Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Robust Stability Criteria for Markovian Jump Singular Systems with Time-Varying Delays

A. Haidar and E. K. Boukas

***Abstract*— This paper deals with the class of continuous-time Markovian jump singular systems with time-varying delay. The stability problem of this class of systems is addressed and delay-dependent sufficient conditions such that the system is regular, impulse free and asymptotically mean square stable are developed in the linear matrix inequality (LMI) setting. A numerical example is employed to show the usefulness of the proposed results.**

Keywords: Singular time-delay systems, Markovian jump systems, Delay-dependent, Stability, Linear matrix inequality.

I. INTRODUCTION

In the past decades, standard state-space theory has been extensively studied and the theory is well developed. Standard state-space system is described by a set of ordinary differential equations (ODEs). However, in many physical systems such as chemical processes, circuit systems and economic systems, the state variables may be related algebraically, resulting in a more general class of systems, called singular systems [1]. Thus, a singular system model is a set of coupled differential and algebraic equations, which include information on the static as well as dynamic constraints of a real plant.

Delays are one of the most important causes of instability and are encountered in many physical systems such as chemical processes, rolling mills, nuclear reactors, long transmission lines, and microwave oscillators [4]. Therefore, time-delay systems has been extensively studied and many results has been published in the literature.

Singular time-delay systems, which have both delay and algebraic constraints, may in fact be systems of advanced type [5]. These systems often appear in various engineering systems, including aircraft stabilization, chemical engineering systems, lossless transition lines, etc. (see [5] and the references therein). It is worth noting that this class of systems are also referred to in the literature as delay differential-algebraic equations, implicit systems with delay or descriptor systems with delay.

In the last decades, a class of stochastic systems driven by continuous-time Markov chains has been used to model many practical systems, where random failures and repairs and sudden environment changes may occur. This class of systems is referred to in the literature as Markovian jump systems. Applications of Markovian jump systems include failures and repairs of machine in manufacturing systems, modifications of the operating point of a linearized model of a nonlinear system, power systems, networked control systems and economics systems. For more details on what has been done on this class of systems, we refer the reader to the book by Boukas [12]. For the class of Markovian jump singular systems, we refer the reader also to the recent book by Boukas [11].

There is only few work on Markovian jump singular systems with delays [2], [3]. In [2], delay-independent conditions were proposed in terms of LMIs to check the stability and to design a state feedback controller for Markovian jump singular systems with constant time delay. The stability conditions proposed in [2] guarantee the stability of the slow subsystem only and there is no guarantee for the stability of the fast subsystem. Also, the bounding approach of the derivative of the Lyapunov functional in [2] is conservative due to the ignorance of some useful terms. In [10] the robust stabilization problem for singular systems with time varying delay has been tackled. LMIs conditions were proposed to design the state feedback controller.

This paper is concerned with the problems of robust stability analysis for singular Markovian jump systems with time-varying delays. In terms of a set of linear matrix inequalities (LMIs), we first present a delay-dependent sufficient condition which guarantees the regularity, absence of impulses, and asymptotic mean square stability of such systems. Based on this, we extend the results to the case of uncertain system. The method used is based on the Lyapunov-Krasovskii approach and the free-weighting matrices method is used to get a less conservative results. The Lyapunov functional and some inequalities from [9] are adopted, with some modifications, in order to prove the stability of the slow subsystem. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed method.

Notation: Throughout this paper, the notation $X \geq Y (X > Y)$ where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (positive definite). $C_\tau = C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. $\|\cdot\|$ refers to the Euclidean norm whereas $\|\phi\|_c = sup_{-\tau \leq t \leq 0} \|\phi(t)\|$ stands for the norm of a function $\phi \in C_\tau$.

This work was supported by NSERC-Canada, Grant OPG0035444.

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978-1-4244-3124-3/08/$25.00 ©2008 IEEE

II. PROBLEM STATEMENT AND DEFINITIONS

Consider a continuous-time uncertain singular system with random abrupt changes that has N_m modes, i.e., $\mathscr{S} = \{1,2,\cdots,N_m\}$. The mode switching is assumed to be governed by a continuous-time Markov process $\{r_t, t \geq 0\}$ taking values in the state space $\mathscr{S}$ and having the following infinitesimal generator

$$\Lambda = (\lambda_{ij}), i, j \in \mathscr{S},$$

where $\lambda_{ij} \geq 0, \forall j \neq i, \lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$.

The mode transition probabilities are described as follows:

$$P[r_{t+\Delta} = j | r_t = i] = \begin{cases} \lambda_{ij}\Delta + o(\Delta), & j \neq i \\ 1 + \lambda_{ii}\Delta + o(\Delta), & j = i \end{cases} \quad (1)$$

where $\lim_{\Delta \to 0} o(\Delta)/\Delta = 0$.

