Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Delay compensation in packet-switching networked controlled systems

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Abstract— **In this paper, we consider the problem of stabilizing sufficiently smooth nonlinear time-invariant plants over a network whereby feedback is closed through a limited-bandwidth digital channel. Reliable packet switching networks are explicitly considered, for which both the time between consecutive accesses to each node (MATI) and the delay by which each data packet is received, processed, and fed back to the plant are unknown but bounded. For what concerns networked feedback control, the main difference between a packet-switching and a circuit-switching network with the same bandwidth is that packets can convey larger amounts of feedback data (measurements and control inputs) with much higher latency and jitter than a conventional communication channel. To compensate the unpredictably varying delays in packet switching networked control systems, we propose a model-based strategy that exploits the relatively high payload which can be associated to each packet. A bound on the tolerable delays and access frequency is explicitly provided.**

I. INTRODUCTION

For network controlled systems (NCS), the communication between the plant and its controller is transmitted over a digital network [18]. A digital communication, whether based on a wired or a wireless device, yields important alterations of the sent information: sampling and quantization, simultaneous access to only part of the nodes, delays, packet losses, *etc.* This influence cannot be neglected in applications that involve a great number of sensors and actuators, or when the nodes are physically distant (distributed), or in case of a particularly limited bandwidth, The effects induced by these phenomena may drastically hamper the performances of the closed-loop plant, and even result in an unstable behavior. New strategies are required to guarantee an acceptable behavior of the closed-loop system under these particular constraints.

Several recent results have aimed at analyzing or compensating these effects: see for instance [16], [3], [2], [18], [13], [11], [8], [15], [17], [6]. Let us underline that the recent accession of *hybrid systems*, *i.e.* systems whose dynamics is both discrete and continuous, offer an ideal framework to address the questions relative to control over network in a both general, realistic and powerful formalism. Indeed, most of the considered plants are continuous by nature, whereas networks are intrinsically discrete. In their recent work [8], Nešić and Teel propose a general formulation for network controlled systems that takes into account sampling, quantization, packet losses and scheduling. However, rare are the analysis tools or the control approaches that allow to take into account all these phenomena, including delays. Communication delays in network-controlled problems constitute a crucial issue that cannot be left apart. The present article aims at using the hybrid formalism recently proposed by Nešić and Teel in order to study a control strategy that compensates the effect of sufficiently small delays, while taking into account the possible physical distribution of nodes (making impossible to have an instantaneous global knowledge of the plant), that data access to the network is ruled by a protocol (thus constraining the access times), and an unperfect model of the plant. To the best of our knowledge, this constitutes the first bridge between two analysis approaches: that combining sampling and delays, such as [2], [6], [5], [1], [17], [9], and that based on a hybrid formalism oriented more towards limited nodes simultaneous access, quantization, *etc.* such as [12], [8], [7]. Following the general formalism introduced in [8], our results apply to nonlinear plants, while most of the literature in this field addresses linear dynamics, *e.g.* [12], [2], [5], [17], [6], [14].

In the spirit of [9], [10], [5], the approach presented here consists in exploiting the possibly large payload available on each packet by sending, whenever possible, not only the value of the control law to be applied at this instant, but also a prediction of the control law that will be applied, obtained based on an (imprecise) model of the plant. The so-obtained control-packet, thus containing a *sequence* of control values valid on a given time-horizon, is then stocked in an embedded memory. Based on a local re-synchronization, made possible by a time-stamping of measurements, we are then able to compensate the effect of sufficiently small communication delays in the control loop. Contrarily to [10] which uses model predictive control, our result exploits the controller that would stabilize the plant if network effects were not present.

After having presented the context and formulated the problem, we present, in Section III, an explicit bound on the maximum tolerable delay, as well as on the Maximum Allowable Time Interval (MATI [12], *i.e.* the maximum duration between two successive communications) are given under which global exponential stability is guaranteed. These bounds depend on the bandwidth of the network, as well as on the precision of the available model of the plant (from which the control prediction is derived). The result is obtained in a nonlinear context, and then adapted to linear time-invariant plants. As a second step, in Section IV, we exhibit the trade-off existing between embedded computation capabilities and communication bandwidth, by showing that, thanks to a simplest embedded computer, the restriction of the domain of attraction and the resulting steady-state error of the closed loop system can be reduce at will provided a sufficiently large size of the packets sent. Proofs are provided in Section V.

II. PROBLEM STATEMENT

A. Notations and Assumptions

In the context of this work, the digital network is seen as a communication channel which allows the sending of a limited information of a *unique* node information at a time (*e.g.*, only partial instantaneous knowledge of the plant state if several sensors are present), at given instants of time and with variable delays.

This work was supported by a fellowship from Università di Pisa, Italy.

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More precisely, we assume that sensors measurements on the plant are sent at instants $\{\tau_i^m\}_{i\in\mathbb{N}}$. These measurements are assumed to be coded with sufficient precision to neglect quantization effects, but do not transit instantaneously; we denote by $\{T_i^m\}_{i\in\mathbb{N}}$ the (variable) measurements data delays, which cover delays resulting from both processing and transmission. Based on the measurements received, the controller then computes control signals which are then encoded into packets that are sent over the networks at time instants $\{\tau_j^c\}_{j\in\mathbb{N}}$. Depending on the quantity of information to be coded into these control packets, imprecision resulting from sampling may need to be considered. These control packet finally reach the plant after a variable delay $\{T_j^c\}_{j\in\mathbb{N}}$ which covers both transmission and computation delays. In this setting, measurements thus arrive at the controller at instants $\{\tau_i^m+T_i^m\}_{i\in\mathbb{N}}$, while control packets reach the plant at instants $\{\tau_j^c+T_j^c\}_{j\in\mathbb{N}}$.

