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Robust Linear Filtering for Continuous-Time Hybrid Markov Linear Systems

O.L.V. Costa and M. D. Fragoso

Abstract— **We consider a class of hybrid systems which is modelled by continuous-time linear systems with Markovian jumps in the parameters (LSMJP). We assume that only an output of the system is available, and therefore the values of the jump parameter are not known. It is desired to design a dynamic linear filter such that the closed loop system is mean square stable and minimizes the stationary expected value of the square error. We consider uncertainties on the parameters of the possible modes of operation of the system. A Linear Matrix Inequalities (LMI) formulation is proposed to solve the problem.**

I. Introduction

There has been by now a steadily rising level of activity with linear systems which are subject to abrupt changes in their structures. This is particularly true for the case in which the abrupt changes are modelled by a Markov chain. In this scenario, linear systems with Markovian jump parameters (LSMJP) outstand with a coherent body of theory. This class have been the subject of extensive research over the last few years and the associated literature is now fairly extensive (see, e.g., [1], [2], [3], [10], [12] and the references therein). This, in turn, has led to a litany of applications in a variety of fields (see, e.g., [3], [10], and references therein). The application of these models includes, for instance, safety-critical and high-integrity systems (e.g., aircraft, chemical plants, nuclear power station, robotic manipulator systems, large scale flexible structures for space stations such as antenna, solar arrays, etc.).

An enormous impetus to the theory of filtering was given with the appearance of the seminal papers [7], [9] and [13], which have been widely celebrated as a great achievement in stochastic systems theory and of fundamental importance in applications. One of the challenging questions that remains in this area is that the description of the optimal nonlinear filter can rarely be given in terms of a closed finite system of stochastic differential equations, i.e., the so-called finite filters (the exceptions are the classical Kalman filter and those described, for instance, in [11]). Unfortunately, this is what happens with the optimal nonlinear filter for the LSMJP model when the jump and state variables are not available. In [4] it was derived the best linear mean square estimator for such systems. The approach adopted there produced a filter which bears those desirable properties of the Kalman filter: a recursive scheme suitable for computer implementation which allows some offline computation that alleviates the computational burden. This filter has dimension Nn, with n denoting the dimension of the state vector and N the number of states of the Markov chain. In [5] it was derived a stationary version of this filter, which was written in terms of a Riccati filter equation, leading to a linear time-invariant filter. In particular it was proved that the covariance matrix of the error converges to a stationary value, which coincides with the unique positive semi-definite solution of the Riccati filter equation.

As in [4] and [5] we consider in this paper that only an output of the system is available, and therefore the values of the jump parameter are not known. We consider uncertainties on the parameters of the possible modes of operation of the system. An LMI approach is proposed to obtain a robust linear filter for the LSMJP, that is, a dynamic linear filter such that the closed loop system is mean square stable and minimizes the stationary expected value of the square error.

A brief outline of the content of this paper is as follows. In section II it is presented some assumptions, notation, the model we will consider and some preliminary results. The dynamic filter that will be considered and some auxiliary results are introduced in section III. In section IV it is presented some mean square stability results and the dynamic filter problem formulation. Finally in section V the LMI filter formulation for the case in which there are uncertainties on the parameters of the possible modes of operation of the system is presented.

II. Notations, Preliminaries, and Auxiliary Results

We shall denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by $\mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$ the normed bounded linear space of all $m \times n$ matrices with $\mathbb{B}(\mathbb{R}^n) := \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$. The $n \times n$ identity matrix will be denoted by I_n or simply by I whenever the dimension is clearly defined. For $L \in \mathbb{B}(\mathbb{R}^n)$, we denote $\sigma(L)$ the spectrum of L, L' will indicate the transpose of L, and $\text{tr}(L)$ the trace of L. As usual, $L \geq 0\,(L > 0)$ will mean that the symmetric matrix $L \in \mathbb{B}(\mathbb{R}^n)$ is positive semi-definite (positive definite), respectively. In addition, we set $\mathbb{B}(\mathbb{R}^n)^+ := \{L \in \mathbb{B}(\mathbb{R}^n); L = L' \geq 0\}$. We use $\mathbb{R}^+$ to denote the interval $[0, \infty)$ and define by $L \otimes K \in \mathbb{B}(\mathbb{R}^{sn}, \mathbb{R}^{rm})$, the Kronecker product for any $L \in \mathbb{B}(\mathbb{R}^s, \mathbb{R}^r)$ and $K \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$. For $D_i \in \mathbb{B}(\mathbb{R}^n)$, $i = 1, \ldots, N$, $diag(D_i)$ stands for an $Nn \times Nn$ matrix where

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O.L.V. Costa is with Departamento de Engenharia de Telecomunicações e Controle, Escola Politécnica da Universidade de São Paulo, CEP: 05508 900-São Paulo, Brazil oswaldo@lac.usp.br

M. D. Fragoso is with National Laboratory for Scientific Computing - LNCC/CNPq, Av. Getulio Vargas 333, Quitandinha, CEP: 25651-070, Petrópolis - RJ, Brazil frag@lncc.br

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the matrices D_i, $i = 1, \ldots, N$, are put together corner-to-corner diagonally, with all other entries being zero, and $\mathrm{dg}(D_j)$ stands for an $Nn \times Nn$ matrix where only D_j is put together corner-to-corner diagonally with all other entries being zero. In addition, set $\mathbb{H}^{n,m}$ the linear space made up of all N-sequences of matrices $V = (V_1, \ldots, V_N)$ with $V_i \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $i = 1, \ldots, N$ and, for simplicity, set $\mathbb{H}^n := \mathbb{H}^{n,n}$. We denote by $I\!R_e\{\lambda\}$ the real part of a complex number λ. For $\mathcal{L} \in \mathbb{B}(\mathbb{H}^n, \mathbb{H}^n)$ we set $I\!R_e\{\lambda(\mathcal{L})\} := \sup\{I\!R_e\{\lambda\}; \lambda \in \sigma(\mathcal{L})\}$. Finally $1_{\{.\}}$ stands for the Dirac measure.

