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Robust Discrete-Time Simple Adaptive Model Following with Guaranteed H_∞ Performance

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***Abstract*— Robust output-feedback simplified adaptive control model following is considered for discrete-time systems with uncertainties and disturbances. Sufficient conditions for closed-loop stability, model following performance and prescribed H_∞ disturbance attenuation level are introduced, under an almost-strictly-positive-realness requirement on the plant. A numerical example, taken from the field of flight control, demonstrates the proposed method.**

I. Introduction

A class of direct adaptive controller schemes for continuous-time systems, known as Simplified Adaptive Control (SAC), has received considerable attention in the literature ([1],[2],[3]). Robustness of SAC controllers facing polytopic uncertainties has been established ([3]-[6]) allowing application to real engineering problems (see e.g. reference [6]). The stability of continuous-time SAC is related to the Strictly Positive Real (SPR) property of the controlled plant. The stability of closed-loop SAC is related to the Almost Strictly Positive Real (ASPR) property of the controlled plant. Namely, if a plant is ASPR there exists a static output-feedback gain (possibly parameter-dependent) which stabilizes the plant and makes it SPR. In such a case, SAC stabilizes the closed-loop dynamics and consequently leads to zero tracking errors. The ASPR property can be verified, under quadratic stability assumption, by solving a set of LMIs or by using a parameter-dependent Lyapunov function ([5]-[6]).

Bar-Kana [3] has recently provided a proof to the fact that any proper minimum-phase linear system with positive definite input-output feed-through matrix D is Almost Strictly Positive Real (ASPR). In addition, any strictly minimum-phase transfer function with minimal realization A, B, C where $CB > 0$ is also ASPR. In [7], a Lyapunov-based framework for SAC stabilization of discrete-time uncertain systems has been developed and sufficient conditions have been derived for the stability of the closed-loop dynamics of the SAC scheme. It has also been shown that the Minimum Phase (MP) property (together with the requirement for a positive definite D) is a necessary and sufficient condition for the the system to be ASPR. In [8], a Lyapunov-based framework for the tracking and stabilization of discrete-time uncertain systems by SAC output-feedback controllers has been developed, and sufficient conditions have been derived for the stability of the closed-loop and the gain adaptation formula. In [9], a framework for the combination of optimal H_∞ control and SAC has been developed. The objective is to use SAC while satisfying some H_∞ -norm bound on the disturbance attenuation level. Sufficient conditions have been derived for the stability of the closed-loop dynamics of the SAC scheme with a prescribed disturbance attenuation level γ.

In the present paper the relationship between optimal H_∞ control and SAC model following will be investigated for discrete-time systems. The objective is to use SAC while satisfying some prescribed H_∞-norm bound γ. Note that SAC can stabilize and follow an output of an uncertain system without knowing the explicit system dynamics. Sufficient conditions are derived for the stability and the model following of the closed-loop dynamics of the SAC scheme with prescribed disturbance attenuation level. These sufficient conditions are expressed in terms of Bilinear Matrix Inequalities (BMI), which can be solved using local iterations. A numerical example is given which illustrates the method.

A. Notation

Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space, $\mathcal{R}^{n\times m}$ is the set of all $n \times m$ real matrices, and the notation $P>0$, for $P \in \mathcal{R}^{n\times n}$, means that P is symmetric and positive definite. T_{zw} denotes the transference from the exogenous disturbance w to the objective function z, $\|T_{zw}\|_\infty$ is its H_∞ -norm and $\|T_{zw}\|_2$ is its H_2-norm. $col\{a,\ b\}$ stands for $[a^T\ b^T]^T$ and $trace\{H\}$ denotes the trace of the matrix H and $\|T_k\|_2^2 = \sum_0^\infty T_k^T T_k$.