We assume that the maximum time interval (MATI) between two consecutive successful accesses to the network is bounded, both on the measurement side (*i.e.*, from the plant to the controller) and on the control side (*i.e.*, from controller to the plant.

Assumption 1 (MATI) *There exist two constants $h_m, h_c \geq 0$ such that $\tau_{i+1}^m - \tau_i^m \leq h_m$ and $\tau_{j+1}^c - \tau_j^c \leq h_c$ for all $i, j \in \mathbb{N}$.*

We also assume that the delays cannot overpass a certain limit.

Assumption 2 (Bounded delays) *There exist two constants $T_m, T_c \geq 0$ such that $T_i^m \leq T_m$ and $T_j^c \leq T_c$ for all $i, j \in \mathbb{N}$.*

At each instant τ_i^m at which a measurement can transit over the network, the error z between the state estimate $\hat{x}$ and the actual state measure x of the system is somehow evaluated (relying on the measurements available from the sensors) and, based on this, a protocol decides which node (sensor) should communicate its data among all nodes data available at this time instant. The fact that a unique node may communicate at a time generates a dynamical error between the information sent by the plant and the data actually received by the controller, even without considering delays and sampling. In the spirit of [8], we model the network protocol as a time-varying discrete system involving the error $z \in \mathbb{R}^n$ that this type of communication generates:

$$z(i+1) = h_i(z(i)), \quad \forall i \in \mathbb{N},$$

If the network was able to send the measurement of the whole state x at each time instant τ_i^m, then the function h_i would be identically zero; this is an assumption commonly posed in the literature on network controlled systems (see for instance [2], [19], [5], [1], [17], [6], [10], [9]) which may not be justified when numerous sensors are involved or when they are physically distributed.

Note that some protocols are purely static, such as the Round Robin protocol which executes a cyclic inspection of each node, in which case the function h_i does not depend on $z = \hat{x} - x$ but purely on i. On the opposite, some networks purely relies on the present value of the network error, in which case h is independent of i: this is the case of the Try-Once-Discard protocol [12].

We will see in the sequel that the fact of considering network communication on both sides of the feedback loop may impede the exploitation of some measurement sent. We therefore introduce the following definition, which is more restrictive than the original definition of a UGES protocol [8, Definition 7].

Definition 1 (Invariably UGES protocol) *Given positive constants $\underline{a}$, $\overline{a}$, c and $\rho \in (0;1)$, the protocol defined by the discrete dynamics*

$$\begin{cases} z(0) &= z_0 \\ z(k+1) &= h_k(z_k), \quad \forall k \in \mathbb{N} \end{cases}$$

is said to be invariably uniformly globally exponentially stable with parameters $(\underline{a}, \overline{a}, \rho, c)$ *if, for any increasing sequence $\{\sigma_k\}_{k\in\mathbb{N}} \in \mathbb{N}$, there exists a differentiable function $W : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that, for all $z \in \mathbb{R}^n$, $k \in \mathbb{N}$,*

$$\underline{a}\,|z| \leq W(k,z) \leq \overline{a}\,|z| \tag{1}$$

$$W(k+1, h_{\sigma_k}(z)) \leq \rho\, W(k,z) \tag{2}$$

$$\left|\frac{\partial W}{\partial z}(k,z)\right| \leq c\,. \tag{3}$$

Not only the above definition imposes an exponential convergence of the discrete update law induced by the protocol, but it also requires that this property remains valid when the update is not made at each step but according to an arbitrary increasing sequence $\{\sigma_k\}_{k\in\mathbb{N}} \in \mathbb{N}$. It can easily be shown that the Round Robin protocol is *not* invariably GES (just consider two nodes and the sequence $\{\sigma_k\}_{k\in\mathbb{N}} = \{1,3,5,7,\ldots\}$ which corresponds to the update of the same unique node, regardless of the data provided by the other node). On the opposite, for time-invariant protocols (that is, $z(k+1) = h(z(k))$), (1) and (2) are equivalent to GES. In view of [8, Proposition 5], it can easily be shown that the TOD protocol is invariably (U)GES with $W(k,z) = |z|$.

In the sequel, we consider invariably UGES protocols:

Assumption 3 *The protocol $z(k+1) = h_k(z(k))$ is invariably UGES with some parameters $(\underline{a}, \overline{a}, \rho_0, c)$.*

III. A PACKET-BASED STRATEGY

Although communication networks do limit the data load that may transit between the plant and its controller, the size of each packet is often relatively large. In the spirit of [10], [9], the strategy here consists in exploiting this characteristic to provide a feedforward control signal between two transmission instants. Roughly speaking, at each reception of a new measurement, the controller updates an internal model-based estimate of the current state of the plant. We assume here that the whole state is measured, but the ideas presented here may be extended to partial measurements (in that case, the controller would have to run an internal observer for this state estimation). Based on this estimate, the controller computes[1] a prediction of the control signal on a fixed time horizon by running the plant model. This signal is then coded and sent in a single packet at the next network access. When received by the plant, it is decoded and resynchronized by an embedded computer, based on the time-stamping of the original measurement.

A. State estimation

In order to guarantee that a reasonable control signal is always available, the fixed time horizon on which each state prediction is achieved is chosen, in view of Assumptions 1 and 2, as

$$T_0 \geq T_c + T_m + h_m + h_c\,. \tag{4}$$

[1]The time required for this computation is included in the delay T_j^c.

