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Delayed Input and State Observability

Carlos Bielsa Campos and Harish J. Palanthandalam-Madapusi

***Abstract*— This paper considers the concept of input and state observability, that is, conditions under which both the unknown input and initial state of a known model can be determined from output measurements. We extend previous results on input and state observability for discrete-time systems by considering a more general framework. Particularly, we introduce a delay in the input estimation, that is, we consider estimation of delayed input instead of the current input. Furthermore, we derive results for cases in which partial inputs may be available. Finally, we present several illustrative examples, including a system in which partial inputs are available.**

I. Introduction

Systems with unknown inputs have received considerable attention [4–26, 28–31]. The unknown inputs may represent unknown external drivers, input uncertainty, or instrument faults. An active research area is state reconstruction with known model equations and unknown inputs. Approaches include full-order observers [6, 8, 11, 18, 19, 31], reduced-order observers [9, 10, 22, 24], geometric techniques [4], and trial-and-error methods [29]. A widely used approach is to model the unknown inputs as outputs of a known dynamic system and incorporate the input dynamics with the plant dynamics [1, 16]. However, this approach increases the dimension of the observer and is limited to specific types of inputs.

In [26, 28] input reconstruction is achieved by inverting the known transfer function. More recently, methods for input reconstruction using optimal filters are developed in [6, 11, 13, 17, 30]. Finally, [12, 28] considers input reconstruction with an inherent delay.

A related problem is the concept of input and state observability, which is the ability to reconstruct the inputs and states using only output measurements. Necessary and sufficient conditions for input and state observability for continuous-time systems in terms of the invariant zeros of the system are presented in [6, 10, 15, 17, 22]. Input and state observability for discrete-time systems is considered in [17, 27], while [11] uses a constructive algorithm to determine the observability of the unknown input and state.

In this paper, we examine conditions under which both the input and state can be estimated from the output measurements, under a more general framework than in [27]. Specifically, we consider using all avaliable output measurements to estimate the state and all inputs except the inputs in the last L time steps. This approach differs from the approach of [12] where a bank of future outputs is used to estimate the current input.

The authors are with the Department of Mechanical and Aerospace Engineering, Syracuse University, Syracuse, NY 13244, {cbielsac,hjpalant}@syr.edu

We discuss necessary and sufficient conditions for a discrete-time system to be L-delay input and state observable and derive tests for the same. This approach includes the results of [27] as a special case. Furthermore, since no assumptions on the input are made, the unknown input can be either an unmodeled exogenous signal or a consequence of an unknown endogenous nonlinear function of the states.

Next, we derive conditions under which the state and the inputs can be estimated from output measurements, when the inputs for a brief period of time-steps are known. This class of systems are referred to as M-delay input and state estimable systems, and constitute a broader class of systems than L-delay input and state observable systems. These results are especially useful for applications in which partial input information is available, such as fault detection or systems with a brief sensor failure.

Finally, we present several illustrative examples. For a linear example with a known model and an unknown exogenous input, we estimate the unknown input based on noisy output measurements. Furthermore, we present examples to illustrate the fact that the class of systems that are L-delay input and state observable is much broader than the class of system considered in [12, 27]. Finally, we consider an example in which partial input information is available. We first use the results derived in the current paper to estimate the unknown inputs, and then we subsequently estimate the states using a Kalman filter.

II. L-delay Input and State Observability

Consider the system

$$x_{k+1} = Ax_k + He_k, \tag{II.1}$$

$$y_k = Cx_k + Ge_k, \tag{II.2}$$

where $x_k \in \mathbb{R}^n$, $e_k \in \mathbb{R}^p$, $y_k \in \mathbb{R}^l$, $A \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{l\times n}$, and $G \in \mathbb{R}^{l\times p}$. Without loss of generality, we assume $l \le n$, $\text{rank}(C) = l > 0$, and $\text{rank}\begin{bmatrix} H \\ G \end{bmatrix} = p > 0$. No assumptions on the unmeasured signal e_k are made. Hence, e_k can be either an exogenous input or a consequence of nonlinear, time-varying function of the states.