Let $(\Omega, \mathcal{F}, I\!P)$ be a complete probability space carrying its natural filtration $\{\mathcal{F}_t, t \in \mathbb{R}^+\}$, as usual augmented by all null sets in the $I\!P$–completion of $\mathcal{F}$, and consider the class of hybrid dynamical systems modelled by the following Markovian jump linear system. For $t \in \mathbb{R}^+$:

$$dx(t) = A_{\theta_t} x(t)dt + C_{\theta_t} dw(t), \quad (1)$$
$$dy(t) = H_{\theta_t} x(t)dt + G_{\theta_t} dw(t), \quad (2)$$
$$v(t) = L_{\theta_t} x(t), \quad (3)$$

where $\{x(t)\}$ denotes the state vector in $\mathbb{R}^n$ (*signal process*), $\{y(t)\}$ the output process in $\mathbb{R}^m$, which generates the observational information that is available at time t, and $\{v(t)\}$ denotes the vector in $\mathbb{R}^r$ that it is desired to estimate. Furthermore, we assume that:

A.1) $W = \{(w(t), \mathcal{F}_t), t \in \mathbb{R}^+\}$ is a standard Wiener process in $\mathbb{R}^p$.

A.2) θ=$\{(\theta_t, \mathcal{F}_t), t \in \mathbb{R}^+\}$ is a homogeneous ergodic Markov process with right continuous trajectories and taking values on the finite set $\mathcal{S} := \{1, 2, \ldots, N\}$. In addition

$$I\!P(\theta_{t+h} = j | \theta_t = i) = \begin{cases} \lambda_{ij} h + o(h), & i \neq j \\ 1 + \lambda_{ii} h + o(h), & i = j \end{cases}. \quad (4)$$

where $[(\lambda_{ij})]$ is the stationary $N \times N$ transition rate matrix of $\{\theta\}$ with $\lambda_{ij} \geq 0$, $i \neq j$ and $\lambda_i = -\lambda_{ii} = \sum_{j:\ j \neq i} \lambda_{ij} < \infty$. We define $p_{ij}(t)$:=$I\!P(\theta_{t+s} = j | \theta_s = i)$, $i, j = 1, \ldots, N$ and denote $p_i(t)$:=$I\!P(\theta_t = i)$, for any $i \in \mathcal{S}$. Notice that, in this setting, P_t:=$(p_1(t), \ldots, p_N(t))'$, satisfies the Kolmogorov forward differential equation $dP_t/dt = \Lambda P_t$; $P_0 = P$, $t \in \mathbb{R}^+$, where Λ:=$[(\lambda_{ij})]$. We set $p_i = \lim_{t \to \infty} p_i(t) > 0$.

A.3) x and $\{\theta_t\}$ are independent of $\{w(t)\}$.

We set $A = (A_1, \ldots, A_N) \in \mathbb{H}^n$, $C = (C_1, \ldots, C_N) \in \mathbb{H}^{p,n}$, $H = (H_1, \ldots, H_N) \in \mathbb{H}^{n,m}$ and $G = (G_1, \ldots, G_N) \in \mathbb{H}^{p,m}$, $L = (L_1, \ldots, L_N) \in \mathbb{H}^{n,r}$. We define $z_i(t) := x(t) 1_{\{\theta_t = i\}}$ and $Z_i(t) := E\big(z_i(t) z_i(t)'\big)$ $i \in \mathcal{S}$, in $\mathbb{R}^n$ and $\mathbb{B}(\mathbb{R}^n)^+$, respectively, $\mathcal{Z}(t) = (Z_1(t), \ldots, Z_N(t)) \in \mathbb{H}^{n+}$, and $z(t) := (z_1(t)', \ldots, z_N(t)')' \in \mathbb{R}^{Nn}$. Set $\mathcal{R}(t) = (\mathcal{R}_1(t), \ldots, \mathcal{R}_N(t)) \in \mathbb{H}^{n+}$, $\mathcal{R}_i(t) = p_i(t) C_i C_i'$ and $\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_N) \in \mathbb{H}^{n+}$, $\mathcal{R}_i = p_i C_i C_i'$. Set the operator $\mathcal{L} \in \mathbb{B}(\mathbb{H}^n, \mathbb{H}^n)$ as follows: for $P = (P_1, \ldots, P_N) \in \mathbb{H}^n$,

$$\mathcal{L}_j(P) := A_j P_j + P_j A_j' + \textstyle\sum_{i=1}^N \lambda_{ij} P_i. \quad (5)$$

For system (1) it has been shown in [6] that:

Proposition 2.1: For t $\in \mathbb{R}^+$ and i=1,...,N, we have:

$$\dot{\mathcal{Z}}(t) = \mathcal{L}(\mathcal{Z}(t)) + \mathcal{R}(t). \quad (6)$$

If $I\!R_e\{\lambda(\mathcal{L})\} < 0$ then $\mathcal{Z}(t) \to -\mathcal{L}^{-1}(\mathcal{R}) \geq 0$ as $t \to \infty$.

We recall now the following definition of mean square stability for system (1).

Definition 2.2: System (1) is mean square stable (MSS), if with $w(t) = 0$ we have that $E(\|x(t)\|^2) \to 0$ whenever $t \to \infty$ for any initial condition $x(0)$ and θ_0.

The following result was proved in Proposition 5.4 and Theorem 5.2 in [6].

Proposition 2.3: System (1) is MSS if and only if $I\!R_e\{\lambda(\mathcal{L})\} < 0$.