For all $j \in \mathbb{N}$, we let $\gamma(j) := \max\left\{i \in \mathbb{N} : \tau_i^m + T_i^m < \tau_j^c\right\}$. The estimate $\hat{x}_j(t)$ is obtained based on the knowledge of a (possibly unperfect) model $\hat{f}$ of the plant f. At the instant τ_j^c, at which a new control packet can be sent, we run the model of the plant initialized with the latest measurement received by the controller, *i.e.*

$$\left\{\begin{array}{l} \dot{\hat{x}}_j(t) = \hat{f}(\hat{x}_j(t), \kappa(\hat{x}_j(t))), \\ \qquad \forall t \in [\tau_{\gamma(j)}^m + T_{\gamma(j)}^m; \tau_{\gamma(j)}^m + T_{\gamma(j)}^m + T_0], \\ \\ \hat{x}_j(\tau_{\gamma(j)}^m + T_{\gamma(j)}^m) = \\ \qquad x(\tau_{\gamma(j)}^m) + h_{\gamma(j)}\left(\hat{x}_{\gamma(j-1)}(\tau_{\gamma(j)}^m) - x(\tau_{\gamma(j)}^m)\right), \end{array}\right. \tag{5}$$

where k is a stabilizing control law for the nominal plant, (*i.e.*, in the ideal case when the plant and its controller are wired analogically), arising from the following assumption:

Assumption 4 (Nominal GES) *The nominal closed-loop system $\dot{x} = f(x, \kappa(x))$ is globally exponentially stable with a Lyapunov function V satisfying, for all $x \in \mathbb{R}^n$,*

$$\underline{\alpha}\,|x|^2 \leq V(x) \leq \overline{\alpha}\,|x|^2$$

$$\frac{\partial V}{\partial x}(x) f(x, \kappa(x)) \leq -\alpha\,|x|^2$$

$$\left|\frac{\partial V}{\partial x}(x)\right| \leq d\,|x|\ ,$$

where $\underline{\alpha}, \overline{\alpha}, \alpha$ and d denote positive constants.

The proofs of the results exploit the robustness introduced by this nominal property. They require sufficient regularity of the plant:

Assumption 5 *The vector fields f and $\hat{f}$ and the feedback law k are continuously differentiable and their gradients are bounded by positive constants λ_f, $\lambda_{\hat{f}}$ and λ_k respectively. In addition, $\hat{f}(0, \kappa(0)) = 0$.*

B. Hybrid formulation

In this section, we neglect the sampling induced by the digitalization of the control signal sent from the controller to the plant, to focus first on the phenomena resulting from single node access, delays and model imprecision.

Due to the communication delays and the ensured "continuity of control", the corresponding control laws are seen from the plant as

$$u_j(t + T_j^c) = \kappa(\hat{x}_j(t)), \quad \forall t \geq \tau_{\gamma(j)}^m + T_{\gamma(j)}^m, \forall j \in \mathbb{N}.$$

With our approach, the model-based estimate (5) of the state of the plant not only allows to provide a control law at the instants between two accesses to the network and to attenuate the effect of the parsimonious access to the network, but also to compensate for the transmission delays. Indeed, by the time-stamping of the transmitted data, it is possible to locally re-synchronize the control law on the plant. Mathematically, this boils down to anticipating the jth control packet of the whole accumulated delay, that is $T_j^c + T_{\gamma(j)}^m$. The control law actually applied is then, for each $j \in \mathbb{N}$,

$$u(t) = u_j(t + T_j^c + T_{\gamma(j)}^m), \quad \forall t \in [\tau_j^c + T_j^c; \tau_{j+1}^c + T_{j+1}^c]. \tag{6}$$

Using the time-invariance of (5), it holds that

$$\hat{x}_j(t + T_{\gamma(j)}^m) = \bar{x}_j(t), \qquad \forall t \geq \tau_{\gamma(j)}^m, \tag{7}$$

where $\bar{x}_j(\cdot)$ is the solution of

$$\begin{array}{l} \dot{\bar{x}}_j = \hat{f}(\bar{x}_j, \kappa(\bar{x}_j)), \quad \forall t \geq \tau_{\gamma(j)}^m \\ \bar{x}_j(\tau_{\gamma(j)}^m) = x(\tau_{\gamma(j)}^m) + h_{\gamma(j)}\left(\bar{x}_{\gamma(j-1)}(\tau_{\gamma(j)}^m) - x(\tau_{\gamma(j)}^m)\right). \end{array} \tag{8}$$

The above re-formulation simply states that the delayed state estimates $\hat{x}_j$ coincide with the state estimate $\bar{x}_j$ up to the corresponding delay $T_{\gamma(j)}^m$. This writing possesses the advantage of simplifying the expression of the overall closed-loop system as, in view of (6)-(7), the applied input becomes

$$u(t) = \kappa(\bar{x}_j(t)), \qquad \forall t \in [\tau_j^c + T_j^c; \tau_{j+1}^c + T_{j+1}^c). \tag{9}$$