Throughout this paper, r denotes a nonnegative integer. Furthermore, for convenience, every vector or matrix with zero rows or zero columns is an empty matrix, and $0_{s\times t}$ is the zero matrix with s rows and t columns. Let span P denote the space spanned by the columns of matrix P, and let $(P)_s$ denote the first s rows of matrix P. Define $\mathcal{Y}_r \in \mathbb{R}^{(r+1)l}$ and

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$\mathcal{E}_r \in \mathbb{R}^{(r+1)p}$ as

$$\mathcal{Y}_r \triangleq \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_r \end{bmatrix}, \quad \mathcal{E}_r \triangleq \begin{bmatrix} e_0 \\ e_1 \\ \vdots \\ e_r \end{bmatrix}. \tag{II.3}$$

Definition II.1. Let $r \geq 1$. Then the *input and state unobservable subspace* $\mathfrak{U}_r$ of (II.1), (II.2) is the subspace

$$\mathfrak{U}_r \triangleq \left\{ \begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix} \in \mathbb{R}^{n+(r+1)p} \colon \mathcal{Y}_r = 0 \right\}. \tag{II.4}$$

We define $\Gamma_r \in \mathbb{R}^{(r+1)l\times n}$, $M_r \in \mathbb{R}^{(r+1)l\times(r+1)p}$, and $\Psi_r \in \mathbb{R}^{(r+1)l\times(n+(r+1)p)}$ by

$$\Gamma_r \triangleq \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^r \end{bmatrix},$$

$$M_r \triangleq \begin{bmatrix} G & 0 & 0 & \cdots & 0 \\ CH & G & 0 & \cdots & 0 \\ CAH & CH & G & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{r-1}H & CA^{r-2}H & CA^{r-3}H & \cdots & G \end{bmatrix}, \tag{II.5}$$

and

$$\Psi_r \triangleq \begin{bmatrix} \Gamma_r & M_r \end{bmatrix}. \tag{II.6}$$

Note that $M_0 = G$, $\Gamma_0 = C$ and thus $\Psi_0 = \begin{bmatrix} C & G \end{bmatrix}$. Next, from (II.1), (II.2), we can write

$$\mathcal{Y}_r = \Gamma_r x_0 + M_r \mathcal{E}_r = \Psi_r \begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix}, \tag{II.7}$$

so that

$$\mathfrak{U}_r = \mathcal{N}(\Psi_r). \tag{II.8}$$

Let $\mathcal{R}(\Psi_r)$ denote the range space of matrix Ψ_r. We then have the following definition.

Definition II.2. The system (II.1), (II.2) is *L-delay input and state observable* if there exists $r_0 \geq 1$ such that $\mathfrak{U}_r \subseteq \mathcal{R} \begin{bmatrix} 0_{n+(r-L+1)l\times Lp} \\ I_{Lp} \end{bmatrix}$ for all $r \geq r_0$.

Definition II.2 implies that if (II.1), (II.2) is L-delay input and state observable, then, for all $r \geq r_0$, the initial condition x_0 and input sequence $\{e_i\}_{i=0}^{r-L}$ are uniquely determined from the measured output sequence $\{y_i\}_{i=0}^{r}$.

Definition II.3. The system (II.1), (II.2) is *input and state observable* if (II.1), (II.2) is 0-delay input and state observable.

If (II.1), (II.2) is input and state observable, then the initial state and inputs for all time steps can be estimated from the output measurements.

Let $\Psi^\dagger$ represents the Moore-Penrose generalized inverse of Ψ, $\Psi_r^\dagger = (\Psi_r^{\mathrm{T}}\Psi_r)^{-1}\Psi_r^{\mathrm{T}}$. Then, the following result gives necessary and sufficient conditions for (II.1), (II.1) to be L-delay input and state observable.