We will need a matricial representation of (6) to get the LMI formulation of the robust filter. In order to do that we consider the following notation: For matrices $Z_i \in \mathbb{B}(\mathbb{R}^n)$, $i \in \mathcal{S}$, and $Z = diag(Z_i) \in \mathbb{B}(\mathbb{R}^{Nn})$, set the operator $\mathcal{V}(Z)$ as follows

$$\mathcal{V}(Z) := diag\left(\sum_{i=1}^N \lambda_{ij} Z_i\right) - \{(\Lambda' \otimes I_n) Z + Z(\Lambda' \otimes I_n)'\}. \quad (7)$$

Define also

$$\mathcal{A} := \Lambda' \otimes I_n + diag(A_i) \in \mathbb{B}\big(\mathbb{R}^{Nn}\big), \quad (8)$$
$$\mathcal{G}_t := \begin{bmatrix} \sqrt{p_1(t)} G_1 & \ldots & \sqrt{p_N(t)} G_N \end{bmatrix} \in \mathbb{B}\big(\mathbb{R}^{Np}, \mathbb{R}^m\big), \quad (9)$$
$$\mathcal{G} := \begin{bmatrix} \sqrt{p_1} G_1 & \ldots & \sqrt{p_N} G_N \end{bmatrix} \in \mathbb{B}\big(\mathbb{R}^{Np}, \mathbb{R}^m\big), \quad (10)$$
$$\mathcal{C}_t := diag(\sqrt{p_i(t)} C_i) \in \mathbb{B}\big(\mathbb{R}^{Np}, \mathbb{R}^{Nn}\big), \quad (11)$$
$$\mathcal{C} := diag(\sqrt{p_i} C_i) \in \mathbb{B}\big(\mathbb{R}^{Np}, \mathbb{R}^{Nn}\big), \quad (12)$$
$$H := \begin{bmatrix} H_1 & \ldots & H_N \end{bmatrix} \in \mathbb{B}\big(\mathbb{R}^{Nn}, \mathbb{R}^m\big), \quad (13)$$
$$L := \begin{bmatrix} L_1 & \ldots & L_N \end{bmatrix} \in \mathbb{B}\big(\mathbb{R}^{Nn}, \mathbb{R}^r\big). \quad (14)$$

Set $Z(t) := E\big(z(t) z(t)'\big) = diag(Z_i(t))$. From (6) it can be easily shown the following matricial equation for $Z(t)$:

$$\dot{Z}(t) = \mathcal{A} Z(t) + Z(t) \mathcal{A}' + \mathcal{V}(Z(t)) + \mathcal{C}_t \mathcal{C}_t'. \quad (15)$$

III. Dynamic Filter

It is desired to design a dynamic estimator $\hat{v}(t)$ for $v(t)$ given in (3) of the following form:

$$d\hat{z}(t) = A_f \hat{z}(t) dt + B_f dy(t), \quad (16)$$
$$\hat{v}(t) = L_f \hat{z}(t), \quad (17)$$
$$e(t) = v(t) - \hat{v}(t), \quad (18)$$

where $A_f \in \mathbb{B}(\mathbb{R}^{n_f}, \mathbb{R}^{n_f})$, $B_f \in \mathbb{B}(\mathbb{R}^m, \mathbb{R}^{n_f})$, $L_f \in \mathbb{B}(\mathbb{R}^{n_f}, \mathbb{R}^r)$, and $e(t)$ denotes the estimation error. Defining $x_e(t)' = \begin{bmatrix} x(t)' & \hat{z}(t)' \end{bmatrix}$, we have from (1), (2) and (16)-(18) that

$$dx_e(t) = \begin{bmatrix} A_{\theta_t} & 0 \\ B_f H_{\theta_t} & A_f \end{bmatrix} x_e(t) dt + \begin{bmatrix} C_{\theta_t} \\ B_f G_{\theta_t} \end{bmatrix} dw(t) \quad (19)$$
$$e(t) = [L_{\theta_t} - L_f] x_e(t)$$

which is a continuous-time Markovian jump linear system. We will be interested in filters such that (19) is mean square stable. Define

$$\hat{Z}(t) := E\big(\hat{z}(t)\hat{z}(t)'\big),$$
$$U(t) := \begin{bmatrix} U_1(t) \\ \vdots \\ U_N(t) \end{bmatrix},\ U_i(t) := E\big(z_i(t)\hat{z}(t)'\big),$$
$$P(t) := E\Big(\begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix}\begin{bmatrix} z(t)' & \hat{z}(t)' \end{bmatrix}\Big) = \begin{bmatrix} Z(t) & U(t) \\ U(t)' & \hat{Z}(t) \end{bmatrix}. \quad (20)$$
$$\widetilde{L} := \begin{bmatrix} L & -L_f \end{bmatrix},\ \widetilde{\mathcal{A}} := \begin{bmatrix} \mathcal{A} & 0 \\ B_f H & A_f \end{bmatrix},$$
$$\widetilde{\mathcal{C}} := \begin{bmatrix} \mathcal{C} \\ B_f\mathcal{G} \end{bmatrix},\ \widetilde{\mathcal{C}}_t := \begin{bmatrix} \mathcal{C}_t \\ B_f\mathcal{G}_t \end{bmatrix}.$$

We have that:

Proposition 3.1: For $t \in \mathbb{R}^+$,

$$\dot{P}(t) = \widetilde{\mathcal{A}}P(t) + P(t)\widetilde{\mathcal{A}}' + \begin{bmatrix} \mathcal{V}(Z(t)) & 0 \\ 0 & 0 \end{bmatrix} + \widetilde{\mathcal{C}}_t\widetilde{\mathcal{C}}_t'. \quad (21)$$

Proof: By Ito's calculus and noting that $B_f H_{\theta_t} x(t) = B_f H z(t)$, we have from (19) that:

$$\dot{\hat{Z}}(t) = B_f H U(t) + A_f\hat{Z}(t) + U(t)'H'B_f' + \hat{Z}(t)A_f' + B_f\mathcal{G}_t\mathcal{G}_t'B_f'. \quad (22)$$