II. Preliminaries

Consider the following discrete-time linear system:

$$\begin{aligned} x^*(k+1) &= A\,x^*(k) + B_2\,u^*(k) \\ y^*(k) &= C_2\,x^*(k) + D_{22}\,u^*(k) \end{aligned} \tag{1a,b}$$

where $x^*(k) \in \mathcal{R}^n$ is the system state, $y^*(k) \in \mathcal{R}^m$ is the measured plant output and $u^*(k) \in \mathcal{R}^m$ is the control input. A, B_2, C_2 and D_{22} are constant matrices of appropriate dimensions. We assume that $D_{22} > 0$.

Remark 1: In the general case, for any proper but not strictly proper system, when D_{22} is not symmetric but satisfies $D_{22} + D_{22}^T > 0$, we define $u^*(k) = D_{22}^T\widehat{u^*}(k)$ and obtain the following representation for (1)

$$\begin{aligned} x^*(k+1) &= A\,x^*(k) + \widehat{B}_2\,\widehat{u}^*(k) \\ y^*(k) &= C_2\,x^*(k) + \widehat{D}_{22}\,\widehat{u}^*(k) \end{aligned} \tag{2a,b}$$

where $\widehat{B}_2 = B_2 D_{22}^T$ and $\widehat{D}_{22} = D_{22} D_{22}^T > 0$. We therefore, assume in the sequel, without loss of generality, that $D_{22}{>}0$.

The output of the plant (1) is required to follow the output of the asymptotically stable model:

$$\begin{aligned} x_m(k+1) &= A_m\, x_m(k) + B_m\, u_m(k), \quad x(0) = 0 \\ y_m(k) &= C_m\, x_m(k) + D_m\, u_m(k) \end{aligned} \tag{3a,b}$$

where $x_m(k) \in \mathcal{R}^q$ is the system state, $y_m(k) \in \mathcal{R}^m$ is the plant output, $u_m(k) \in \mathcal{R}^m$ is the control input and A_m, B_m, C_m and D_m are constant matrices of appropriate dimensions. The reference model (3) is used to define the desired input-output behavior of the plant. It is important to note that the dimension of the reference model state may be less than the dimension of the plant state. However, since $y^*(k)$ is to track $y_m(k)$, the number of the model outputs must be equal to number of the plant outputs.

Perfect Following (PF) is defined as following with zero tracking error, namely

$$y^*(k) = y_m(k)$$

The next lemma presents a necessary condition for PF.

Lemma 1: PF is possible only if the system (1) is ASPR.

Proof : In PF the relation $y^*(k) = y_m(k)$ holds, and equation (1b) can then be written in the form

$$u^*(k) = D_{22}^{-1}(y_m(k) - C_2 x^*(k))\,. \tag{4}$$

Substituting (4) in (1a), one gets:

$$x^*(k+1) = (A - B_2 D_{22}^{-1} C_2)\, x^*(k) + B_2 D_{22}^{-1} y_m(k)$$

Therefore, since the reference model is stable, we need $A - B_2 D_{22}^{-1} C_2$ to be stable; this is guaranteed by the fact that (A, B_2, C_2, D_{22}) is ASPR. [7] QED

The next lemma determines the relation that exists between the plant's and the model's state vectors.

Lemma 2: There exist $F(k) \in \mathcal{R}^{n\times q}$ and $G(k) \in \mathcal{R}^{n\times m}$ such that the trajectories of (1) are of the form:

$$x^*(k) = F(k) x_m(k) + G(k) u_m(k) \tag{5}$$

Proof : (5) describes n equations with $n\times(q{+}m)$ variables, thus the existence of $F(k)$ and $G(k)$ is guaranteed for all k. Moreover, since the number of variables is greater than the number of equations we can impose additional constraints on $F(k)$ and $G(k)$. Defining $f(k) = C_2 F(k)$ and $g(k) = C_2 G(k)$ we, require $f(k) = \beta C_m + (1-\beta) f(k-1)$ and $g(k) = \beta D_m + (1-\beta) g(k-1)$ where $\beta \in (0,1)$ is a scalar. In the common case where $n > m$ these additional requirements can be fulfilled by choosing a large enough model order q. QED