Let $x(t) \in \mathbb{R}^n$ be the physical state of the system, which satisfies the following dynamics:

$$\begin{cases} E\dot{x}(t) = (A(r_t) + \Delta A(r_t))x(t) + (A_1(r_t) + \Delta A_1(r_t))x(t-d(t)) \\ x(s) = \phi(s), -h \leq s \leq 0 \end{cases} \quad (2)$$

where $A(i)$ and $A_1(i)$ are known real matrices with appropriate dimensions for each $i \in \mathscr{S}$, the matrix E may be singular with $0 \leqslant \text{rank}(E) \leqslant n$, the initial condition of the system is specified as $(r_0, \phi(\cdot))$ with r_0 is the initial mode and $\phi(.)$ is the initial functional such that $x(s) = \phi(s) \in C_h$ and $d(t)$ is a time-varying differentiable function that satisfies

$$0 < d(t) \leq h \quad \text{and} \quad \|\dot{d}(t)\| \leq \mu < 1$$

where $h > 0$ and μ are constants. In addition, the matrices $\Delta A(r_t)$ and $\Delta A_1(r_t)$ denotes the uncertainties in the system and take the form of

$$[\Delta A(r_t) \ \Delta A_1(r_t)] = DF(r_t)\,[E_1(r_t) \ E_2(r_t)] \quad (3)$$

where D, $E_1(i)$ and $E_2(i)$ are known constant matrices and $F(i)$ is an unknown matrix function with Lesbesgue measurable elements bounded by:

$$F^\top(i)F(i) \leq \mathbb{I} \quad (4)$$

Definition 2.1: [11]

i. System (2) is said to be regular if the characteristic polynomial, $\det(sE - A(i))$ is not identically zero for each mode $i \in \mathscr{S}$.
ii. System (2) is said to be impulse-free, i.e. the $\deg(\det(sE - A(i))) = rank(E)$ for each mode $i \in \mathscr{S}$.

It has been discussed in [7] that singular Markovian jump systems admits jump discontinuities in its solution even if the system is impulse-free and has consistence initial conditions. These discontinuities appear only at instances where the system changes its mode. However, when delayed solution terms exist, these jumps can propagate in the solution. This fact will be discussed in what follows.

If system (2) is impulse-free, then it can be written in the following form [8], [11]:

$$\begin{aligned} \dot{x}_1(t) &= A_1(r_t)x_1(t) + A_2(r_t)x_2(t) + A_{d1}(r_t)x_1(t-d(t)) \\ &\quad + A_{d2}(r_t)x_2(t-d(t)) \qquad (5) \\ 0 &= A_3(r_t)x_1(t) + A_4(r_t)x_2(t) + A_{d3}(r_t)x_1(t-d(t)) \\ &\quad + A_{d4}(r_t)x_2(t-d(t)) \qquad (6) \end{aligned}$$

with $A_4(i)$ invertible, where $A_j(i)$ and $A_{dj}(i)$, $j = 1, \ldots 4$, are real known matrices with appropriate dimensions. Let τ_k, $k = 1,2,\cdots$, be the instances when the system jumps from mode i to mode j. Then, from (6), we have:

$$\begin{aligned} x_2(\tau_k^-) = &-A_4^{-1}(i)A_3(i)x_1(\tau_k^-) \\ &- A_4^{-1}(i)A_{d3}(i)x_1(\tau_k^- - d(\tau_k^-)) \\ &- A_4^{-1}(i)A_{d4}(i)x_2(\tau_k^- - d(\tau_k^-)) \end{aligned}$$

but at the same time, we have

$$\begin{aligned} x_2(\tau_k^+) = &-A_4^{-1}(j)A_3(j)x_1(\tau_k^+) \\ &- A_4^{-1}(j)A_{d3}(j)x_1(\tau_k^+ - d(\tau_k^+)) \\ &- A_4^{-1}(j)A_{d4}(j)x_2(\tau_k^+ - d(\tau_k^+)) \\ &\neq x_2(\tau_k^-) \end{aligned}$$

which implies a finite jump discontinuity at $t = \tau_k$. Now, if at $\hat{t} > \tau_k$, the argument $(\hat{t} - d(\hat{t}))$ crosses τ_k, this jump discontinuity will propagate to the instance $\hat{t}$, and so on with the following instances.

Remark 2.1: Although the system exhibits inevitable jump discontinuities due to the changing in its mode, it is still important to insure that the system is impulse-free with respect to Definition 2.1. Those impulses is due to the singular structure of the systems and can be avoided in impulse-free systems. Also, if the systems is not impulse-free, the system may exhibit many impulses due to smooth inputs.

For system (2), we have also the following definitions:

Definition 2.2: System (2) is said to be asymptotically mean square stable if

$$\lim_{t \to \infty} \mathbb{E}\left[\|x(t)\|^2 \middle| r_0, x(s) = \phi(s), s \in [-h, 0] \right] = 0 \quad (7)$$

Definition 2.3: System (2) is said to be robustly asymptotically mean square stable if

$$\lim_{t \to \infty} \mathbb{E}\left[\|x(t)\|^2 \middle| r_0, x(s) = \phi(s), s \in [-h, 0] \right] = 0 \quad (8)$$

for all admissible uncertainties.

In order to obtain the main results, the following lemma is needed.

Lemma 2.1: Given matrices Ω, Γ and Ξ of appropriate dimensions and with Ω symmetrical, then

$$\Omega + \Gamma F \Xi + (\Gamma F \Xi)^\top < 0$$

for all F satisfies $F^\top F \leq \mathbb{I}$, if and only if there exists a scalar $\sigma > 0$ such that

$$\Omega + \sigma \Gamma \Gamma^\top + \sigma^{-1} \Xi^\top \Xi < 0$$

III. Main Results

In this section, we will develop results that assure that system (2) is regular, impulse-free and robustly asymptotically mean square stable. Our first result in this paper deals with the stability of (2) with $\Delta A(i) = 0$ and $\Delta A_1(i) = 0$. The following theorem gives such result.