Similarly by Ito's calculus,

$$\dot{U}(t) = \mathcal{A}U(t) + Z(t)H'B_f' + U(t)A_f' + \mathcal{C}_t\mathcal{G}_t'B_f'. \quad (23)$$

By combining (15), (22) and (23) we get (21). ■

We want now to re-write (15) so that the term $\mathcal{V}(Z(t))$ can be decomposed as a sum of matrices. In order to do that we first define $\Gamma_\ell := \big[\varphi_{\ell,i,j}\big]$, $\ell \in \mathcal{S}$ where

$$\varphi_{\ell,i,j} = \begin{cases} |\lambda_{\ell i}| & i = j, \\ -\lambda_{\ell i} & i \neq j,\ j = \ell, \\ -\lambda_{\ell j} & i \neq j,\ i = \ell, \\ 0 & \text{otherwise} \end{cases}.$$

After some straightforward calculations we have that (7) can be re-written, for $Z = diag(Z_i)$, as

$$\mathcal{V}(Z) = \sum_{\ell=1}^{N}\Big(\Gamma_\ell \otimes I_n\Big)\mathrm{dg}(Z_\ell).$$

We have the following result:

Proposition 3.2: $\Gamma_\ell \geq 0$.

Proof: For any vector $v' = \begin{bmatrix} v_1 & \cdots & v_N \end{bmatrix}$, we have that

$$v'\Gamma_\ell v = \sum_{i=1}^{N}\sum_{j=1}^{N}\varphi_{\ell,i,j}v_iv_j = |\lambda_{\ell\ell}|v_\ell^2 + \sum_{i\neq\ell}\lambda_{\ell i}v_i^2 - v_\ell\Big(\sum_{j\neq\ell}\lambda_{\ell j}v_j\Big) - \Big(\sum_{i\neq\ell}v_i\lambda_{\ell i}\Big)v_\ell.$$

But recalling that $-\lambda_{\ell\ell} = \sum_{j\neq\ell}\lambda_{\ell j}$, we get that

$$v'\Gamma_\ell v = \sum_{i\neq\ell}\lambda_{\ell i}v_\ell^2 + \sum_{i\neq\ell}\lambda_{\ell i}v_i^2 - 2\Big(\sum_{i\neq\ell}\lambda_{\ell i}v_iv_\ell\Big) = \sum_{i\neq\ell}\lambda_{\ell i}\Big(v_\ell^2 + v_i^2 - 2v_iv_\ell\Big) = \sum_{i\neq\ell}\lambda_{\ell i}\Big(v_\ell - v_i\Big)^2 \geq 0$$

showing the desired result. ■

It follows from Proposition 3.2 that for $Z = diag(Z_i) \geq 0$, we have that

$$\mathcal{V}(Z) = \sum_{\ell=1}^{N}(\Gamma_\ell^{1/2}\otimes I_n)(\Gamma_\ell^{1/2}\otimes I_n)\mathrm{dg}(Z_\ell) = \sum_{\ell=1}^{N}(\Gamma_\ell^{1/2}\otimes I_n)\mathrm{dg}(Z_\ell)(\Gamma_\ell^{1/2}\otimes I_n) \geq 0. \quad (24)$$

Therefore, writing for $\ell \in \mathcal{S}$, $\Psi_\ell = \Gamma_\ell^{1/2}\otimes I_n$ and $\Upsilon_\ell := \begin{bmatrix} \Psi_\ell \\ 0 \end{bmatrix}$ we have from (21) that

$$\dot{P}(t) = \widetilde{\mathcal{A}}P(t) + P(t)\widetilde{\mathcal{A}}' + \sum_{\ell=1}^{N}\Upsilon_\ell\mathrm{dg}(Z_\ell(t))\Upsilon_\ell' + \widetilde{\mathcal{C}}_t\widetilde{\mathcal{C}}_t'. \quad (25)$$

IV. Mean Square Stability Conditions

In this section we present some conditions to get the mean square stability of the system (19). First we present a necessary and sufficient condition based on the spectrum of the operator $\mathcal{L}$ and the matrix A_f.

Proposition 4.1: System (19) is MSS if and only if $\mathbb{R}_e\{\lambda(\mathcal{L})\} < 0$ and A_f is a stable matrix.

Proof: ($\Longrightarrow$) If (19) is MSS then, considering $w(t) = 0$ in (19), we have for any initial condition $x_e(0)$, θ_0, that

$$E\big(\|x_e(t)\|^2\big) = E\big(\|x(t)\|^2\big) + E\big(\|\hat{z}(t)\|^2\big) \overset{t\to\infty}{\to} 0,$$

that is, $E\big(\|x(t)\|^2\big) \overset{t\to\infty}{\to} 0$ and $E\big(\|\hat{z}(t)\|^2\big) \overset{t\to\infty}{\to} 0$. From [6], $\mathbb{R}_e\{\lambda(\mathcal{L})\} < 0$. Consider now an initial condition $x_e(0) = \begin{pmatrix} 0' & \hat{z}(0)' \end{pmatrix}'$. Then clearly $\hat{z}(t) = e^{A_f t}\hat{z}(0)$ and since $\hat{z}(t) \overset{t\to\infty}{\to} 0$ for any initial condition $\hat{z}(0)$, it follows that A_f is a stable matrix.