Theorem II.1. The following statements are equivalent:

$i)$ (II.1), (II.2) is L-delay input and state observable.

$ii)$ $\mathcal{N}(\Psi_r) \subseteq \mathcal{R} \begin{bmatrix} 0_{n+(r-L+1)l\times Lp} \\ I_{Lp} \end{bmatrix}$.

$iii)$ $\begin{bmatrix} x_0 \\ \mathcal{E}_{r-L} \end{bmatrix} = (\Psi_r^+ \mathcal{Y}_r)_{n+(r-L+1)p}$.

Proof. The proof of $i) \Leftrightarrow ii)$ follows from (II.8) and Definition II.2. Next, to show that $ii) \Leftrightarrow iii)$, from (II.7) and Proposition 6.1.7 in [2], it follows that

$$\begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix} = \Psi_r^+ \mathcal{Y}_r + (I - \Psi_r^+ \Psi_r) \begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix}. \tag{II.9}$$

Furthermore, noting that $\mathcal{R}(I - \Psi_r^+\Psi_r) = \mathcal{N}(\Psi_r)$, it follows from (II.9) that $iii)$ holds for all $\begin{bmatrix} x_0 \\ \mathcal{E}_r \end{bmatrix}$, if and only if $ii)$ holds. □

From Theorem II.1, it follows that if (II.1), (II.1) is L-delay input and state observable, then for $r \geq r_0$, the first $n + (r - L + 1)p$ columns of Ψ_r have full column rank. The first $n + (r - L + 1)p$ columns of Ψ_r is denoted by $\Psi_{r,n+(r-L+1)p}$. Finally, it follows from Theorem II.1 that if (II.1), (II.2) is L-delay input and state observable, then estimates of the states and inputs can be obtained as

$$\begin{bmatrix} x_0 \\ \mathcal{E}_{r-L} \end{bmatrix} = \left(\Psi_r^+ \mathcal{Y}_r\right)_{n+(r-L+1)p}. \tag{II.10}$$

The following result follows from the structure of matrix Ψ_r.

Proposition II.1. If

$$\mathrm{rank}(\Psi_{r_0,n+(r_0-L+1)p}) = \mathrm{rank}(\Psi_{r_0+1,n+(r_0+1-L+1)}) + p \tag{II.11}$$

and

$$\mathcal{N}(\Psi_{r_0}) \subseteq \mathcal{R} \begin{bmatrix} 0_{n+(r_0-L+1)p\times Lp} \\ I_{Lp} \end{bmatrix}, \tag{II.12}$$

then for all $r \geq r_0$,

$$\mathcal{N}(\Psi_r) \subseteq \mathcal{R} \begin{bmatrix} 0_{n+(r-L+1)l\times Lp} \\ I_{Lp} \end{bmatrix}. \tag{II.13}$$

The following result follows from Proposition II.1.

Corollary II.1. Let $p \leq l$. Then if (II.1), (II.2) is L-delay input and state observable, then (II.1), (II.2) is L^*-delay input and state observable for all $L^* \geq L$.

Corollary II.2. If $p \leq l$ and there exists $r_0 \geq 1$ such that $\mathcal{N}(\Psi_{r_0}) \subseteq \mathcal{R} \begin{bmatrix} 0_{n+(r_0-L+1)p\times Lp} \\ I_{Lp} \end{bmatrix}$, then (II.1), (II.2) is L-delay input and state observable.

However, a similar result for $p > l$ cannot be derived since (II.11) cannot hold for any r_0. Therefore, for $p > l$, the ability to estimate inputs and state from the output measurements depends on the particular r, and general results for L-delay input and state observability cannot be derived. Therefore, for the remainder of the paper, we assume that $p \leq l$.

The following two results describe necessary conditions for (II.1), (II.2) to be L-delay input and state observable. Proposition II.2 describes necessary conditions related to the rank of Ψ_r, while Proposition II.3 describes necessary conditions related to various dimensions.