Theorem 3.1: The singular Markovian jump system (2) is regular, impulse-free and asymptotically mean square stable if there exist a set of nonsingular matrices $P = (P(1),\cdots,P(N))$ and symmetric and positive-definite matrices $Q > 0$, $R > 0$, $Z_1 > 0$, $Z_2 > 0$, and a matrices

$$N = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix},\ S = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix}, M = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix}$$

such that the following set of coupled LMIs holds for each $i \in \mathscr{S}$:

$$\Phi(i) = \begin{bmatrix} \psi(i) & hN & hS & hM & hA_{cl}^\top(i)(Z_1+Z_2) \\ \star & -hZ_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -hZ_1 & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -hZ_2 & \mathbf{0} \\ \star & \star & \star & \star & -h(Z_1+Z_2) \end{bmatrix} < 0 \tag{9}$$

with the following constraints:

$$E^\top P(i) = P^\top(i) E \geq 0 \tag{10}$$

where

$$\psi(i) = \psi_1(i) + \psi_2 + \psi_2^\top$$
$$\psi_1(i) = \begin{bmatrix} \Pi & P^\top(i)A_d(i) & \mathbf{0} \\ \star & -(1-\mu)Q & \mathbf{0} \\ \star & \star & -R \end{bmatrix}$$
$$\psi_2 = \begin{bmatrix} NE+ME & -NE+SE & -ME-SE \end{bmatrix}$$
$$A_{cl}(i) = \begin{bmatrix} A(i) & A_d(i) & \mathbf{0} \end{bmatrix}$$
$$\Pi = P^\top(i)A(i) + A^\top(i)P(i) + Q + R + \sum_{j=1}^{N} \lambda_{ij} E^\top P(j)$$

Proof: Let us first of all show that system (2) is regular and impulse-free (2). In fact from (9), it is easy to see that the following holds for each $i \in \mathscr{S}$:

$$P^\top(i)A(i) + A^\top(i)P(i) + \sum_{j=1}^{N} \lambda_{ij} E^\top P(j) < 0 \tag{11}$$

Now, choose two nonsingular matrices $\widehat{M}$ and $\widehat{N}$ such that

$$\widehat{M}E\widehat{N} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix}$$

and write

$$\widehat{M}A(i)\widehat{N} = \begin{bmatrix} \widehat{A}_1(i) & \widehat{A}_2(i) \\ \widehat{A}_3(i) & \widehat{A}_4(i) \end{bmatrix},\ \widehat{M}^{-\top}P(i)\widehat{N} = \begin{bmatrix} \widehat{P}_1(i) & \widehat{P}_2(i) \\ \widehat{P}_3(i) & \widehat{P}_4(i) \end{bmatrix}.$$

Then, by (10), it can be shown that $\widehat{P}_2(i) = 0$. Pre- and post-multiplying (11) by $\widehat{N}^\top$ and $\widehat{N}$, respectively, we have

$$\begin{bmatrix} * & * \\ * & \widehat{A}_4^\top(i)\widehat{P}_4(i) + \widehat{P}_4^\top(i)\widehat{A}_4(i) \end{bmatrix} < 0,$$

where $*$ will not be used in the following development. Then, from this, we get:

$$\widehat{A}_4^\top(i)\widehat{P}_4(i) + \widehat{P}_4^\top(i)\widehat{A}_4(i) < 0$$

which implies that $\widehat{A}_4(i)$ is nonsingular. Therefore, system (2) is regular and impulse-free.

Next, we will show the stochastic stability. Since system (2) is regular and impulse-free, for any $i \in \mathscr{S}$, we can choose nonsingular matrices $\check{M}(i)$ and $\check{N}$ such that [8]

$$\widetilde{E} = \check{M}(i)E\check{N} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\widetilde{A}(i) = \check{M}(i)A(i)\check{N} = \begin{bmatrix} \widehat{A}(i) & 0 \\ \widehat{A}_3(i) & \widehat{A}_4(i) \end{bmatrix}.$$

where $\widehat{A}_4(i)$ is nonsingular. Now, write

$$\widetilde{P}(i) = \check{M}^{-\top}(i)P(i)\check{N} = \begin{bmatrix} \widetilde{P}_1(i) & \widetilde{P}_2(i) \\ \widetilde{P}_3(i) & \widetilde{P}_4(i) \end{bmatrix},$$
$$\widetilde{Q} = \check{N}^\top Q\check{N} = \begin{bmatrix} \widetilde{Q}_1 & \widetilde{Q}_2 \\ \widetilde{Q}_2^\top & \widetilde{Q}_4 \end{bmatrix},$$
$$\widetilde{A}_1(i) = \check{M}(i)A_1(i)\check{N} = \begin{bmatrix} \widetilde{A}_{11}(i) & \widetilde{A}_{12}(i) \\ \widetilde{A}_{13}(i) & \widetilde{A}_{14}(i) \end{bmatrix}.$$
$$\widetilde{R} = \check{N}^\top R\check{N} = \begin{bmatrix} \widetilde{R}_1 & \widetilde{R}_2 \\ \widetilde{R}_2^\top & \widetilde{R}_4 \end{bmatrix},$$
$$\widetilde{Z}_j = \check{M}^\top(i)Z_j\check{M}^{-1}(i),\ j = 1,2$$
$$\widetilde{S}(i) = \operatorname{diag}\left(\check{N}^\top, \check{N}^\top, \check{N}^\top\right) S\check{M}^{-1}(i)$$