($\Longleftarrow$) Suppose now that $\mathbb{R}_e\{\lambda(\mathcal{L})\} < 0$ and A_f is a stable matrix. From [6], $\mathbb{R}_e\{\lambda(\mathcal{L})\} < 0$ implies that $\|e^{\mathcal{L}t}\| \leq ae^{-bt}$ for some $a > 0$, $b > 0$ and $t \in \mathbb{R}^+$. From (19) with $w(t) = 0$ and according [6], $E\big(\|x(t)\|^2\big) \leq ce^{-bt}\|x(0)\|^2$ for some $c > 0$. Moreover,

$$d\hat{z}(t) = A_f\hat{z}(t)dt + B_fH_{\theta(t)}x(t)dt$$

and thus

$$\hat{z}(t) = e^{A_f t}\hat{z}(0) + \int_0^t e^{A_f(t-s)}B_fH_{\theta(s)}x(s)ds.$$

From the triangular inequality,

$$E\big(\|\hat{z}(t)\|^2\big)^{1/2} \leq E\big(\|e^{A_f t}\hat{z}(0)\|^2\big)^{1/2} + \int_0^t E\big(\|e^{A_f(t-s)}B_fH_{\theta(s)}x(s)\|^2\big)^{1/2}ds$$

$$\leq \|e^{A_f t}\| E\big(\|\hat{z}(0)\|^2\big)^{1/2} + \|B_f\| \|H\|_{max} \int_0^t \|e^{A_f(t-s)}\| E\big(\|x(s)\|^2\big)^{1/2} ds$$

where $\|H\|_{max} = max\{\|H_i\|, i = 1,\dots,N\}$. Since A_f is stable, we can find $a' > 0$, $b' > 0$, such that $\|e^{A_f t}\| \leq a' e^{-b' t}$. Then, for some $\bar{a} > 0$, $\bar{b} > 0$,

$$E\big(\|\hat{z}(t)\|^2\big)^{1/2} \leq \bar{a}\Big(e^{-b' t}\|\hat{z}(0)\| + e^{-\bar{b}t} t\|x(0)\|\Big)$$

showing that $E\big(\|x_e(t)\|^2\big) = E\big(\|x(t)\|^2\big) + E\big(\|\hat{z}(t)\|^2\big) \overset{t\to\infty}{\to} 0$ for any initial condition $x_e(0)$, θ_0, so that, from [6], system (19) is MSS. ■

The next proposition presents a necessary and sufficient condition for MSS of system (19) based on an LMI representation.

Proposition 4.2: System (19) is MSS if and only if there exists $P > 0$, with

$$P = \begin{bmatrix} diag(Q_i) & U \\ U' & \hat{Z} \end{bmatrix}$$

such that

$$\widetilde{\mathcal{A}}P + P\widetilde{\mathcal{A}}' + \sum_{\ell=1}^{N} \Upsilon_\ell \text{dg}(Q_\ell)\Upsilon_\ell' < 0. \tag{26}$$

Proof: ($\Longrightarrow$) Consider the operator $\tilde{\mathcal{L}} \in \mathbb{B}(\mathbb{H}^{n+n_f}, \mathbb{H}^{n+n_f})$ as follows: for $\tilde{V} = (\tilde{V}_1,\dots,\tilde{V}_N) \in \mathbb{H}^{n+n_f}$, $\tilde{\mathcal{L}}(\tilde{V}) = (\tilde{\mathcal{L}}_1(\tilde{V}),\dots,\tilde{\mathcal{L}}_N(\tilde{V}))$ is given by:

$$\tilde{\mathcal{L}}_j(V) := \tilde{A}_j V_j + V_j \tilde{A}_j' + \sum_{i=1}^{N} \lambda_{ij} V_i. \tag{27}$$

where

$$\tilde{A}_i = \begin{bmatrix} A_i & 0 \\ B_f H_i & A_f \end{bmatrix}.$$

Consider model (19) with $w(t) = 0$, and for $t \in \mathbb{R}^+$,

$$\tilde{P}_i(t) = \begin{bmatrix} E(z_i(t)z_i(t))' & E(z_i(t)\hat{z}(t)') \\ E(\hat{z}(t)z_i(t)') & E(\hat{z}(t)\hat{z}(t)' 1_{\theta(t)=i}) \end{bmatrix}, \tag{28}$$

and $\tilde{P}(t) = (\tilde{P}_1(t),\dots,\tilde{P}_N(t)) \in \mathbb{H}^{n+n_f}$. From Proposition 2.1, $\dot{\tilde{P}}(t) = \tilde{\mathcal{L}}(\tilde{P}(t))$, and from [6] system (19) is MSS if and only if there exists $\tilde{P} = (\tilde{P}_1,\dots,\tilde{P}_N) \in \mathbb{H}^{n+n_f}$, $\tilde{P}_i > 0$, $i = 1,\dots,N$, such that (see [6])

$$\tilde{\mathcal{L}}_j(\tilde{P}) < 0, \qquad j \in \mathcal{S}. \tag{29}$$

Partitionate $\tilde{P}_j$ as follows

$$\tilde{P}_j = \begin{bmatrix} Q_j & U_j \\ U_j' & \hat{Z}_j \end{bmatrix}, \quad j \in \mathcal{S}$$

where $Q_j \in \mathbb{B}(\mathbb{R}^n)$, and $\hat{Z}_j \in \mathbb{B}(\mathbb{R}^{n_f})$, and define $\hat{Z} = \sum_{j=1}^{N} \hat{Z}_j$, $Z = diag(Q_i)$, and

$$U = \begin{bmatrix} U_1 \\ \vdots \\ U_N \end{bmatrix}, \qquad P = \begin{bmatrix} Z & U \\ U' & \hat{Z} \end{bmatrix}.$$

Notice that $P > 0$. Indeed since for each $j \in \mathcal{S}$, $\tilde{P}_j > 0$, it follows from Schur's complement that $\hat{Z}_j > U_j' Z_j^{-1} U_j$, for $j \in \mathcal{S}$, so that

$$\hat{Z} = \sum_{j=1}^{N} \hat{Z}_j > \sum_{j=1}^{N} U_j' Q_j^{-1} U_j = U' Z^{-1} U.$$

From Schur's complement again it follows that

$$\begin{bmatrix} Z & U \\ U' & \hat{Z} \end{bmatrix} = P > 0.$$

Re-organizing equations (29) and using (24) we obtain that (26) holds.