The above notation still uselessly involves an infinite number of state variables $\bar{x}_j$, $j \in \mathbb{N}$. In order to condense the notation, we use the fact that each signal $\bar{x}_j(t)$ is only used over $[\tau_{\gamma(j)}^m; \tau_{j+1}^c + T_{j+1}^c)$. This allows us to make use of only two variables updated alternatively: one (x_{c1}) being reset at instants of time $\tau_{\gamma(j)}^m$ corresponding to an *odd* integer j and the other one (x_{c_2}) initialized at each $\tau_{\gamma(j)}^m$ for *even* j's. However, in order to ensure that none of these variables is reset while still in use in the control, we first extract the following integer subsequence, defined in a recursive way: $\mu(j+1) = \min\left\{k \geq \mu(j) : \exists i \in \mathbb{N}, \tau_k^c + T_k^c \geq \tau_i^m\right\}$ starting with $\mu(0) = 0$. Note that, with this notation, it holds that $\tau_{\gamma\circ\mu(j+1)}^m > \tau_{\mu(j)}^c + T_{\mu(j)}^c$, meaning that the measurement instants considered by this subsequence $\tau_{\gamma\circ\mu(j+1)}^m$ are always greater than the last time instant $\tau_{\mu(j)}^c + T_{\mu(j)}^c$ at which the corresponding variable was in use in the control law at the previous step. For simplicity, we let $\tilde{\gamma}(j) := \gamma \circ \mu(j)$ for all $j \in \mathbb{N}$. With this sequence of times, the dynamics of the variables x_{c1} and x_{c2} is given by the hybrid system

$$\left\{\begin{array}{rcl} \dot{x}_{c1} & = & \hat{f}(x_{c1}, \kappa(x_{c1})) \\ \dot{x}_{c2} & = & \hat{f}(x_{c2}, \kappa(x_{c2})) \end{array}\right. \tag{10}$$

$$\left\{\begin{array}{rcl} x_{c1}({\tau_{\tilde{\gamma}(j)}^m}^+) & = & x(\tau_{\tilde{\gamma}(j)}^m) + h_{\tilde{\gamma}(j)}\left(\tilde{x}_{c2}(\tau_{\tilde{\gamma}(j)}^m)\right) \\ & & + \left[\tilde{x}_{c1}(\tau_{\tilde{\gamma}(j)}^m) - h_{\tilde{\gamma}(j)}\left(\tilde{x}_{c2}(\tau_{\tilde{\gamma}(j)}^m)\right)\right]\eta(j) \\ \\ x_{c2}({\tau_{\tilde{\gamma}(j)}^m}^+) & = & x(\tau_{\tilde{\gamma}(j)}^m) + h_{\tilde{\gamma}(j)}\left(\tilde{x}_{c1}(\tau_{\tilde{\gamma}(j)}^m)\right) \\ & & + \left[\tilde{x}_{c2}(\tau_{\tilde{\gamma}(j)}^m) - h_{\tilde{\gamma}(j)}\left(\tilde{x}_{c1}(\tau_{\tilde{\gamma}(j)}^m)\right)\right](1 - \eta(j)) \end{array}\right.$$

where, for notation compactness, we have introduced

$$\begin{array}{rcl} \tilde{x}_{c1}(\tau_{\tilde{\gamma}(j)}^m) & := & x_{c1}(\tau_{\tilde{\gamma}(j)}^m) - x(\tau_{\tilde{\gamma}(j)}^m) \\ \tilde{x}_{c2}(\tau_{\tilde{\gamma}(j)}^m) & := & x_{c2}(\tau_{\tilde{\gamma}(j)}^m) - x(\tau_{\tilde{\gamma}(j)}^m) \\ \eta(j) & := & \left\{\begin{array}{ll} 1 & \text{if} \quad j \in 2\mathbb{N} \\ 0 & \text{otherwise.} \end{array}\right. \end{array} \tag{11}$$

The applied input (9) can then be written as $u(t) = u_a(t, x_c(t))$ where, for all $t \in \mathbb{R}_{\geq 0}$, $u_a(t, x_c) := \kappa(x_{c1})\mathbb{P}(t) + \kappa(x_{c2})(1 - \mathbb{P}(t))$, with $x_c := (x_{c1}^T, x_{c2}^T)^T$ and[2], for all $t \in \mathbb{R}_{\geq 0}$,

$$\mathbb{P}(t) := \left\{\begin{array}{ll} 1 & \text{if} \quad \exists j \in 2\mathbb{N} : t \in [\tau_j^c + T_j^c; \tau_{j+1}^c + T_{j+1}^c) \\ 0 & \text{otherwise.} \end{array}\right.$$

It is worth noting that, in (10), the update of x_{c1} depends, through $h_{\tilde{\gamma}(j)}$, on the current value of x_{c_2}, and vice versa. This comes from the fact that each of these variables is alternatively updated, and that the value of the *current* state estimate is used by the protocol to decide which node should be updated. So the state estimate that matters is the one of the *previous step*, in conformity with (5)-(8).

[2] Note that, with this definition, $\mathbb{P}(t) = 0$ if and only if there exists an odd integer j such that $t \in [\tau_j^c + T_j^c; \tau_{j+1}^c + T_{j+1}^c)$.

Remark 1 *For the sake of clarity, the above formulation does* not *take into account the fact that more than one measurement may be received between two packet sendings, which would allow for a more precise state estimate. This can however be addressed by simply considering a more involved update function* $h_{\tilde{\gamma}(j)}$.