and $\widetilde{N}(i)$ and $\widetilde{M}(i)$ similar to $\widetilde{S}(i)$. Then, for any $i \in \mathscr{S}$, system (2) becomes equivalent to the following one:

$$\begin{aligned} \dot{\xi}_1(t) &= \widehat{A}(r_t)\xi_1(t) + \widetilde{A}_{11}(r_t)\xi_1(t-d(t)) + \widetilde{A}_{12}(r_t)\xi_2(t-d(t)), \\ 0 &= \widehat{A}_3(r_t)\xi_1(t) + \widehat{A}_4(r_t)\xi_2(t) + \widetilde{A}_{13}(r_t)\xi_1(t-d(t)) \\ &\quad + \widetilde{A}_{14}(r_t)\xi_2(t-d(t)). \end{aligned} \tag{12}$$

where

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \check{N}^{-1}x(t).$$

By (9), it can be shown that

$$\begin{bmatrix} \widehat{A}_4^\top(i)\widetilde{P}_4(i) + \widetilde{P}_4^\top(i)\widehat{A}_4(i) + \widetilde{Q}_4 + \widetilde{R}_4 & \widetilde{P}_4^\top(i)\widetilde{A}_{14}(i) \\ \widetilde{A}_{14}^\top(i)P_4(i) & -\widetilde{Q}_4 \end{bmatrix} < 0.$$

Using the fact that $\widetilde{R} > 0$ and pre- and post-multiply this by $\begin{bmatrix} -\widetilde{A}_{14}^\top(i)\widehat{A}_4^{-\top}(i) & \mathbb{I} \end{bmatrix}$ and $\begin{bmatrix} -\widehat{A}_4^{-1}(i)\widetilde{A}_{14}(i) \\ \mathbb{I} \end{bmatrix}$ respectively, we get:

$$\widetilde{A}_{14}^\top(i)\widehat{A}_4^{-\top}(i)Q_4\widehat{A}_4^{-1}\widetilde{A}_{14}(i) - Q_4 < 0$$

Therefore

$$\rho\left(\widehat{A}_4^{-1}(i)\widetilde{A}_{14}(i)\right) < 1. \tag{13}$$

where $\rho(\widehat{A}_4^{-1}(i)\widetilde{A}_{14}(i))$ is the spectral radius of the matrix $\widehat{A}_4^{-1}(i)\widetilde{A}_{14}(i)$. Now, let us choose the following Lyapunov functional [9]:

$$V(\xi_t, r_t) = \xi^\top(t)\widetilde{E}^\top\widetilde{P}(r_t)\xi(t) + \int_{t-d(t)}^{t} \xi^\top(s)\widetilde{Q}\xi(s)ds$$

$$+\int_{-h}^{0}\int_{t+v}^{t}(\widetilde{E}\dot{\xi}(s))^{\top}(\widetilde{Z}_1(r_s)+\widetilde{Z}_2(r_s))\widetilde{E}\dot{\xi}(s)dsdv + \int_{t-h}^{t}\xi^{\top}(s)\widetilde{R}\xi(s)ds \quad (14)$$

with $\xi_t(t)$ taking values in $\mathscr{C}[-h,0]$ and defined by $\xi_t(t)=\xi(s+t), t-h\le s\le t$. Let $\mathscr{L}$ be the weak infinitesimal generator of the random process $\{(\xi_t(t),\ r_t), t\ge 0\}$. Then, for each $r_t=i$, $i\in\mathscr{S}$, we have