($\Longleftarrow$) From (26) and (24), it follows that $\mathcal{L}(Q) < 0$ where $Q = (Q_1,\dots,Q_N) \in \mathbb{H}^n$ which means, from [6], that $\mathbb{R}_e\{\lambda(\mathcal{L})\} < 0$. From the Lyapunov equation (26), it is easy to see that $\widetilde{\mathcal{A}}$ is stable, and thus in particular A_f is stable. Since $\mathbb{R}_e\{\lambda(\mathcal{L})\} < 0$, A_f is stable it follows from Proposition 4.1 that system (19) is MSS. ■

The next result guarantees the convergence of $P(t)$ as defined in (20), (21) to a $P \geq 0$ when $t \to \infty$.

Proposition 4.3: Consider $P(t)$ given by (20), (21) and that $\mathbb{R}_e\{\lambda(\mathcal{L})\} < 0$, A_f is stable. Then $P(t) \overset{t\to\infty}{\to} P \geq 0$, with P of the following form:

$$P = \begin{bmatrix} Z & U \\ U' & \hat{Z} \end{bmatrix}, \qquad Z = diag(Q_i) \geq 0.$$

Moreover, P is the only solution of the equation in V

$$0 = \widetilde{\mathcal{A}}V + V\widetilde{\mathcal{A}}' + \sum_{\ell=1}^{N} \Upsilon_\ell \text{dg}(X_\ell)\Upsilon_\ell' + \mathcal{C}\mathcal{C}' \tag{30}$$

$$V = \begin{bmatrix} X & R \\ R' & \hat{X} \end{bmatrix}, \qquad X = diag(X_i). \tag{31}$$

Furthermore if V satisfies

$$0 \geq \widetilde{\mathcal{A}}V + V\widetilde{\mathcal{A}}' + \sum_{\ell=1}^{N} \Upsilon_\ell \text{dg}(X_\ell)\Upsilon_\ell' + \widetilde{\mathcal{C}}\widetilde{\mathcal{C}}' \tag{32}$$

then $V \geq P$.

Proof: Consider the operator $\tilde{\mathcal{L}}(\cdot)$ as in (27), $\tilde{P}_i(t)$ as in (28). As shown in [6],

$$\dot{\tilde{P}}_j(t) = \tilde{\mathcal{L}}_j(\tilde{P}(t)) + p_j(t) \begin{bmatrix} C_j C_j' & C_j G_j' B_f' \\ B_f G_j C_j' & B_f G_j G_j' B_f' \end{bmatrix}$$

and $\tilde{P}_j(t) \overset{t\to\infty}{\to} \tilde{P}_j \geq 0$, with $\tilde{P} = (\tilde{P}_1,\dots,\tilde{P}_N)$ satisfying

$$0 = \tilde{\mathcal{L}}_j(\tilde{P}) + p_i \begin{bmatrix} C_j C_j' & C_j G_j' B_f' \\ B_f G_j C_j' & B_f G_j G_j' B_f' \end{bmatrix}. \tag{33}$$

Note that $\tilde{P}_i(t) = \begin{bmatrix} Q_i(t) & U_i(t) \\ U_i'(t) & \hat{Q}_i(t) \end{bmatrix} \overset{k\to\infty}{\to} \begin{bmatrix} Q_i & U_i \\ U_i' & \hat{Q}_i \end{bmatrix}$ where $\hat{Q}_i(t) = E(\hat{z}(t)\hat{z}(t)' 1_{\{\theta(t)=i\}})$. Moreover

$$\hat{Z}(t) = \sum_{i=1}^{N} \hat{Q}_i(t) \overset{t\to\infty}{\to} \sum_{i=1}^{N} \hat{Q}_i = \hat{Z}.$$

Defining $P = \begin{bmatrix} Z & U \\ U' & \hat{Z} \end{bmatrix}$, $Z = diag(Q_i)$, $U = \begin{bmatrix} U_1 \\ \vdots \\ U_N \end{bmatrix}$ it follows that $P(t) \stackrel{t\to\infty}{\to} P$. Furthermore from (33) we have that P satisfies (30), (31). Suppose that V also satisfies (30), (31). Then $0 = A_jX_j + X_jA'_j + \sum_{i=1}^{N} \lambda_{ij}X_i$, $0 = A_jQ_j + Q_jA'_j + \sum_{i=1}^{N} \lambda_{ij}Q_i$, and since $\mathbb{R}e\{\lambda(\mathcal{L})\} < 0$ we have from [6] that $Q_j = X_j$, $j \in \mathcal{S}$. This yields that $0 = \widetilde{\mathcal{A}}(V - P) + (V - P)\widetilde{\mathcal{A}}'$. From [6], Proposition 4.3, $\mathbb{R}e\{\lambda(\mathcal{L})\} < 0$ implies that $\mathcal{A}$ is stable and thus that the block diagonal matrix $\widetilde{\mathcal{A}}$ is also stable, yielding that $V = P$. Finally suppose that V is such that (31), (32) are satisfied. Then $0 \geq A_jX_j + X_jA'_j + \sum_{i=1}^{N} \lambda_{ij}X_i$ and it follows that $0 \geq \mathcal{L}(X - Q)$ where $X = (X_1, \ldots, X_N)$. This implies from Proposition 5.6 in [6] that $X \geq Q$. Using this fact, we conclude that $0 \geq \widetilde{\mathcal{A}}(V - P) + (V - P)\widetilde{\mathcal{A}}'$ and again from stability of $\widetilde{\mathcal{A}}$ it follows that $V - P \geq 0$. ■