Based on this formulation, the closed-loop dynamics can be summarized as the following hybrid dynamical system:

$$\begin{aligned} \dot{x} &= F(t,x,e) && (12a)\\ \dot{e} &= G(t,x,e) && (12b)\\ e(t_i^+) &= H(i, e(t_i))\,, && (12c)\end{aligned}$$

where, omitting to write the arguments of some functions,

$$e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} := \begin{pmatrix} x_{c1} - x \\ x_{c2} - x \end{pmatrix} \quad (13a)$$

$$F := f(x, u_a(t, e + J_n x)) \quad (13b)$$

$$G := \begin{pmatrix} \hat{f}(e_1 + x, \kappa(e_1 + x)) - f(x, u_a(t, e + J_n x)) \\ \hat{f}(e_2 + x, \kappa(e_2 + x)) - f(x, u_a(t, e + J_n x)) \end{pmatrix} \quad (13c)$$

$$H := \begin{pmatrix} h_{\nu(i)}(e_2) + \left[e_1 - h_{\nu(i)}(e_2)\right]\eta(i) \\ h_{\nu(i)}(e_1) + \left[e_2 - h_{\nu(i)}(e_1)\right](1 - \eta(i)) \end{pmatrix}. \quad (13d)$$

The matrix $J_n := (I_n, I_n)^T$ is just to adapt the dimensions between $x \in \mathbb{R}^n$ and $e \in \mathbb{R}^{2n}$. The time-sequence $\{t_i\}_{i\in\mathbb{N}}$ is defined as

$$\begin{cases} t_0 & := \tau^m_{\tilde{\gamma}(0)} \\ t_{i+1} & := \min_{j\in\mathbb{N}} \left\{\tau^m_{\tilde{\gamma}(j)} \,:\, \tau^m_{\tilde{\gamma}(j)} > t_i\right\}, \quad \forall i \in \mathbb{N}_{\geq 1}\,, \end{cases} \quad (14)$$

and the index $\nu(i)$ is defined as

$$\nu(i) := \min\left\{j \in \mathbb{N} \,:\, t_i = \tau^m_{\tilde{\gamma}(j)}\right\}, \qquad \forall i \in \mathbb{N}. \quad (15)$$

The hybrid system (12) is in the form of the NCS studied in [8], which is at the basis of all developments in the sequel.

C. Main result

Based on these preliminaries, we are now ready to present our main result. It states that, when sampling effects on the transmitted control signal are neglected, our strategy preserves global exponential stability under network communication, provided that the MATI and the delays involved remain below a certain limit. The latter bound is given explicitly, based on the parameters characterizing the quality of the model and the stability of the protocol and of the nominal closed-loop plant. Its proof in given in Section V.

Theorem 1 *Assume that Assumptions 1-5 hold and let* h_m, h_c, T_m, T_c, ρ_0, $\underline{a}$, $\overline{a}$, c, α, d, λ_f, $\lambda_{\hat{f}}$, λ_k *be generated by them. Assume that there exists a constant* $\varepsilon > 0$ *such that*[3]

$$h_m + h_c + T_m + T_c < \frac{1}{L} ln\left(\frac{L+\gamma}{(\rho_0+\varepsilon)L+\gamma}\right), \quad (16)$$

where

$$\begin{aligned} L &:= \frac{\sqrt{2}(1+\varepsilon)c}{\underline{a}\min\{1,\varepsilon\}} \max\left\{\lambda_{\hat{f}}(1+\lambda_k); 2\lambda_f\lambda_k\right\} && (17)\\ \gamma &:= \frac{2d\lambda_f\lambda_k c(1+\varepsilon)(\lambda_f+\lambda_{\hat{f}})(1+\lambda_k)}{\alpha\sqrt{\underline{a}\min\{1,\varepsilon\}}}\,. && (18)\end{aligned}$$

Then, the origin of the NCS (12) is uniformly globally exponentially stable.

[3] We also implicitly assume that there exists $\underline{\varepsilon} > 0$ such that $t_{i+1} - t_i \geq \underline{\varepsilon}$ for all $i \in \mathbb{N}$, in order to avoid Zeno solutions.

The upper bound (16) gives an explicit sufficient condition on the the maximum tolerable delays and times between two transmissions to guarantee the preservation of exponential stability. This bound can easily be computed based on the information available on the stability properties of the nominal closed-loop system (parameters α and d), on the protocol characteristics (parameters $\underline{a}$, $\overline{a}$, ρ_0 and c) and on the plant, controller and model's dynamics regularity (parameters λ_k, λ_f and $\lambda_{\hat{f}}$). The proposed approach unifies the effects due to sequential access to the network (MATI) and the effects of delays. If access to the network is frequent enough, then larger delays can be tolerated, and *vice-versa*.

Other strategies exploiting a model-based prediction of the control signal to be applied between two accesses to the network have recently been proposed in the literature; see *e.g.* [19], [5], [10], [9]. However, beyond the fact that [19], [5] considered only linear time-invariant dynamics, they did not address the possible distribution of the nodes nor the delays induced by processing and transmission. Also, Theorem 1 tolerates (possibly large) imprecision between the plant f and its model $\hat{f}$, which was not the case of [9]. Of course, this is at the price of not guaranteeing full performance recovery with respect to the nominal plant.

D. The linear case

In the particular case of linear time-invariant dynamics, as addressed in *e.g.* [12], [2], [19], [5], [17], [6], [14], that is $f(x,u) = Ax + Bu$, $\hat{f}(x,u) = \hat{A}x + \hat{B}u$ and $\kappa(x) = Kx$, the above system boils down to

$$\dot{x} = (A+BK)x + BK\left(e_1\mathbb{P}(t) + e_2(1-\mathbb{P}(t))\right) \quad (19a)$$

$$\dot{e} = \begin{pmatrix} (\tilde{A}+\tilde{B}K)x + (\hat{A}+\tilde{B}K)e_1\mathbb{P}(t) \\ +\left[(\hat{A}+\hat{B}K)e_1 - BKe_2\right](1-\mathbb{P}(t)) \\ \\ (\tilde{A}+\tilde{B}K)x + (\hat{A}+\tilde{B}K)e_2(1-\mathbb{P}(t)) \\ +\left[(\hat{A}+\hat{B}K)e_2 - BKe_1\right]\mathbb{P}(t) \end{pmatrix} \quad (19b)$$

$$e(t_i^+) = \begin{pmatrix} h_{\nu(i)}(e_2) + \left[e_1 - h_{\nu(i)}(e_2)\right]\eta(i) \\ h_{\nu(i)}(e_1) + \left[e_2 - h_{\nu(i)}(e_1)\right](1-\eta(i)) \end{pmatrix} \quad (19c)$$

where $\tilde{A} := \hat{A} - A$ and $\tilde{B} := \hat{B} - B$. We assume that a control gain K has be selected to stabilize the nominal dynamics:

Assumption 6 *There exists a symmetric positive definite matrix* $P \in \mathbb{R}^{n\times n}$ *satisfying*

$$(A+BK)^T P + P(A+BK) = -I\,. \quad (20)$$

Following the prooflines of Theorem 1, we obtain the following result, that better fits the particular LTI context.