Proposition II.2. If (II.1), (II.2) is L-delay input and state observable, then the following statements hold:

1) $\mathrm{rank}(\Psi_r) \geq n + (r - L + 1)p$.
2) $\mathrm{rank}(\Psi_{r,n+(r-L+1)p}) = n + (r - L + 1)p$.
3) (A, C) is observable, that is, $\mathrm{rank}(\Gamma_{n-1}) = n$.
4) $\mathrm{rank}(\Psi_r) = \mathrm{rank}(\Psi_{r-1}) + p$ for all $r \geq r_0$.

The following result follows from the fact that if $\Psi_{r,n+r-l+1}$ is full column rank, then the number of columns is less than or equal to the number of rows.

Proposition II.3. If (II.1), (II.2) is L-delay input and state observable, then the following statements hold:

1) $n + (r - L + 1)p \leq (r + 1)l$.
2) $L \geq \frac{n-r(l-p)-r}{p} + 1$.
3) If $p = l$, then $L \geq \frac{n}{p}$.
4) If $p = l = 1$, then $L \geq n$.
5) If $p < l$, then $r \geq \frac{n-(l-p)-Lp}{l-p}$.

From Proposition II.3, point 2, it follows that since $p \leq l$, when $p = l$, $L \geq n\partial$ is the largest lower bound for L. Moreover, if $L \geq n/p$, it follows from the size of $\Psi_{r,n+(r-L+1)p}$ that, if (II.1), (II.2) is L-delay input and state observable then $r_0 \geq$. Also, if a SISO system ($p = l = 1$) is L-delay input and state observable, then $L \geq n$.

For a lower bound on r_0, we note that to estimate x_0 and at least one unknown input e_0, we need $n+(r_0-L+1)p \geq n + p$. Therefore, $r_0 \geq L$.

Finally, the following result provides a test for L-delay input and state observability that is independent of r.

Proposition II.4. (II.1), (II.2) is L-delay input and state observable if and only if

$$\mathcal{N}(\Psi_n) \subseteq \mathcal{R} \begin{bmatrix} 0_{n+(n-L+1)p \times Lp} \\ I_{Lp} \end{bmatrix}. \tag{II.14}$$

Note that if no unknown inputs are present, that is, $p = 0$, then $\Psi_r = \Gamma_r$, and Theorem II.1, $ii)$ becomes the standard rank test for observability.

III. M-delay Input and State Estimability

In this section, we derive conditions under which the state and input sequence can be estimated when a part of the input sequence is known. Specifically, we consider (II.1), (II.2) with known outputs, and inputs known for the last last M time-steps. That is, y_k, $k = 0, 1, \ldots, r$, and e_k, $k = r - M+1, \ldots, r$ are known. If $M = 0$, this is the same as Input and state observability, throughout this section we assume $M > 0$.

First, we consider the case in which $e_k = 0$, $k = r - M + 1, \ldots, r$. In this case, it follows from (II.7) that

$$\mathcal{Y}_r = \Psi_{r,n+(r-M+1)p} \begin{bmatrix} x_0 \\ \mathcal{E}_{r-M} \end{bmatrix}. \tag{III.1}$$

Therefore,

$$\begin{bmatrix} x_0 \\ \mathcal{E}_{r-M} \end{bmatrix} = \Psi^+_{r,n+(r-M+1)p} \mathcal{Y}_r + v_M, \tag{III.2}$$

where $v_M \in \mathcal{N}(\Psi_{r,n+(r-M+1)p})$. Therefore, if $\mathcal{N}(\Psi_{r,n+(r-M+1)p}) = \{0\}$, it follows that

$$\begin{bmatrix} x_0 \\ \mathcal{E}_{r-M} \end{bmatrix} = \Psi^+_{r,n+(r-M+1)p} \mathcal{Y}_r, \tag{III.3}$$

and hence $\begin{bmatrix} x_0 \\ \mathcal{E}_{r-M} \end{bmatrix}$ can be estimated from $\mathcal{Y}_r$.