Recall that $v(t) = Lz(t)$ and that $e(t) = v(t) - \hat{v}(t) = Lz(t) - L_f\hat{z}(t)$. We will be interested in finding (A_f, B_f, L_f) such that A_f is stable and minimizes $\lim_{t\to\infty} E(\|e(t)\|^2)$, that is, minimizes

$$\lim_{t\to\infty} E(\|e(t)\|^2) = \text{tr}\Big(\lim_{t\to\infty} E\big(e(t)e(t)'\big)\Big) = \text{tr}\Big(\widetilde{L} \lim_{t\to\infty} P(t)\widetilde{L}'\Big) = \text{tr}\Big(\widetilde{L}P\widetilde{L}'\Big) \tag{34}$$

where the last equality follows from Proposition 4.3 and

$$P = \begin{bmatrix} Z & U \\ U' & \hat{Z} \end{bmatrix}, \qquad Z = diag(Q_i)$$

which satisfies, according to Proposition 4.3, the equation

$$0 = \widetilde{\mathcal{A}}P + P\widetilde{\mathcal{A}}' + \sum_{\ell=1}^{N} \Upsilon_\ell \text{dg}(Q_\ell)\Upsilon'_\ell + \widetilde{\mathcal{C}}\widetilde{\mathcal{C}}'. \tag{35}$$

V. LMI Formulation

In this section we shall formulate the filter problem seen in the previous section as an LMI optimization problem. In the sequel we present the robust version of the filter. From the results of the previous section we have that the problem we want to solve is:

$$\min \ \text{tr}(W) \quad \text{subject to:}$$

$$P = \begin{bmatrix} Z & U \\ U' & \hat{Z} \end{bmatrix} > 0, \ Z = diag(Q_i), \tag{36}$$

$$\begin{bmatrix} P & P\widetilde{L}' \\ \widetilde{L}P & W \end{bmatrix} > 0 \tag{37}$$

$$\begin{bmatrix} \widetilde{\mathcal{A}}P + P\widetilde{\mathcal{A}}' & \Upsilon_1\text{dg}(Q_1) & \ldots & \Upsilon_N\text{dg}(Q_N) & \widetilde{\mathcal{C}} \\ \text{dg}(Q_1)\Upsilon'_1 & -\text{dg}(Q_1) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{dg}(Q_N)\Upsilon'_N & 0 & \ldots & -\text{dg}(Q_N) & 0 \\ \widetilde{\mathcal{C}}' & 0 & \ldots & 0 & -I \end{bmatrix} < 0. \tag{38}$$

Indeed, suppose that $\bar{P} > 0$, $\bar{W} > 0$, (A_f, B_f, L_f) satisfy (36), (37) and (38). Then from Schur's complement,

$$\text{tr}(\bar{W}) > \text{tr}(\widetilde{L}\bar{P}\widetilde{L}') \tag{39}$$

$$0 > \widetilde{\mathcal{A}}\bar{P} + \bar{P}\widetilde{\mathcal{A}}' + \sum_{\ell=1}^{N} \Upsilon_\ell \text{dg}(\bar{Q}_\ell)\Upsilon'_\ell + \widetilde{\mathcal{C}}\widetilde{\mathcal{C}}'. \tag{40}$$

From Proposition 4.2 we have that system (19) is MSS. From Proposition 4.3 we know that $P(t) \stackrel{t\to\infty}{\to} P$, and (34) e (35) hold. Furthermore, from Proposition 4.3, we have that $\bar{P} \geq P$. Since we want to minimize tr(W), it is clear that for (A_f, B_f, L_f) fixed, the best solution would be, in the limit, P, W satisfying (39), (40) with equality. However, as it will be shown next, it is more convenient to work with the strict inequality restrictions (36)-(38). We consider from now on that $n_f = Nn$.

Theorem 5.1: The problem of finding P, W, and (A_f, B_f, L_f) such that minimizes tr(W) and satisfies (36)-(38) is equivalent to:

$$\min \ \text{tr}(W) \quad \text{subject to}$$

$$X = diag(X_i), \tag{41}$$

$$\begin{bmatrix} X & X & L' - J' \\ X & Y & L' \\ L - J & L & W \end{bmatrix} > 0, \tag{42}$$

$$\begin{bmatrix} P(X, Y, F, R, H, \mathcal{A}) & S(X, Y) & T(X, Y, \mathcal{C}, \mathcal{G}) \\ S(X, Y)' & -D(X) & 0 \\ T(X, Y, \mathcal{C}, \mathcal{G})' & 0 & -I \end{bmatrix} < 0, \tag{43}$$

where

$$P(X, Y, F, R, H, \mathcal{A}) = \begin{bmatrix} X\mathcal{A} + \mathcal{A}'X & X\mathcal{A} + \mathcal{A}'Y + H'F' + R \\ \mathcal{A}'X + Y\mathcal{A} + FH + R & \mathcal{A}'Y + Y\mathcal{A} + FH + H'F' \end{bmatrix},$$

$$S(X, Y) = \begin{bmatrix} X\Psi_1 & \ldots & X\Psi_N \\ Y\Psi_1 & \ldots & Y\Psi_N \end{bmatrix},$$

$$T(X, Y, \mathcal{C}, \mathcal{G}) = \begin{bmatrix} X\mathcal{C} \\ Y\mathcal{C} + F\mathcal{G} \end{bmatrix},$$

$$D(X) = \begin{bmatrix} \text{dg}(X_1) & \ldots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \ldots & \text{dg}(X_N) \end{bmatrix},$$

which are now LMI since the variables are X_i, $i = 1, \ldots, N$, Y, W, R, F, J. Once we have X, Y, R, F, we recover P, A_f, B_f, L_f as follows. Choose a non-singular $(Nn \times Nn)$ matrix U, make $Z = X^{-1} = diag(X_i^{-1}) = diag(Q_i)$, and choose $\hat{Z} > U'Z^{-1}U$. Define $V = Y(Y^{-1} - Z)(U')^{-1}$ (which is non-singular since from (42), $X > XY^{-1}X \Rightarrow Z = X^{-1} > Y^{-1}$). Then

