s(f+u_k g)} h(x_k))^T u_k ds \\ = \int_0^\delta h^T(x^+(su_k))ds u_k &= \int_0^{\delta u_k} h^T(x^+(v))dv \\ = y_{av}^T(\delta u_k)\delta u_k. \end{aligned}$$

■

Remark. According to such a definition $H^\delta(x_k, u_k) = \int_0^\delta h(x^+(\sigma u_k)d\sigma = \delta y_{av}(\delta u_k)$ and the continuous-time energy function is preserved at the sampling instants.

These concepts yield to the achieved results below.

Theorem 5.1: Assuming Σ_c passive (resp. lossless or positive real) with storage function V, then for all $\delta \in]0, T^*]$, its sampled equivalent dynamics Σ^δ is *sampled average passive (resp. sampled average lossless or sampled average positive real)* with the same energy function at the sampling instants.

Remark. The terminology *sampled average passivity* emphasizes that both the drift and the controlled vector field of sampled dynamics Σ^δ are τ-dependent over $]0, \delta]$ while they are computed for a given fixed sampling time δ when interpreted as discrete-time maps.

A. Systems with direct input-output link

The analysis can be generalized to output mapping of the form $\bar{y} = \bar{h}(x, u)$ transformed under constant control into $\bar{h}(x, u_k)$ with average output over time interval of length δ given by $\bar{y}_{av}(\delta u_k) := \frac{1}{\delta u_k}\int_0^{\delta u_k} \bar{h}(x^+(v), u_k)dv$.

Proposition 5.2: Assuming Σ_c passive with output mapping $\bar{y} = \bar{h}(x, u)$ and storage V, then for all $\delta \in]0, T^*]$, its sampled equivalent dynamics Σ^δ is *sampled average passive* with storage V and supply rate $< \delta u_k, \bar{y}_{av}(\delta u_k >$.

Proof: A simple calculus shows that

$$\begin{aligned} \int_0^\delta \bar{h}^T(x(s), u_k)ds u_k &= \int_0^\delta (e^{s(f+u_k g)}\bar{h}(x_k, u_k))^T ds u_k \\ = \Big(\int_0^\delta \bar{h}^T(x^+(su_k), u_k)ds\Big)u_k &= \int_0^{\delta u_k} \bar{h}^T(x^+(v), u_k)dv \\ = \bar{y}_{av}^T(\delta u_k)\delta u_k. \end{aligned}$$

■

Remark. We note that when dynamics under zero-order sampling are considered, the continuous-time output mapping $\bar{h}(x, u)$ is transformed into $\bar{h}(x, u_k)$ so getting a dissipation inequality which differs from (34) where the direct input-output link affects the average.

B. The linear case as an example

Let (A, B, C) be the minimal realization of a LTI continuous-time system on R^n assumed passive with quadratic storage $V = \frac{1}{2}x^T Px$ ($P \geq 0$ a symmetric positive matrix). The DDR over $]0, \delta]$ of its sampled equivalent (A^δ, B^δ, C) under zero-order-holding device is given by

$$x^+ = A^\tau x; \quad \frac{dx^+(\tau u)}{d(\tau u)} = \frac{1}{\tau}B^\tau; \quad y(\tau u_k) = Cx^+(\tau u_k)$$

It is a matter of computations to verify that the supply rates associated with the output map (21) proposed in [5] for a fixed given u_k and the average over $]0, \delta u_k]$ of $y(\tau u_k)$ coincide; i.e.

$$< u_k, y^\delta(x_k, u_k) > = < \delta u_k, y_{av}(\delta u_k) > . \tag{38}$$

In fact, an easy calculus shows that $\delta y_{av}(\delta u_k) = C^\delta x_k + D^\delta u_k$ because, for any $\tau \in]0, \delta]$

$$Cx^+(\tau u_k) = CA^\tau x_k + C\int_0^{\tau u_k} \frac{1}{\tau}B^\tau dv = CA^\tau x_k + CB^\tau u_k.$$

When considering $\bar{y} = Cx + Du_k$, one has

$$\begin{aligned} \delta\bar{y}_{av}(\delta u_k) &:= \frac{1}{u_k}\int_0^{\delta u_k} (Cx^+(v) + Du_k)dv \\ &= C\int_0^\delta A^\tau x_k d\tau + u_k C\int_0^\delta B^\tau d\tau + \delta Du_k \end{aligned}$$

so recovering the result in [5] presently rephrased in terms of average output mapping; i.e.

$$\begin{aligned} &< \delta u_k, \bar{y}_{av}(\delta u_k) > = < u_k, C\int_0^\delta A^\tau x_k d\tau > \\ &+ < u_k, C\int_0^\delta B^\tau d\tau u_k > + < u_k, \delta Du_k > . \end{aligned}$$

C. Counter clockwise - CCW - input-output dynamics

In [1], a new input-output property, the counter clockwise input-output dynamics - CCW - is proposed for the study of positive feedback interconnections in both linear and nonlinear contexts. Reinterpreting the CCW property as the classical definition of passivity with respect to the derivative of the output mapping, the following definition can be set.