Define $K_x^*(k) \equiv D_{22}^{-1}(C_m - C_2\, F(k))$ and $K_u^*(k) \equiv D_{22}^{-1}(D_m - C_2 G(k))$. We note that it follows from the above equations for $f(k), g(k)$ that:

$$K_x^*(k) = (1-\beta) K_x^*(k-1) \tag{6}$$

$$K_u^*(k) = (1-\beta) K_u^*(k-1) \tag{7}$$

We also define the ideal control $u^*(k)$ by

$$u^*(k) = K_x^*(k) x_m(k) + K_u^*(k) u_m(k). \tag{8}$$

Then considering the ideal system (1) and substituting (8) in (1b) we obtain:

$$y^*(k) = y_m(k). \tag{9}$$

Therefore, the ideal control (8) and the ideal system (1) allow PF.

III. Problem formulation

Consider the following continuous-time linear system:

$$\begin{aligned} x(k+1) &= A\, x(k) + B_1\, w(k) + B_2 u(k), \quad x(0) = x_0 \\ y(k) &= C_2\, x(k) + D_{21}\, w(k) + D_{22}\, u(k) \end{aligned} \tag{10a,b}$$

where $x(k) \in \mathcal{R}^n$ is the system state, $y(k) \in \mathcal{R}^m$ is the plant output which can be measured, $w(k) \in \mathcal{R}^m$ is the exogenous disturbance which is energy bounded and $w(k) \in \ell_2$ and $u(k) \in \mathcal{R}^m$ is the control input. A, B_1, B_2, C_2, D_{21} and $D_{22} > 0$ are constant matrices of appropriate dimensions.

The output of plant (10) is required to follow the output of the asymptotically stable model (3). We define the following objective vector:

$$z(k) = C_1\, e_y(k) + D_{12} e_u(k) \tag{11}$$

where, following [3], we define

$$e_y(k) = y_m(k) - y(k) = y^*(k) - y(k) \tag{12}$$

$$e_u(k) = u^*(k) - u(k) \tag{13}$$

The matrices C_1 and D_{12} are weights used to shape the control objective (11). It is required to assure that the plant (10) follows the output of the asymptotically stable model (3) so that the standard H_∞ cost J satisfies

$$J \stackrel{\Delta}{=} ||z||_2^2 - \gamma^2 ||w||_2^2 < 0 \tag{14}$$

for any $w(k) \neq 0$ and $w(k) \in \ell_2$, by employing a SAC controller.

IV. Solution

A. Control Law

ASPR is a necessary condition for PF, however for systems with an exogenous disturbance it is impossible to achieve PF. Suppose that the plant (10) is ASPR and therefore can be closed-loop stabilized and made SPR by the non-empty set of output-feedback gain controllers $K_e(k) \in U$ [3],[7]. In case $w(k) \not\equiv 0$, we consider a controller of the form:

$$u(k) = K^*(k) r(k) - \widetilde{u}(k) \tag{15}$$

where

$$K^*(k) = \left[\begin{array}{ccc} K_e(k) & K_x^*(k) & K_u^*(k) \end{array}\right], \tag{16}$$

$$r(k) = col\{e_y(k),\, x_m(k),\, u_m(k)\}, \tag{17}$$

where $K_e(k) \in \mathcal{R}^m$, $K_x^*(k) \in \mathcal{R}^{m\times q}$ and $K_u^*(k) \in \mathcal{R}^m$ are stabilizing and bounded gains and where $\widetilde{u}(k)$ is an auxiliary input signal which will be defined later. Note that when $e_y(k) = 0$, the controller (15-17) reduces to (8) for $\widetilde{u}(k) = 0$. This control, however, requires calculation of $F(k)$ and $G(k)$ for all $k = 1, 2, ...$ and the explicit knowledge of the system dynamics.

Instead, we use the direct SAC scheme [3] to calculate the gains which lead, in the steady state, to the same control signal that would have been achieved by $K_e(k)$, $K_x^*(k)$ and $K_u^*(k)$. The application of SAC requires the explicit knowledge of neither the gains matrix nor the system dynamics or the exogenous disturbance $w(k)$.