Next, when e_k, $k = r - M + 1, \ldots, r$ are not zero but are known, we can write

$$\mathcal{Y}_r = \Psi_{r,n+(r-M+1)p} \begin{bmatrix} x_0 \\ \mathcal{E}_{r-M} \end{bmatrix} + \Phi^M_r E^M_r, \tag{III.4}$$

where

$$E^M_r \triangleq \begin{bmatrix} e_{r-M+1} \\ \vdots \\ e_r \end{bmatrix} \in \mathbb{R}^{(M-1)p}, \tag{III.5}$$

is the vector of known inputs, and Φ^M_r denotes the last $(M-1)p$ columns of Ψ_r. Therefore, if $\mathcal{N}(\Psi_{r,n+(r-M+1)p}) = \{0\}$, it follows that

$$\begin{bmatrix} x_0 \\ \mathcal{E}_{r-M} \end{bmatrix} = \Psi^+_{r,n+(r-M+1)p} (\mathcal{Y}_r - \Psi^M_r E^M_r), \tag{III.6}$$

and thus $\begin{bmatrix} x_0 \\ \mathcal{E}_{r-M} \end{bmatrix}$ can be estimated if $\Psi_{r,n+(r-M+1)p}$ has full column rank and e_k, $k = r - M, \ldots, r$ are known.

Theorem III.1. (II.1), (II.2) is M-delay input and state estimable if and only if $\mathrm{rank}(\Psi_{r,n+(r-M+1)p}) = n + (r - M + 1)p$.

Proposition III.1. If (II.1), (II.2) is M-delay input and state estimable, then (II.1), (II.2) is M^*-delay input and state estimable for all $M^* \geq M$.

Proposition III.2. If (II.1), (II.2) is M-delay input and state estimable, then the following statements hold:

1) rank $\Psi_r \geq n + (r - M + 1)p$.
2) rank $(\Psi^{\mathrm{T}}_r)_{n+(r-M+1)p} = n + (r - M + 1)p$.
3) (A, C) is observable, that is, $\mathrm{rank}(\Gamma_{n-1}) = n$.
4) $\mathrm{rank}(\Psi_r) = \mathrm{rank}(\Psi_{r-1}) + p$ for all $r \geq r_0$.

Proposition III.3. If (II.1), (II.2) is M-delay input and state estimable, then the following statements hold:

1) $n + (r - M + 1)p \leq (r + 1)l$.
2) $M \geq \frac{n-r(l-p)-r}{p} + 1$.
3) If $p = l$, then $M \geq \frac{n}{p}$.
4) If $p = l = 1$, then $M \geq n$.
5) If $p < l$, then $r \geq \frac{n-(l-p)-Mp}{l-p}$.

IV. Noise Analysis

To analyze the sensitivity of the estimate (II.10) to noise, consider (II.1), (II.2) with additive measurement and process noise so that

$$x_{k+1} = Ax_k + He_k + w_k, \quad \text{(IV.1)}$$
$$y_k = Cx_k + Ge_k + v_k, \quad \text{(IV.2)}$$

where $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^l$ are zero mean, uncorrelated, white noise sequences. Then

$$\mathcal{Y}_r = \Psi_r \begin{bmatrix} x_0 \\ \mathcal{E}_{r-1} \end{bmatrix} + N_r \mathcal{W}_{r-1} + \mathcal{V}_r, \quad \text{(IV.3)}$$

where

$$N_r \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ C & 0 & \cdots & 0 \\ CA & C & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{r-1} & CA^{r-2} & \cdots & C \end{bmatrix} \in \mathbb{R}^{(r+1)l \times rn},$$

$$\mathcal{W}_r \triangleq \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_r \end{bmatrix} \in \mathbb{R}^{(r+1)n}, \quad \mathcal{V}_r \triangleq \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_r \end{bmatrix} \in \mathbb{R}^{(r+1)l}.$$