$$\begin{aligned}
&\mathscr{L}V(\xi_t(t),i)\\
&=\xi^{\top}(t)\Big[\widetilde{A}^{\top}(i)\widetilde{P}(i)+\widetilde{P}^{\top}(i)\widetilde{A}(i)+\widetilde{Q}+\widetilde{R}\\
&\quad+\sum_{j=1}^{N}\lambda_{ij}\widetilde{E}^{\top}\widetilde{P}(j)\Big]\xi(t)+2\xi^{\top}(t)\Big[P^{\top}(i)A_1(i)\Big]\xi^{\top}(t-d(t))\\
&\quad+\xi^{\top}(t-d(t))\Big[-(1-\dot{d}(t))\widetilde{Q}\Big]\xi(t-d(t))\\
&\quad+\xi^{\top}(t-h)\Big[-\widetilde{R}\Big]\xi(t-h)+h(\widetilde{E}\dot{\xi}(t))^{\top}\Big[\widetilde{Z}_1(i)+\widetilde{Z}_2(i)\Big](\widetilde{E}\dot{\xi}(t))\\
&\quad-\int_{t-h}^{t}(\widetilde{E}\dot{\xi}(s))^{\top}\Big[\widetilde{Z}_1(r_s)+\widetilde{Z}_2(r_s)\Big](\widetilde{E}\dot{\xi}(s))ds\\
&\le\xi^{\top}(t)\Big[\widetilde{A}^{\top}(i)\widetilde{P}(i)+\widetilde{P}^{\top}(i)\widetilde{A}(i)+\widetilde{Q}+\widetilde{R}\\
&\quad+\sum_{j=1}^{N}\lambda_{ij}\widetilde{E}^{\top}\widetilde{P}(j)\Big]\xi(t)+2\xi^{\top}(t)\Big[P^{\top}(i)A_1(i)\Big]\xi^{\top}(t-d(t))\\
&\quad+\xi^{\top}(t-d(t))\Big[-(1-\mu)\widetilde{Q}\Big]\xi(t-d(t))\\
&\quad+\xi^{\top}(t-h)\Big[-\widetilde{R}\Big]\xi(t-h)+h(\widetilde{E}\dot{\xi})^{\top}(t)\Big[\widetilde{Z}_1(i)+\widetilde{Z}_2(i)\Big](\widetilde{E}\dot{\xi})\\
&\quad-\int_{t-d(t)}^{t}(\widetilde{E}\dot{\xi}(s))^{\top}\Big[\widetilde{Z}_1(r_s)\Big](\widetilde{E}\dot{\xi}(s))ds\\
&\quad-\int_{t-h}^{t-d(t)}(\widetilde{E}\dot{\xi}(s))^{\top}\Big[\widetilde{Z}_1(r_s)\Big](\widetilde{E}\dot{\xi}(s))ds\\
&\quad-\int_{t-h}^{t}(\widetilde{E}\dot{\xi}(s))^{\top}\Big[\widetilde{Z}_2(r_s)\Big](\widetilde{E}\dot{\xi}(s))ds\\
&\quad+2\chi^{\top}(t)\widetilde{N}(i)\Big[\widetilde{E}\xi(t)-\widetilde{E}\xi(t-d(t))-\int_{t-d(t)}^{t}\widetilde{E}\dot{\xi}(s)ds\Big]\\
&\quad+2\chi^{\top}(t)\widetilde{S}(i)\Big[\widetilde{E}\xi(t-d(t))-\widetilde{E}\xi(t-h)-\int_{t-h}^{t-d(t)}\widetilde{E}\dot{\xi}(s)ds\Big]\\
&\quad+2\chi^{\top}(t)\widetilde{M}(i)\Big[\widetilde{E}\xi(t)-\widetilde{E}\xi(t-h)-\int_{t-h}^{t}\widetilde{E}\dot{\xi}(s)ds\Big]\\
&\le\chi^{\top}(t)\Big[\Psi(i)+h\widetilde{A}_{cl}^{\top}(i)(\widetilde{Z}_1(i)+\widetilde{Z}_2(i))\widetilde{A}_{cl}(i)\\
&\quad+h\mathrm{diag}\left(\check{N}^{\top},\check{N}^{\top},\check{N}^{\top}\right)NZ_1^{-1}N^{\top}\mathrm{diag}\left(\check{N},\check{N},\check{N}\right)\\
&\quad+h\mathrm{diag}\left(\check{N}^{\top},\check{N}^{\top},\check{N}^{\top}\right)SZ_1^{-1}S^{\top}\mathrm{diag}\left(\check{N},\check{N},\check{N}\right)\\
&\quad+h\mathrm{diag}\left(\check{N}^{\top},\check{N}^{\top},\check{N}^{\top}\right)MZ_2^{-1}M^{\top}\mathrm{diag}\left(\check{N},\check{N},\check{N}\right)\Big]\chi(t)\\
&\quad-\int_{t-d(t)}^{t}\Big[\chi^{\top}(t)\widetilde{N}+(E\dot{\xi}(s))^{\top}\widetilde{Z}_1(r_s)\Big]\widetilde{Z}_1^{-1}(r_s)\\
&\quad.\Big[\chi^{\top}(t)\widetilde{N}+(E\dot{\xi}(s))^{\top}\widetilde{Z}_1(r_s)\Big]^{\top}ds\\
&\quad-\int_{t-h}^{t-d(t)}\Big[\chi^{\top}(t)\widetilde{S}+(E\dot{\xi}(s))^{\top}\widetilde{Z}_1(r_s)\Big]\widetilde{Z}_1^{-1}(r_s)\\
&\quad.\Big[\chi^{\top}(t)\widetilde{S}+(E\dot{\xi}(s))^{\top}\widetilde{Z}_1(r_s)\Big]^{\top}ds\\
&\quad-\int_{t-h}^{t}\Big[\chi^{\top}(t)\widetilde{M}+(E\dot{\xi}(s))^{\top}\widetilde{Z}_2(r_s)\Big]\widetilde{Z}_2^{-1}(r_s)\\
&\quad.\Big[\chi^{\top}(t)\widetilde{M}+(E\dot{\xi}(s))^{\top}\widetilde{Z}_2(r_s)\Big]^{\top}ds
\end{aligned}\quad (15)$$

where

$$\begin{aligned}
\chi(t)&=\left[\begin{array}{ccc}\xi^{\top}(t) & \xi^{\top}(t-d(t)) & \xi^{\top}(t-h)\end{array}\right]^{\top},\\
\Psi(i)&=\mathrm{diag}\left(\check{N}^{\top},\check{N}^{\top},\check{N}^{\top}\right)\psi(i)\mathrm{diag}\left(\check{N},\check{N},\check{N}\right)\\
\widetilde{A}_{cl}(i)&=\left[\begin{array}{ccc}\widetilde{A}(i) & \widetilde{A}_d(i) & \mathbf{0}\end{array}\right]
\end{aligned}$$