Definition 5.1: Σ_c is CCW if it is passive with respect to the inner product $< u(t), \dot{y}(t) >$; i.e. for any pair (x, u) so that the output mapping is differentiable, there exists a C^∞ nonnegative function $V : X \to R$ with $V(0) = 0$ such that

$$V(x(t)) - V(x_0) \leq \int_0^t \dot{y}^T(s)u(s)ds. \tag{39}$$

It is a matter of computation to verify that under sampling, the dissipation inequality (39) specializes for any $u_k \in \mathcal{U}_d$, $x_k \in X$, $k \geq 0$ as

$$V(x_{k+1}) - V(x_k) \leq u_k(y_{k+1} - y_k) \tag{40}$$

so providing a dissipativity notion which makes sense in the discrete-time context as it does not require a direct input-output link. The following result holds in a linear context.

Theorem 5.2: Assuming (A, B, C) a minimal realization of a LTI continuous-time system passive with respect to $< u, \dot{y} >=< u, CAx + CBu >$ - equivalently CCW - , then its sampled equivalent is passive with respect to the inner product $< u_k, CA^\delta x_k - Cx_k + CB^\delta u_k >$ with the same energy function at the sampling instants and supply rate $V(x) = \frac{1}{2}x^T Px$; i. e. the matrix inequality

$$\begin{pmatrix} (A^\delta)^T PA^\delta - P & (A^\delta)^T PB^\delta - (A^\delta)^T C^T + C^T \\ (B^\delta)^T PA^\delta - CA^\delta + C & -2CB^\delta + (B^\delta)^T PB^\delta \end{pmatrix} \leq 0$$

has a solution for a symmetric matrix $P \geq 0$.

It is interesting to compare the supply rate $< u_k, CA^\delta x_k - Cx_k + CB^\delta u_k >$ in Theorem 5.2 with the supply rate $< u_k, C^\delta x_k + D^\delta u_k >$ in (21) and the supply rate $< u_k, CA^\delta x_k + \frac{1}{2}CB^\delta u_k >$ setting $A_d = A^\delta, B_d = B^\delta, C_d = C$ in Theorem 4.1.

D. A linear simulated example

We refer to the simple RC series connection with capacity voltage V_{out} as output, to compare sampled average passivity, average passivity and the CCW property with the continuous-time and discrete-time criteria. One has

$$\dot{V}_{out} = -1/RCV_{out} + 1/RCV_{in}$$

with equivalent sampled dynamics

$$V_{k+1 out} = e^{-\delta/RC} V_{k out} + (1 - e^{-\delta/RC}) V_{k in}$$

Under storage function $\vartheta = \frac{1}{2}(V_{cap})^2 RC$, one computes

$$\vartheta(x(t)) - \vartheta(x_0) \leq \int_0^t u(s)y(s)ds.$$

so getting under sampling

$$\vartheta(x_{k+1}) - \vartheta(x_k) \leq u_k \int_0^\delta y(s)ds$$
$$\delta y_{av}(\delta u_k) = \delta u_k e^{-\delta/RC} + RC(1 - e^{-\delta/RC})(y_k - u_k)$$
$$y_{av}^d(x_k, u_k) = e^{-\delta/RC} y_k + \frac{1}{2}(1 - e^{-\delta/RC})u_k.$$

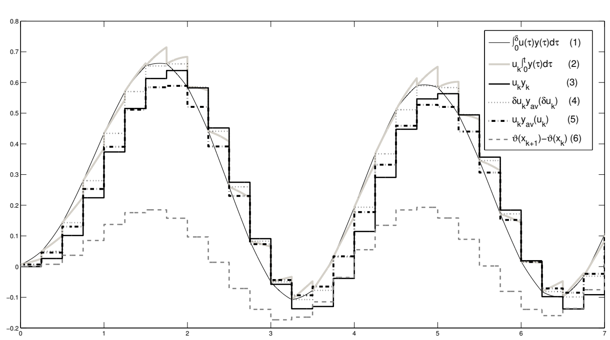

Fig. 1. RC Circuit Supply Rates and Storage Function

Figure 1 illustrates the performances of average dissipativity concepts for preserving passivity under sampling with the same storage function. While the usual passivity inequality is lost (6-3), the discrete-time average output and the sampled average output are associated with supply rates which respect this inequality, (4)-(5).

VI. CONCLUSIONS

The notion of average passivity has been introduced in discrete time; it has been shown to restitute the continuous-time criterium when applied under sampling. Its use for discrete-time design might be investigated.

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