Next, let $L = 0$, that is, let (II.1), (II.1) be input and state observable. We thus consider the least-squares estimate

$$\begin{bmatrix} \hat{x}_0 \\ \hat{\mathcal{E}}_{r-1} \end{bmatrix} \triangleq \Psi_r^\dagger \mathcal{Y}_r = \begin{bmatrix} x_0 \\ \mathcal{E}_{r-1} \end{bmatrix} + \Psi_r^\dagger N_r \mathcal{W}_{r-1} + \Psi_r^\dagger \mathcal{V}_r. \quad \text{(IV.4)}$$

Since w_k and v_k are zero mean noise sequences, (IV.4) implies

$$\mathbb{E} \begin{bmatrix} \hat{x}_0 \\ \hat{\mathcal{E}}_{r-1} \end{bmatrix} = \begin{bmatrix} x_0 \\ \mathcal{E}_{r-1} \end{bmatrix}, \quad \text{(IV.5)}$$

and thus (IV.4) is an unbiased estimate of $\begin{bmatrix} x_0 \\ \mathcal{E}_{r-1} \end{bmatrix}$. Finally, the variance of the estimate (IV.4) is given by

$$\operatorname{var} \begin{bmatrix} \hat{x}_0 \\ \hat{\mathcal{E}}_{r-1} \end{bmatrix} = \Psi_r^\dagger N_r R_w N_r^{\mathrm{T}} (\Psi_r^\dagger)^{\mathrm{T}} + \Psi_r^\dagger R_v (\Psi_r^\dagger)^{\mathrm{T}}, \quad \text{(IV.6)}$$

where $R_w \triangleq \mathbb{E}\left[\mathcal{W}_{r-1}\mathcal{W}_{r-1}^{\mathrm{T}}\right]$ and $R_v \triangleq \mathbb{E}\left[\mathcal{V}_r \mathcal{V}_r^{\mathrm{T}}\right]$.

The above analysis can be extended to the case of L-delay state and input estimation.

V. Compartmental Model Example

To illustrate input and state observability with noisy data, we consider a system comprised of $n = 6$ compartments that exchange mass or energy through mutual interaction [3]. Applying conservation yields

$$x_{1,k+1} = x_{1,k} - \beta x_{1,k} + \alpha(x_{2,k} - x_{1,k}), \quad \text{(V.1)}$$
$$x_{i,k+1} = x_{i,k} - \beta x_{i,k} + \alpha(x_{i+1,k} - x_{i,k}) - \alpha(x_{i,k} - x_{i-1,k}),$$
$$i = 2, \ldots, n-1, \quad \text{(V.2)}$$
$$x_{n,k+1} = x_{n,k} - \beta x_{n,k} - \alpha(x_{n,k} - x_{n-1,k}), \quad \text{(V.3)}$$

where $0 < \beta < 1$ is the loss coefficient and $0 < \alpha < 1$ is the flow coefficient. In addition, an unknown input enters compartment 2. The outputs are the energy states in compartments 2 and 3, and therefore $l = 2$. It then follows that

$$x_{k+1} = Ax_k + He_k, \quad \text{(V.4)}$$
$$y_k = Cx_k, \quad \text{(V.5)}$$

where $A \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{2 \times n}$ are defined as

$$A \triangleq \begin{bmatrix} 1-\beta-\alpha & \alpha & 0 & \cdots & 0 \\ \alpha & 1-\beta-\alpha & \alpha & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha & 1-\beta-\alpha \end{bmatrix}, \quad \text{(V.6)}$$

$$H \triangleq \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{(V.7)}$$

$$C \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \end{bmatrix}. \quad \text{(V.8)}$$

For simulations, we set $\alpha = 0.3$ and $\beta = 0.1$. It can be verified that

$$\mathcal{N}(\Psi_r) = \begin{bmatrix} 0_{2r+1 \times 1} \\ 1 \end{bmatrix}, \quad \text{(V.9)}$$

and thus (II.1), (II.2) is 1-delay input and state observable. Hence using measurements of y_k, $k = 0, \ldots, r$, we estimate the initial state x_0 and the unknonw inputs e_k, $k = 0, \ldots, r-1$.