Now, if

$$\begin{aligned}
\Sigma&=\Psi(i)+h\widetilde{A}_{cl}^{\top}(i)(\widetilde{Z}_1(i)+\widetilde{Z}_2(i))\widetilde{A}_{cl}(i)\\
&\quad+h\mathrm{diag}\left(\check{N}^{\top},\check{N}^{\top},\check{N}^{\top}\right)NZ_1^{-1}N^{\top}\mathrm{diag}\left(\check{N},\check{N},\check{N}\right)\\
&\quad+h\mathrm{diag}\left(\check{N}^{\top},\check{N}^{\top},\check{N}^{\top}\right)SZ_1^{-1}S^{\top}\mathrm{diag}\left(\check{N},\check{N},\check{N}\right)\\
&\quad+h\mathrm{diag}\left(\check{N}^{\top},\check{N}^{\top},\check{N}^{\top}\right)MZ_2^{-1}M^{\top}\mathrm{diag}\left(\check{N},\check{N},\check{N}\right)<0
\end{aligned}$$

which is equivalent to (9) and by basic algebraic manipulations and Schur complements, implies that there is a scalar $a>0$ such that for $i=\{1,2,\cdots,N\}$,

$$\Sigma+a\mathbb{I}<0$$

This together with (15) gives

$$\mathscr{L}V(\xi_t(t),i)\le -a\xi_1^{\top}(t)\xi_1(t) \quad (16)$$

for $i=\{1,2,\cdots,N\}$. Now, applying Dynkin's formula [12], we have that for each $r_t=i$, $i\in\mathscr{S}$, $t>0$,

$$\begin{aligned}
&\mathbb{E}\left\{V(\xi_t,r_t)\big|\xi(0),r_0\right\}-V(\xi_0,r_0)\\
&=\mathbb{E}\left\{\int_0^t\mathscr{L}V(\xi_\tau,r_t)d\tau\big|\xi(0),r_0\right\}\\
&\le -a\mathbb{E}\left\{\int_0^t\xi_1^{\top}(\tau)\xi_1(\tau)d\tau\big|\xi_1(0),r_0\right\}
\end{aligned}$$

and this implies

$$\mathbb{E}\left\{\int_0^t\xi_1^{\top}(\tau)\xi_1(\tau)d\tau\big|\xi_1(0),r_0\right\}\le\frac{1}{a}V(\xi(\mathbf{0}),r_0)$$

which implies

$$\lim_{t\to\infty}\mathbb{E}\left[\|\xi_1(t)\|^2\Big|\xi_1(0),r_0\right]=0 \quad (17)$$

Define,

$$\begin{aligned}
&t_0=t\\
&t_i=t_{i-1}-d(t_{i-1})\\
&\|\mathbb{A}_{13}\|=\max\left\{\|\widetilde{A}_{13}(1)\|,\cdots,\|\widetilde{A}_{13}(N_m)\|\right\}\\
&\|\mathbb{A}_4\|=\max\left\{\|\widehat{A}_4^{-1}(1)\widetilde{A}_{14}(1)\|,\cdots,\|\widehat{A}_4^{-1}(N_m)\widetilde{A}_{14}(N_m)\|\right\}
\end{aligned}$$

From (12), we get,

$$\begin{aligned}
\xi_2(t)&=-\widehat{A}_4^{-1}(r_{t_0})\widehat{A}_3(r_{t_0})\xi_1(t_0)-\widehat{A}_4^{-1}(r_{t_0})\widetilde{A}_{13}(r_{t_0})\xi_1(t_1)\\
&\quad-\widehat{A}_4^{-1}(r_{t_0})\widetilde{A}_{14}(r_{t_0})\xi_2(t_1)\\
&=-\widehat{A}_4^{-1}(r_{t_0})\widehat{A}_3(r_{t_0})\xi_1(t_0)-\widehat{A}_4^{-1}(r_{t_0})\widetilde{A}_{13}(r_{t_0})\xi_1(t_1)\\
&\quad-\widetilde{A}_{14}(r_{t_0})[-\widehat{A}_4^{-1}(r_{t_1})\widehat{A}_3(r_{t_1})\xi_1(t_1)
\end{aligned}$$

$$
\begin{aligned}
&-\widehat{A}_4^{-1}(r_{t_1})\widetilde{A}_{13}(r_{t_1})\xi_1(t_2)-\widehat{A}_4^{-1}(r_{t_1})\widetilde{A}_{14}(r_{t_1})\xi_2(t_2)]\\
=&-\widehat{A}_4^{-1}(r_{t_0})\widehat{A}_3(r_{t_0})\xi_1(t_0)-\widehat{A}_4^{-1}(r_{t_0})\widetilde{A}_{13}(r_{t_0})\xi_1(t_1)\\
&-\widetilde{A}_{14}(r_{t_0})[-\widehat{A}_4^{-1}(r_{t_1})\widehat{A}_3(r_{t_1})\xi_1(t_1)\\
&-\widehat{A}_4^{-1}(r_{t_1})\widetilde{A}_{13}(r_{t_1})\xi_1(t_2)-\widehat{A}_4^{-1}(r_{t_1})\widetilde{A}_{14}(r_{t_1})\\
&\times[\widehat{A}_4^{-1}(r_{t_2})\widehat{A}_3(r_{t_2})\xi_1(t_2)-\widehat{A}_4^{-1}(r_{t_2})\widetilde{A}_{13}(r_{t_2})\xi_1(t_3)\\
&-\widehat{A}_4^{-1}(r_{t_2})\widetilde{A}_{14}(r_{t_2})\xi_2(t_3)]]
\end{aligned}
$$

and so on.