Corollary 1 *Assume that Assumptions 1, 2, 3 and 6 hold and let* h_m, h_c, T_m, T_c, $\underline{a}$, $\overline{a}$, ρ_0, c *and* P *be generated by them. Assume that there exists a positive constant* $\varepsilon > 0$ *such that the following bound holds*

$$h_m + h_c + T_m + T_c < \frac{1}{L} ln\left(\frac{L+\gamma}{(\rho_0+\varepsilon)L+\gamma}\right), \quad (21)$$

where

$$L := \frac{(1+\varepsilon)c\sqrt{2}}{\min\{1,\varepsilon\}\underline{a}} \max\left\{\left|\hat{A}+\hat{B}K\right|; |BK| + \left|\hat{A}+\tilde{B}K\right|\right\} \quad (22)$$

$$\gamma := \frac{2c(1-\varepsilon)\,|PBK|\,\left|\tilde{A}+\tilde{B}K\right|}{\underline{a}\min\{1,\varepsilon\}}\,. \quad (23)$$

Then, the origin of the NCS (19) is UGES.

IV. Embedded computation Vs. Bandwidth

We next underline the trade-off existing between communication capabilities, embedded computation and performance of the controlled plant by distinguishing two cases: whether the main limitation induced by application is communication or computation.

For applications where the bandwidth of the communication network is the main constraint (*e.g.* distant wireless communication, high number of nodes, multiple interconnected systems, harsh environment constraints, energy limitations, poor communication medium), it is possible to apply Theorem 1 as it is, provided that a sufficient computation capability is embedded on the plant. The main idea is then to transmit only the current state estimate provided by the remote controller, and to achieve both the generation of the control signal and the resynchronization *on-board*.

On the other hand, for applications that allow very limited embedded computation capability (due to constraints in terms of *e.g.* miniaturization, weight, heat, *etc.*) but ensure a reasonable communication between the controller and the plant, it is necessary to limit on-board computation as much as possible. To this end, a possible strategy consists in making the controller pre-compute a control signal valid on a given time horizon, then to encode this signal into a digital format and transmit it through packet(s), and to simply resynchronize it based on the time-stamping on the system. In the sequel, we focus on a simple coding of the transmitted control signals: the constant step-size sample-and-hold. The only required embedded computation, as far as control is concerned, is then very frugal, as its role consists only in evaluating the difference between two quantities (stamped time and present time) and in addressing the corresponding value in the control buffer.

Considering that at least N bytes of data can be effectively sent in each control packet, we intuitively expect that this sampling has an incidence on the performance of the overall closed-loop system, but also that the effect of this imperfect coding diminishes as the available network bandwidth increases (*i.e.* as N gets larger). We formalize this intuition through Theorem 2.

In order to simplify the statement of this result, we first introduce the following notation: given a piecewise continuous signal $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and a positive constant Δ, $u^{\neg\Delta}$ denotes the function defined by (this notation extends to state-dependant signals):

$$u^{\neg\Delta}(t) : \begin{cases} \mathbb{R}_{\geq 0} & \to \mathbb{R}^m \\ t & \mapsto u(k\Delta)\,,\ \forall t \in [k\Delta;(k+1)\Delta)\,,\ \forall k \in \mathbb{N}\,. \end{cases}$$

For the sake of generality, let us focus on a plant with nonlinear dynamics. In this situation, the closed-loop dynamics is given by (12) with (13d) and, omitting some function arguments,

$$F = f(x, u_a^{\neg\Delta}(t, e + J_n x)) \tag{24}$$

$$G = \begin{pmatrix} \hat{f}(e_1 + x, k(e_1 + x)) - f(x, u_a^{\neg\Delta}(t, e + J_n x)) \\ \hat{f}(e_2 + x, k(e_2 + x)) - f(x, u_a^{\neg\Delta}(t, e + J_n x)) \end{pmatrix}. \tag{25}$$

It is important to notice that the original control input u_a affected by the sampling is a *discontinuous* signal. For this reason, the instantaneous difference between the nominal control signal u_a and its Δ-sampled version $u_a^{\neg\Delta}$ maybe large (around the discontinuities) even for a small step-size Δ. In order to avoid this situation, we make the following technical assumption.

Assumption 7 *There exists a positive constant Δ_0 such that $\tau_i^m, \tau_i^c, T_i^m, T_i^c \in \Delta_0\mathbb{N}$, for all $i \in \mathbb{N}$.*

This assumption gives some information on the instants at which the signal u_a may be discontinuous. This knowledge will allow us to choose a step-size Δ that does not induce the above described problem. We actually get the following result.