B. Simple Adaptive Control Law

We consider the following SAC scheme [3]:

$$u(k) \;=\; K(k)r(k) \tag{18}$$

where the definition of the gain $K(k)$ is

$$K(k) \;=\; \begin{bmatrix} K_e(k) & K_x(k) & K_u(k) \end{bmatrix} \tag{19}$$

and the gain adaptation scheme is

$$K(k) \;=\; (1-\beta)K(k-1) + e_y(k)r^T(k), \tag{20}$$

and where $0 < \beta < 1$ is a scalar. The initial condition is

$$K(0) = \begin{bmatrix} K_{e0} & \underline{0} & \underline{0} \end{bmatrix}.$$

A method for calculating K_{e0} is given in the example.

Remark 2: When $\beta = 0$ in (20), $K(k)$ steadily increases while $y(k) \neq 0$. For $0 < \beta < 1$, $K(k)$ is obtained from a first-order filtering of $y(k)y^T(k)$ and thus cannot diverge, unless $y(k)$ diverges [3].

We define $\delta(k) = K^*(k) - K(k)$, namely $\delta(k)$ is the difference between the ideal gain $K^*(k)$ and the current SAC gain $K(k)$. The control law of (18) can now be expressed by the following choice of the auxiliary control signal $\widetilde{u}(k)$ of (15):

$$\widetilde{u}(k) \;=\; \delta(k)r(k) \tag{21}$$

We define

$$\eta(k) = K(k) - K(k-1)$$

and obtain using (20)

$$\eta(k) \;=\; e_y(k)r^T(k) - \beta K(k-1). \tag{22}$$

$\eta(k)$ is the gain increment between steps, and define $\Delta K^*(k) = K^*(k) - K^*(k-1)$, which is the ideal gain increment, we obtain that:

$$\delta(k) - \delta(k-1) = \Delta K^*(k) - \eta(k). \tag{23}$$

and thus the following is obtain using (22)

$$\begin{aligned} \delta(k)\eta^T(k) &= \delta(k)(r(k)e_y^T(k) - \beta K^T(k-1)) \\ &= \widetilde{u}(k)e_y^T(k) - \beta\delta(k)K^T(k-1) \end{aligned} \tag{24}$$

We define the state error:

$$e_x(k) = x^*(k) - x(k)$$

and using (15) and (8), we obtain that $e_u(k)$ of (13) is given by

$$e_u(k) \;=\; -K_e(k)e_y(k) + \widetilde{u}(k). \tag{25}$$

Applying simple algebraic manipulations, we find that:

$$\begin{aligned} e_x(k+1) &= A\,e_x(k) + B_1\,w(k) + B_2 e_u(k) \\ e_y(k) &= C_2\,e_x(k) + D_{21}\,w(k) + D_{22}\,e_u(k) \end{aligned} \tag{26a,b}$$

and substituting (26b) in (25) we obtain:

$$e_u(k) = -K_e(k)\left(C_2 e_x(k) + D_{21}w(k) + D_{22}\,e_u(k)\right) + \widetilde{u}(k) \tag{27}$$

In order to extract $e_u(k)$ from (27), we first define

$$\widehat{K}_e(k) = (I + K_e(k)D_{22})^{-1}K_e(k) \tag{28}$$

and note that $(I + K_e(k)D_{22})^{-1} = I - \widehat{K}_e(k)D_{22}$. An upper-bound on $\widehat{K_e}(k)$ is calculated as follows:

$$\begin{aligned} \widehat{K_e}(k) &= (I + K_e(k)D_{22})^{-1}K_e(k)D_{22}D_{22}^{-1} \\ &= D_{22}^{-1} - (I + K_e(k)D_{22})^{-1}D_{22}^{-1} \\ &= D_{22}^{-1} - \nu(k) \end{aligned}$$

where $\nu(k) = (D_{22} + D_{22}K_e(k)D_{22})^{-1}$ so that $\nu(k) > 0$ and $\nu(k) \leq D_{22}^{-1}$. Namely:

$$0 < \widehat{K}_e(k) \leq D_{22}^{-1}. \tag{29}$$

The algebraic loop for $e_u(k)$ in (27) thus results in

$$\begin{aligned} e_u(k) \;=\; & -\widehat{K}_e(k)(C_2 e_x(k) + D_{21}\,w(k)) + \\ & (I - \widehat{K}_e(k)D_{22})\widetilde{u}(k). \end{aligned} \tag{30}$$