Note that $t_i < t_{i-1}$, therefore, there exists a positive finite integer $k(t)$ such that

$$
\begin{aligned}
\xi_2(t)=\check{A}(k(t)-1)\xi_2(t_{k(t)})-\sum_{i=0}^{k(t)-1}\check{A}(i-1)\widetilde{A}_{13}(r_{t_i})\xi_1(t_{i+1})\\
-\sum_{i=0}^{k(t)-1}\check{A}(i-1)\widehat{A}_3(r_{t_i})\xi_1(t_i)
\end{aligned}
$$

where $t_{k(t)}\in(-d_2,0]$ and $\check{A}(c)=\prod_{j=0}^{c}-\widehat{A}_4^{-1}(r_{t_j})\widetilde{A}_{14}(r_{t_j})$. Therefore,

$$
\begin{aligned}
&\mathbb{E}\left\{\xi_2^\top(t)\xi_2(t)\middle|\xi(0),r_0\right\}\\
&\le\mathbb{E}\left\{\|\check{A}(k(t)-1)\|^2\|\phi\|_c^2\middle|\xi(0),r_0\right\}\\
&+\mathbb{E}\left\{\sum_{i=0}^{k(t)-1}\|\check{A}(i-1)\|^2\|\widetilde{A}_{13}(r_{t_i})\|^2\|\xi_1(t_{i+1})\|^2\middle|\xi(0),r_0\right\}\\
&+\mathbb{E}\left\{\sum_{i=0}^{k(t)-1}\|\check{A}(i-1)\|^2\|\widehat{A}_3(r_{t_i})\|^2\|\xi_1(t_i)\|^2\middle|\xi(0),r_0\right\} \quad (18)
\end{aligned}
$$

Noting that

$$
\|\check{A}(c)\|\le\prod_{j=0}^{c}\|-\widehat{A}_4^{-1}(r_{t_j})\widetilde{A}_{14}(r_{t_j})\|\le\|\mathbb{A}_4\|^{c+1}
$$

and the fact that $k(t)\to\infty$ as $t\to\infty$, this together with (13) implies that the limit of the first term in (18) can be bounded by

$$
\lim_{t\to\infty}\mathbb{E}\left\{\|\check{A}(k(t)-1)\|^2\|\phi\|_c^2\middle|\xi(0),r_0\right\}=0 \qquad (19)
$$

Now, the limit of the second term in (18) can be bounded by

$$
\begin{aligned}
&\mathbb{E}\left\{\sum_{i=0}^{k(t)-1}\|\check{A}(i-1)\|^2\|\widetilde{A}_{13}(r_{t_i})\|^2\|\xi_1(t_{i+1})\|^2\middle|\xi(0),r_0\right\}\\
&\le\|\mathbb{A}_{13}\|^2\mathbb{E}\left\{\sum_{i=0}^{k(t)-1}\|\check{A}(i-1)\|^2\|\xi_1(t_{i+1})\|^2\middle|\xi(0),r_0\right\}\\
&\le\|\mathbb{A}_{13}\|^2\sum_{i=0}^{k(t)-1}\mathbb{E}\left\{\|\mathbb{A}_4\|^{2i}\|\xi_1(t_{i+1})\|^2\middle|\xi(0),r_0\right\}
\end{aligned}
$$

This, together with (13) and (17), implies that

$$
\lim_{t\to\infty}\mathbb{E}\left\{\sum_{i=0}^{k(t)-1}\|\check{A}(i-1)\|^2\|\widetilde{A}_{13}(r_{t_i})\|^2\|\xi_1(t_{i+1})\|^2\middle|\xi(0),r_0\right\}=0 \qquad (20)
$$

Similarly, the limit of the third term in (18) goes to zero as t goes to infinity. Therefore, from (18), (19) and (20), we can deduce that

$$
\lim_{t\to\infty}\mathbb{E}\left[\|\xi_2(t)\|^2\middle|\xi_2(0),r_0\right]=0; \qquad (21)
$$

This together with (17), implies that system (2) is asymptotically mean square stable. This completes the proof. □

Based on Theorem (3.1), we have the following result for uncertain system (2).