Theorem 2 *Assume that Assumptions 1-5 and 7 hold and that the delay bounds T_m and T_c in Assumption 2 satisfy*

$$T_m + T_c < \frac{1}{L} ln\left(\frac{L+\gamma}{(\rho_0+\varepsilon)L+\gamma}\right), \tag{26}$$

where L and γ are defined as in Theorem 1. Assume also that the state estimate is perfectly initialized, such that $e_0 = 0$. Then there exist some positive constants κ_1 and κ_2 such that, given any $M > \delta > 0$, there exists a positive N such that, for all $|x_0| \leq M$, the solution of the NCS with sampling-based packets, that is (12), (13d), (24) and (25), satisfies $|x(t, t_0, x_0, e_0 = 0)| \leq \delta + \kappa_1 |x_0| e^{-\kappa_2(t-t_0)}$ for all $t \geq t_0$.

Remark 2 *The result is stated in the case when the plant has nonlinear dynamics. The linear time-invariant extension follows by simply using the parameters L and γ given in (22) and (23).*

Theorem 2 requires a perfect initialization of the state estimation (*i.e.* $e_0 = 0$). Beyond the fact that this is realizable in practice by an off-line measurement, we stress that this assumption does not strongly damage the qualitative observation of the result. Indeed, the influence of its initial state on the error dynamics fades out with time, yielding a similar asymptotic behavior of the overall plant as when starting from any arbitrary initial value e_0.

The proof of Theorem 2 is omitted here due to space constraints. It mostly relies on that of Theorem 1 and on results on sampled-data systems such as [4] by observing that, once (26) is fulfilled, one can always pick a sufficiently small sampling time Δ that the effects of sampling on the overall behavior be reduced at will, provided that a sufficient quantity N of information can transit in each packet.

V. Proof of Theorem 1

We start by introducing the following result, which establishes UGES of the error dynamics resulting from our approach, based on the assumed GES of the protocol involved.

Proposition 1 *Assume that the discrete dynamical system $z(k+1) = h_0(k, z(k))$ is UGES with a Lyapunov function $W_0 : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying, for all $k \in \mathbb{N}$ and all $z \in \mathbb{R}^n$, $\underline{a}\,|z| \leq W(k,z) \leq \overline{a}\,|z|$, $W(k{+}1, h_k(z)) \leq \rho_0 W(k,z)$ and $\left|\frac{\partial W}{\partial z}(k,z)\right| \leq c$, with some positive constants $\underline{a}$, $\overline{a}$, ρ_0 and c. Consider the $2n$-dimensional system $e(k+1) = h'_k(e(k))$ where*

$$h'_k(e) := \begin{pmatrix} h_k(e_2) + [e_1 - h_k(e_2)]\,\eta(k) \\ h_k(e_1) + [e_2 - h_k(e_1)]\,(1-\eta(k)) \end{pmatrix},$$

with the function η defined in (11). Then, given any $\rho > \rho_0$, the function $W : \mathbb{N} \times \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ defined as $W(k,e) := \big[W_0(k,e_1) + (\rho-\rho_0)W_0(k-1,e_2)\big]\eta(k) + \big[(\rho-\rho_0)W_0(k-1,e_1) + W_0(k,e_2)\big](1-\eta(k))$ satisfies, for all $k \in \mathbb{N}$ and all $e \in \mathbb{R}^{2n}$,

$$\underline{a}\min\{1, \rho-\rho_0\}\,|e| \leq W(k,e) \leq \overline{a}\max\{1,\rho-\rho_0\}\,|e| \tag{27}$$

$$W(k+1, h'_k(e)) \leq \rho W(k,e) \tag{28}$$

$$\left|\frac{\partial W}{\partial e}(k,e)\right| \leq (1+\rho-\rho_0)c\,. \tag{29}$$

Proof: Distinguishing the cases when $k \in 2\mathbb{N}$ and $k \notin 2\mathbb{N}$, the bounds (27) and (29) follow directly. Concerning (28), assume first $k \in 2\mathbb{N}$. Then, writing $h'_k(e)$ as $(h_1(k,e)^T, h_2(k,e)^T)^T$, we have that $W(k+1, h_k(e)) = (\rho - \rho_0)W_0(k, h_1(k,e)) + W_0(k+1, h_2(k,e)) = (\rho - \rho_0)W_0(k, e_1) + W_0(k+1, h_k(e_1)) \leq (\rho - \rho_0)W_0(k, e_1) + \rho_0 W_0(k, e_1) \leq \rho W_0(k, e_1) \leq \rho W(k, e)$. Similarly, we get the same bound for $k \notin 2\mathbb{N}$ and (28) follows. ■

The proof of Theorem 1 consists in applying [8, Corrolary 2]. From the GES of $\dot{x} = f(x, \kappa(x))$, we know that $f(0, \kappa(0)) = 0$. Hence, from Assumption 5 and the mean value theorem, we have that $\left|\hat{f}(e_1 + x, \kappa(e_1 + x))\right| \leq \lambda_{\hat{f}}(1 + \lambda_k)\,|e_1 + x|$, and similarly for f. From this, and considering separately the cases when $\mathbb{P}(t) = 0$ and $\mathbb{P}(t) = 1$, we have from (13c) that, for all $x \in \mathbb{R}^n$ and all $e \in \mathbb{R}^{2n}$, the function G involved in (12) satisfies, for all $t \in \mathbb{R}_{\geq 0}$, $|G(t,x,e)| \leq 2\left(\lambda_f + \lambda_{\hat{f}}\right)(1+\lambda_k)\,|x| + \sqrt{2}\max\left\{\lambda_{\hat{f}}(1+\lambda_k); 2\lambda_f\lambda_k\right\}|e|$. Consequently, in view of Assumption 3 and Proposition 1, it holds that

$$\begin{aligned} \frac{\partial W}{\partial e}G(t,x,e) &\leq \left|\frac{\partial W}{\partial e}\right| |G(t,x,e)| \\ &\leq (1+\varepsilon)c\Big[2\left(\lambda_f + \lambda_{\hat{f}}\right)(1+\lambda_k)\,|x| \\ &\quad + \sqrt{2}\max\left\{\lambda_{\hat{f}}(1+\lambda_k); 2\lambda_f\lambda_k\right\}|e|\Big] \\ &\leq |\tilde{y}| + LW(i,e)\,, \end{aligned}$$

where L is given by (17) and

$$\tilde{y} := 2c(1+\varepsilon)\left(\lambda_f + \lambda_{\hat{f}}\right)(1+\lambda_k)x\,. \tag{30}$$