$$A_f = V^{-1}R(U'X)^{-1}, \tag{44}$$

$$B_f = V^{-1}F, \tag{45}$$

$$L_f = J(U'X)^{-1}. \tag{46}$$

Proof: For (A_f, B_f, L_f) fixed, consider P, W satisfying (36)-(38). Without loss of generality, suppose further that U is non-singular (if not, re-define U as $U + \epsilon I$ so that it is non-singular). As in [8] define

$$P^{-1} = \begin{bmatrix} Y & V \\ V' & \hat{Y} \end{bmatrix} > 0$$

where $Y > 0$ and $\hat{Y} > 0$ are $Nn \times Nn$. We have that $ZY + UV' = I$, $U'Y + \hat{Z}V' = 0$, and thus $Y^{-1} = Z + UV'Y^{-1} = Z - U\hat{Z}^{-1}U' < Z$, $V' = U^{-1} - U^{-1}ZY = U^{-1}(Y^{-1} - Z)Y$ implying that V is a non-singular. Define the non-singular $2Nn \times 2Nn$ matrix

$$T = \begin{bmatrix} Z^{-1} & Y \\ 0 & V' \end{bmatrix}$$

and the non-singular matrices T_1, T_2 as follows:

$$T_1 = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \qquad T_2 = \begin{bmatrix} T & 0 & 0 & 0 & 0 \\ 0 & \mathrm{dg}(Q_1^{-1}) & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \mathrm{dg}(Q_N^{-1}) & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}.$$

Set $X = Z^{-1} = diag(X_i)$, $X_i = Q_i^{-1}$, $i = 1, \ldots, N$, $J = L_f U' Z^{-1} = L_f U' X$, $F = VB_f$ and $R = VA_f U' Z^{-1} = VA_f U' X$. Pre and pos multiply (37) by T_1' and T_1 yields (42). Pre and pos multiply (38) by T_2' and T_2 yields (43), showing the desired result. ■

Assume now that $A = (A_1, \ldots, A_N) \in \mathbb{H}^n$, $C = (C_1, \ldots, C_N) \in \mathbb{H}^{p,n}$, $H = (H_1, \ldots, H_N) \in \mathbb{H}^{n,m}$ and $G = (G_1, \ldots, G_N) \in \mathbb{H}^{p,m}$, are not exactly known but instead there are known matrices $A^j = (A_1^j, \ldots, A_N^j) \in \mathbb{H}^n$, $C^j = (C_1^j, \ldots, C_N^j) \in \mathbb{H}^{p,n}$, $H^j = (H_1^j, \ldots, H_N^j) \in \mathbb{H}^{n,m}$, $G^j = (G_1^j, \ldots, G_N^j) \in \mathbb{H}^{p,m}$, $j = 1, \ldots, M$ such that for $0 \leq \lambda_j \leq 1$, $\sum_{j=1}^M \lambda_j = 1$, we have that

$$A = \sum_{j=1}^M \lambda_j A^j, \qquad C = \sum_{j=1}^M \lambda_j C^j$$
$$H = \sum_{j=1}^M \lambda_j H^j, \qquad G = \sum_{j=1}^M \lambda_j G^j. \tag{47}$$

Define $\mathcal{A}^j$, $\mathcal{C}^j$, $\mathcal{G}^j$ as respectively in (8), (10), (12) replacing A_i, G_i, C_i by A_i^j, G_i^j, C_i^j.

Our final result presents the robust linear filter for system (1)-(3):

Theorem 5.2: Suppose that the following LMI optimization problem has an $(\epsilon-)$ optimal solution $\bar{X}, \bar{Y}, \bar{W}, \bar{R}, \bar{F}, \bar{J}$:

$$\min \ \mathrm{tr}(W) \quad \text{subject to } (41), (42) \text{ and for } j = 1, \ldots, M,$$
$$\begin{bmatrix} P(X, Y, F, R, H^j, \mathcal{A}^j) & S(X, Y) & T(X, Y, \mathcal{C}^j, \mathcal{G}^j) \\ S(X, Y)' & -D(X) & 0 \\ T(X, Y, \mathcal{C}^j, \mathcal{G}^j)' & 0 & -I \end{bmatrix} < 0. \tag{48}$$

Then for the filter given as in (44)-(46) we have that system (19) is MSS, and $\lim_{t \to \infty} E(\|e(t)\|^2) \leq \mathrm{tr}(\bar{W})$.

Proof: Since (48) holds for each $j = 1, \ldots, M$, and A, H, C, G are as in (47), we have that (by taking the sum of (48) multiplied by λ_j, over j from 1 to M) that (48) also holds for $\mathcal{A}$, $\mathcal{C}$, $\mathcal{G}$, H. From Theorem 5.1 we get the result. ■

VI. CONCLUSION

In this paper we have considered the filtering problem for LSMJP in the case in which, both, the base-state $\{x(t)\}$ and the regime $\{\theta(t)\}$, are not accessible. Although much work has been done in this theme, the problem still remains a challenging one, since the optimal nonlinear filter is not finite. In view of this, it is considered either a particular structure for the Markov process or some approximated filters.

This paper is, to some extent, a continuation of previous papers by the authors (see, e.g., [4] and [5]). Besides bridging some gaps in the previous papers, we formulate *the LMI and robust version for the optimal stationary linear filter*. The result is tailored in such a way that it just requires to solve off-line a set of algebraic equations.

It is perhaps noteworthy here that, when compared with the Kalman filter and the optimal filter for the case in which the regime is known, our filter has an additional complexity in the sense that the innovation gain is given in terms of two algebraic equations.

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