Substituting (30) in (26) and defining

$$\widetilde{A} \equiv (A - B_2\widehat{K}_e(k)C_2), \quad \widetilde{B}_2 \equiv B_2(I - \widehat{K}_e(k)D_{22}),$$

$$\widetilde{C}_2 \equiv (I - D_{22}\widehat{K}_e(k))C_2, \quad \widetilde{D}_{22} \equiv D_{22}\,(I - \widehat{K}_e(k)D_{22}),$$

$$\widetilde{B}_1 \equiv (B_1 + B_2\,(I - \widehat{K}_e(k)D_{22})D_{21}),$$

$$\widetilde{D}_{21} \equiv (D_{21} + D_{22}\,(I - \widehat{K}_e(k)D_{22})D_{21})$$

we obtain the closed-loop system

$$\begin{aligned} e_x(k+1) &= \widetilde{A}\,e_x(k) + \widetilde{B}_1\,w(k) + \widetilde{B}_2\widetilde{u}(k) \\ e_y(k) &= \widetilde{C}_2\,e_x(k) + \widetilde{D}_{21}\,w(k) + \widetilde{D}_{22}\widetilde{u}(k). \end{aligned} \tag{31a,b}$$

An expression for $z(k)$ is obtained by substituting (25) and (31b) in (11):

$$z(k) \;=\; \widetilde{C}_1\,e_x(k) + \widetilde{D}_{11}\,w(k) + \widetilde{D}_{12}\widetilde{u}(k) \tag{32}$$

where

$$\widetilde{C}_1 = C_1\widetilde{C}_2 - D_{12}\widehat{K}_e(k)C_2,$$

$$\widetilde{D}_{11} = C_1\widetilde{D}_{21} - D_{12}\widehat{K}_e(k)D_{21},$$

$$\widetilde{D}_{12} = C_1\widetilde{D}_{22} + D_{12}(1 - \widehat{K}_e(k)D_{22})$$

are weights used to shape the control objective (32). We are now in a position to state the main result of this section.

Theorem 1: For an ASPR plant, the adaptive scheme consisting of the plant (10), the control law (18) and the gain adaptation formula (20), creates, for any input command, bounded gains and states and achieves a disturbance attenuation level γ, for any $\beta \in (0,1)$ and any $w(k) \in \ell_2$ if there exist a positive definite matrix $P \in \mathcal{R}^{n\times n}$ and a compact set $\mathcal{K}$ so that the following BMI is feasible for all $\widehat{K}_e(k) \in \mathcal{K}$:

$$\begin{bmatrix} -P & -\widetilde{C}_2^T & 0 & \widetilde{C}_1^T & \widetilde{A}^T P \\ * & -2\widetilde{D}_{22} & \widetilde{D}_{21} & \widetilde{D}_{12}^T & \widetilde{B}_2^T P \\ * & * & -\gamma^2 I & \widetilde{D}_{11}^T & \widetilde{B}_1^T P \\ * & * & * & -I & 0 \\ * & * & * & 0 & -P \end{bmatrix} < 0 \qquad (33)$$

In such a case, the controller that achieves the disturbance attenuation of γ is given by (18) where the gains are restricted to K_e that satisfies (28) for $\widehat{K}_e(k) \in \mathcal{K}$.