Furthermore, in order to show that (12a) is $\mathcal{L}_2$-stable from $\tilde{y}$ to W, we consider the Lyapunov function associated to the nominal controlled plant $\dot{x} = f(x, \kappa(x))$. In view of Assumption 4, considering first the cases when $\mathbb{P}(t) = 0$ and using the mean value theorem together with Assumption 5, the total derivative of V along the solutions of (12a) yields

$$\begin{aligned} &\frac{\partial V}{\partial x}(x)F(t,x,e) = \frac{\partial V}{\partial x}(x)f(x, \kappa(x+e_2)) \\ &\leq \frac{\partial V}{\partial x}(x)f(x,\kappa(x)) + \frac{\partial V}{\partial x}(x)\left[f(x,\kappa(x+e_2)) - f(x,\kappa(x))\right] \\ &\leq -\alpha\,|x|^2 + d\lambda_f\lambda_k\,|e_2|\,|x|\,. \end{aligned}$$

But it can be seen that $d\lambda_f\lambda_k\,|e_2|\,|x| \leq \alpha/2\,|x|^2 + d^2\lambda_f^2\lambda_k^2/2\alpha\,|e_2|^2$. Consequently, we obtain that, whenever $\mathbb{P}(t) = 0$, $\frac{\partial V}{\partial x}(x)F(t,x,e) \leq -\frac{\alpha}{2}|x|^2 + \frac{d^2\lambda_f^2\lambda_k^2}{2\alpha}|e_2|^2$. Similarly, at all time instants where $\mathbb{P}(t) \neq 0$, $\frac{\partial V}{\partial x}(x)F(t,x,e) \leq -\frac{\alpha}{2}|x|^2 + \frac{d^2\lambda_f^2\lambda_k^2}{2\alpha}|e_1|^2$. Thus, using (27), we obtain that, for all $t \in \mathbb{R}_{\geq 0}$, all $x \in \mathbb{R}^n$ and all $e \in \mathbb{R}^{2n}$, $\frac{\partial V}{\partial x}(x)F(t,x,e) \leq -\frac{\alpha}{2}|x|^2 + \frac{d^2\lambda_f^2\lambda_k^2}{2\alpha\underline{a}\min\{1,\varepsilon\}}W(i,e)^2$. Integrating this differential inequality, we get that, for any $t_0 \in \mathbb{R}_{\geq 0}$, any $t \geq t_0$ and any $x_0 \in \mathbb{R}^n$,

$$\int_{t_0}^{t} |x(s)|^2\,ds \leq \frac{2}{\alpha}V(x_0) + \frac{d^2\lambda_f^2\lambda_k^2}{\alpha^2\underline{a}\min\{1,\varepsilon\}}\int_{t_0}^{t} W(i,e(s))^2 ds\,, \tag{31}$$

where we used the shorthand notation $x(\cdot)$ to denote $x(\cdot, t_0, x_0)$. Hence, in view of (30) and Assumption 4,

$$\begin{aligned} &\sqrt{\int_{t_0}^{t} |\tilde{y}(s)|^2\,ds} \leq \frac{2\sqrt{2\overline{\alpha}}c(1+\varepsilon)(\lambda_f + \lambda_{\hat{f}})(1+\lambda_k)}{\sqrt{\alpha}}\,|x_0| \\ &+ \frac{2d\lambda_f\lambda_k c(1+\varepsilon)(\lambda_f + \lambda_{\hat{f}})(1+\lambda_k)}{\alpha\sqrt{\underline{a}\min\{1,\varepsilon\}}}\sqrt{\int_{t_0}^{t} W(i,e(s))^2 ds}\,, \end{aligned}$$

which means that (12a) is $\mathcal{L}_2$-stable from W to $\tilde{y}$ with gain γ given in (18). Furthermore, the NCS (12) is $\mathcal{L}_2$-to-$\mathcal{L}_2$ detectable from $(W, \tilde{y})$ to (e, x). This follows from the fact that (31) implies

$$\begin{aligned} &\sqrt{\int_{t_0}^{t}\left(|x(s)|^2 + |e(s)|^2\right)ds} \leq \sqrt{\frac{2\overline{\alpha}}{\alpha}}\,|x_0| \\ &+ \frac{1}{\underline{a}\min\{1,\varepsilon\}}\left(1 + \frac{d\lambda_f\lambda_k}{\alpha}\right)\sqrt{\int_{t_0}^{t} W(i,e(s))^2 ds}\,, \end{aligned}$$

Finally, invoking the global Lipschitz of the righthand side of (12) ensured by Assumption 5, we see with [8, Proposition 2] that the NCS is UGFTIS. The conclusion then follows by invoking [8, Corollary 2], which guarantees UGES of the overall NCS (12) under the condition that $t_{i+1} - t_i \leq \frac{1}{L}\ln\left(\frac{L+\gamma}{\rho L+\gamma}\right)$ for all $i \in \mathbb{N}$, which is itself ensured by (14).

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