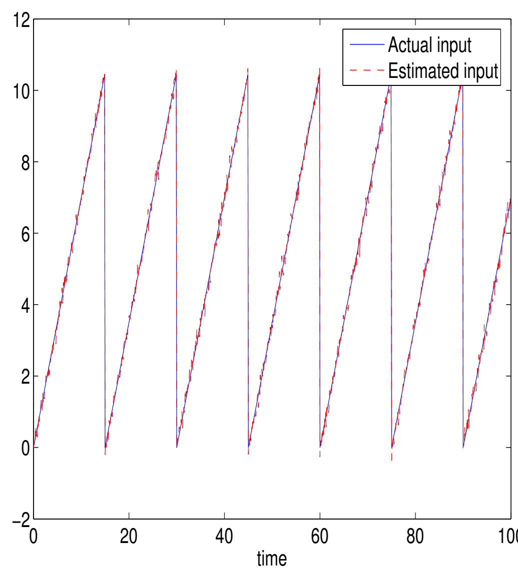

Fig. 1. Compartmental model example. The actual unknown inputs and the estimates of the unknown inputs using measurements of outputs and known model. Measurement and process noise with standard deviation 0.1 is added to the model simulation.

The initial state is chosen to be $x_0 = \begin{bmatrix} 2.0 & 0.1 & -1.0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, and the unknown force is chosen to be a sawtooth signal. Simulations are run with Gaussian process noise w_k and measurement noise v_k with covariances $\mathrm{diag}(0.01, 0.01, 0.01, 0.01, 0.01, 0.01)$ and $\mathrm{diag}(0.01, 0.01, 0.01)$, respectively. Using the measured outputs, the initial state and unknown input are estimated using Theorem II.1 for $r = 1000$. Although (V.4) - (V.8) is input and state observable, poor numerical conditioning of Ψ_r can cause the estimates of the unknown inputs to be inaccurate. In this example, the condition number of Ψ_r is 82.8975 and thus Ψ_r is not ill-conditioned. Figure 1 shows the unknown force and its estimate in the presence of process noise and measurement noise with standard deviation 0.1. In the presence of process noise and measurement noise, the estimate of the initial state is

$$\hat{x}_0 = \begin{bmatrix} 2.0690 \\ 0.1719 \\ -0.9862 \\ -0.0454 \\ 0.0136 \\ -0.6951 \end{bmatrix}. \tag{V.10}$$

The following examples discuss input and state observability/estimability properties for the compartmental system with A given by (V.6), but with different combinations of H and C.

Example V.1. Let A be given by (V.6), and let

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$H = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then, $\mathrm{rank}(CH) = p = 1$, and since $\mathrm{rank}(\Psi_n) = \mathrm{rank}(\Psi_{n,n+1}) = 7 = n + p$, the system is 6-delay input and state estimable. Furthermore, since $\mathcal{N}(\Psi_{n,n+(r-L+1)p}) \not\subseteq \mathcal{R}\begin{bmatrix} 0_{n+(r-L+1)l\times Lp} \\ I_{Lp} \end{bmatrix}$ for any $0 \le L \le n$, it follows that the system is not L-delay input and state observable for any L.

Example V.2. Let A be given by (V.6), and let

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then, $\mathrm{rank}(CH) = 0 \neq p$, and since $\mathrm{rank}(\Psi_n) = \mathrm{rank}(\Psi_{n,n+1}) = 7 = n + p$, the system is 6-delay input and state estimable. Furthermore, since $\mathcal{N}(\Psi_{n,n+(r-6+1)p}) = \mathcal{R}\begin{bmatrix} 0_{n+(r-6+1)l\times 6p} \\ I_{6p} \end{bmatrix}$, it follows that the system is 6-delay input and state observable.