Theorem 3.2: The uncertain singular Markovian jump system (2) is regular, impulse-free and robustly asymptotically mean square stable if there exist a set of nonsingular matrices $P=(P(1),\cdots,P(N))$ and symmetric and positive-definite matrices $Q>0$, $R>0$, $Z_1>0$, $Z_2>0$, and a matrices

$$
N=\begin{bmatrix}N_1\\N_2\\N_3\end{bmatrix},\ S=\begin{bmatrix}S_1\\S_2\\S_3\end{bmatrix},M=\begin{bmatrix}M_1\\M_2\\M_3\end{bmatrix}
$$

and a scalar $\sigma>0$ such that the following set of coupled LMIs holds for each $i\in\mathscr{S}$:

$$
\begin{bmatrix}
\psi(i) & hN & hS & hM & hA_{cl}^\top(i)(Z_1+Z_2) & \Sigma^\top\\
\star & -hZ_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\star & \star & -hZ_1 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\star & \star & \star & -hZ_2 & \mathbf{0} & \mathbf{0}\\
\star & \star & \star & \star & -h(Z_1+Z_2) & h(Z_1+Z_2)D\\
\star & \star & \star & \star & \star & -\sigma\mathbb{I}
\end{bmatrix}<0 \qquad (22)
$$

with the following constraints:

$$
E^\top P(i)=P^\top(i)E\ge 0 \qquad (23)
$$

where

$$
\begin{aligned}
\psi(i)&=\psi_1(i)+\psi_2+\psi_2^\top\\
\psi_1(i)&=\begin{bmatrix}\Theta(i) & P^\top(i)A_d(i)+\sigma E_1^\top(i)E_2(i) & \mathbf{0}\\ \star & -(1-\mu)Q+\sigma E_2^\top(i)E_2(i) & \mathbf{0}\\ \star & \star & -R\end{bmatrix}\\
\psi_2&=\begin{bmatrix}NE+ME & -NE+SE & -ME-SE\end{bmatrix}\\
A_{cl}(i)&=\begin{bmatrix}A(i) & A_1(i) & \mathbf{0}\end{bmatrix}\\
\Sigma&=\begin{bmatrix}D^\top P(i) & \mathbf{0} & \mathbf{0}\end{bmatrix}\\
\Theta(i)&=P(i)A(i)+A^\top(i)P(i)+Q+R+\sum_{j=1}^{N}\lambda_{ij}E^\top P(j)\\
&\quad+\sigma E_1^\top(i)E_1(i)
\end{aligned}
$$

Proof: Replacing $A(i)$ and $A_1(i)$ with $A(i)+DF(i)E_1(i)$ and $A_1(i)+DF(i)E_2(i)$ in (9), respectively, the corresponding formula of (9) of the uncertain system (2) can be written as follows:

$$
\Phi(i)+\begin{bmatrix}P^\top(i)D\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\\ h(Z_1+Z_2)D\end{bmatrix}F(i)\begin{bmatrix}E_1(i) & E_2(i) & \mathbf{0} & \cdots & \mathbf{0}\end{bmatrix}
$$

$$+\begin{bmatrix} E_1^\top(i) \\ E_2^\top(i) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} F^\top(i) \left[\, D^\top P(i) \;\; \mathbf{0} \;\; \cdots \;\; \mathbf{0} \;\; hD^\top(Z_1+Z_2) \,\right] \tag{24}$$

By Lemma (2.1) and Schur complement, (24) holds if there exist a positive number $\sigma > 0$ such that (22) holds. This completes the proof. □

IV. Example

Consider the following singular time-delay system:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A(1) = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}$$
$$A(2) = \begin{bmatrix} -0.5 & -1 \\ 3 & -1 \end{bmatrix}, \qquad A_1(1) = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}$$
$$A_1(2) = \begin{bmatrix} -0.5 & 0 \\ 0.1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$
$$E_1(1) = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \qquad E_2(1) = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}$$
$$E_1(2) = \begin{bmatrix} 0.05 & 0 \end{bmatrix}, \qquad E_2(2) = \begin{bmatrix} 0 & 0.1 \end{bmatrix}$$

with $\mu = 0.5$, $h = 0.4$ and the following transition matrix rates:

$$\Lambda = \begin{bmatrix} -8 & 8 \\ 6 & -6 \end{bmatrix}$$

By Theorem 3.2, it can be confirmed that system (2) is regular, impulse free and asymptotically mean square stable. Figure 1 gives the simulation results of x_1 and x_2 when $d(t) = 0.2 + 0.1\sin(4t)$ and $F(t) = 1$. The initial condition we used for simulation is $(2, 6.2)$.

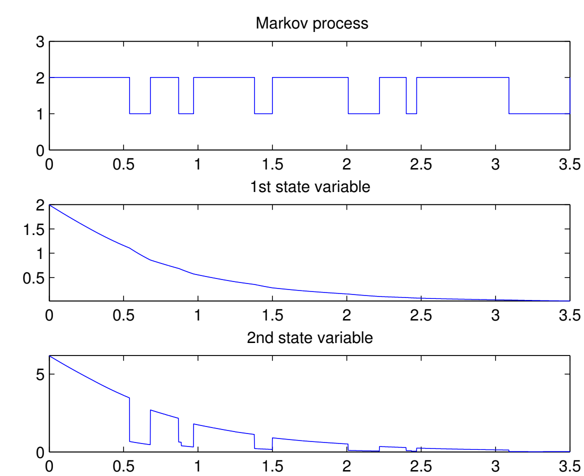

Fig. 1. Simulation results.

As it can be seen from the simulation results, the fast component of the state vector has discontinuities at the jump instants that propagate with time and affect the behavior of this state.

V. Conclusion

This paper dealt with the stability of the class of Markovian jump singular systems with time-varying delay. A delay-dependent stability conditions has been developed for this class of systems. The results we developed in this paper are in the LMI framework which make them tractable using existing tool that handle the convex optimization problems.

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