Example V.3. Let A be given by (V.6), and let

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$
$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, $\mathrm{rank}(CH) = 1 < p$, and $\mathrm{rank}(\Psi_n) = 10$ and $\mathrm{rank}(\Psi_{n,n+1}) = 7 <= n + p$. Therefore, the system is not L-delay input and state observable/estimable for any $0 \le L \le n$. However, note that (A, H) is not controllable.

Example V.4. Let A be given by (V.6), and let

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$
$$H = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, $\mathrm{rank}(CH) = 1 \neq p$, and since $\mathrm{rank}(\Psi_n) = 17$, $\mathrm{rank}(\Psi_{n,(n-2+1)p}) = 16 = n + (n - 2 + 1)p$, the system is 2-delay input and state estimable. Furthermore, since $\mathcal{N}(\Psi_{n,n+(n-2+1)p}) \subset \mathcal{R}\begin{bmatrix} 0_{n+(r-L+1)l\times 2p} \\ I_{2p} \end{bmatrix}$, it follows that the system is 2-delay input and state observable.

VI. State Estimation

For state estimation, we consider Example V.1, which is 6-delay input and state estimable. We use a square wave for the unknown input, However, we assume that the inputs for the last 6 time-steps are known. In this example, since the input is a unit square wave, the last 6 time-steps are just -1.

We first use (III.6) to estimate the unknown inputs and the initial state. Then we build a Kalman filter based on known model equations and the estimated states. It can be shown that if $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^l$ in (IV.1), (IV.2) are zero mean, uncorrelated, white noise sequences, tthen he state estimates obtained in a manner described earlier are unbiased estimates of the states. Figure 2 shows the actual input and the input estimated using (III.6), while Figure 3 shows the actual second state and the estimated second state.

This technique is useful for events such as failure of sensors for a brief time interval, in which case some of the previous inputs are known.

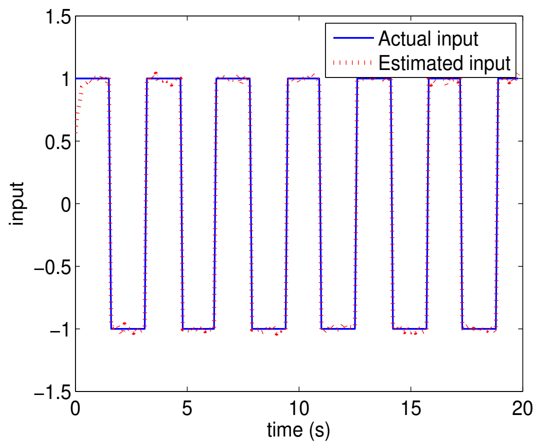

Fig. 2. The actual input and the estimated input for the compartmental model example.

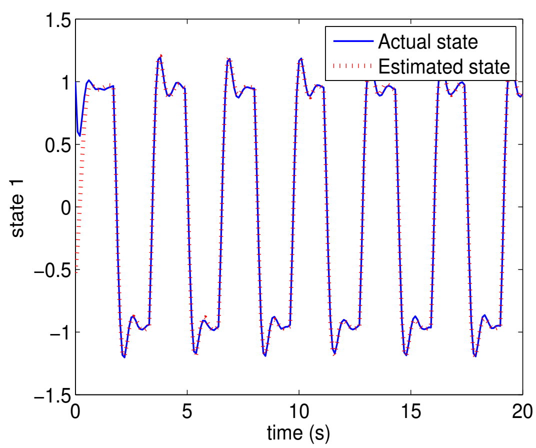

Fig. 3. The actual second state and the estimated second state for the compartmental model example.

VII. Conclusions

In this paper we considered the problem of estimating the state and the input from known output measurements. We derived neccessary and sufficient conditions for which a system is L-delay input and state observable/estimable. Next, we explored the sensitivity of the estimates to additional noise. Finally, we presented a compartmental model with several combinations of input and output matrices to illustrate different scenarios for L-delay input and state observability/estimability. We then assumed known partial input information to estimate the unknown inputs and subsequently estimate the states using a Kalman filter.

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