Proof : In order to establish the desired model following of (10) with a disturbance attenuation level γ, the asymptotic stability of the closed-loop system of (31) with the objective vector (32) should be proven. We consider the following radially-unbounded Lyapunov function candidate

$$\begin{aligned} V(e_x(k), K(k)) \;\; = \;\; & e_x^T(k) P e_x(k) + \\ & tr\{\delta(k-1)\delta^T(k-1)\} > 0. \end{aligned} \qquad (34)$$

Note that $V(0, K^*(k)) = 0$, $V(e_x(k), K(k)) > 0$ for all $(e_x(k), K(k)) \neq (0, K^*(k))$ and $V(e_x(k), K(k)) \to \infty$ as $\|e_x(k)\| \to \infty$ or $\|K(k)\| \to \infty$. To obtain (14) we require that

$$\Delta V_k = V_{k+1} - V_k \leq \; \gamma^2 w^T(k) w(k) - z^T(k) z(k) \qquad (35)$$

where we have suppressed the dependence of V_k on $e_x(k)$ and $K(k)$ for the sake of simplicity. Defining $\mathcal{S} = \Delta V_k - \gamma^2 w(k)^T w(k) + z^T(k) z(k)$, we require that

$$\begin{aligned} \mathcal{S} \;\; = \;\; & e_x(k+1)^T P e_x(k+1) - \\ & e_x(k)^T P e_x(k) + tr\{\delta(k)\delta^T(k)\} - \\ & tr\{\delta(k-1)\delta^T(k-1)\} \\ & -\gamma^2 w(k)^T w(k) + z^T(k) z(k) < 0 \end{aligned} \qquad (36)$$

Substituting (23), we have:

$$\begin{aligned} & e_x(k+1)^T P e_x(k+1) - \\ & e_x(k)^T P e_x(k) + tr\{\delta(k)\delta^T(k)\} - \\ & tr\{(\delta(k) + \eta(k) - \Delta K^*(k)) \\ & (\delta(k) + \eta(k) - \Delta K^*(k))^T\} \\ & -\gamma^2 w(k)^T w(k) + z^T(k) z(k) < 0 \end{aligned}$$

We therefore require that

$$\begin{aligned} & e_x(k+1)^T P e_x(k+1) - \\ & e_x(k)^T P e_x(k) - 2tr\{\delta(k)\eta(k)^T\} \\ & -tr\{(\eta(k) - \Delta K^*(k))(\eta(k) - \Delta K^*(k))^T\} \\ & +2tr\{\delta(k)\Delta K^*(k)^T\} \\ & -\gamma^2 w(k)^T w(k) + z^T(k) z(k) < 0 \end{aligned}$$

Using (31), (32) and (24) we obtain

$$\begin{aligned} & (\widetilde{A}\, e_x(k) + \widetilde{B}_1\, w(k) + \widetilde{B}_2 \widetilde{u}(k))^T(k) P \\ & (\widetilde{A}\, e_x(k) + \widetilde{B}_1\, w(k) + \widetilde{B}_2 \widetilde{u}(k))^T(k) - e_x^T(k) P e_x(k) \\ & -\widetilde{u}(k)^T (\widetilde{C}_2\, e_x(k) + \widetilde{D}_{21}\, w(k) + \widetilde{D}_{22}\, \widetilde{u}(k)) \\ & -(\widetilde{C}_2\, e_x(k) + \widetilde{D}_{21}\, w(k) + \widetilde{D}_{22}\, \widetilde{u}(k))^T \widetilde{u}(k) \\ & -tr\{(\eta(k) - \Delta K^*(k))(\eta(k) - \Delta K^*(k))^T\} \\ & +2tr\{\delta(k)(\Delta K^*(k) + \beta K(k-1))^T\} - \gamma^2 w(k)^T w(k) \\ & +(\widetilde{C}_1\, e_x(k) + \widetilde{D}_{11}\, w(k) + \widetilde{D}_{12}\, \widetilde{u}(k))^T \\ & (\widetilde{C}_1\, e_x(k) + \widetilde{D}_{11}\, w(k) + \widetilde{D}_{12}\, \widetilde{u}(k)) < 0 \end{aligned}$$

where use is made of the fact that $tr(AB) = tr(BA)$. We define:

$$\mathcal{S} \;\; = \;\; \lambda_1(k) + \lambda_2(k)$$

where:

$$\lambda_1(k) \;\; = \;\; \begin{bmatrix} e_x^T(k) & \widetilde{u}^T(k) & w^T(k) \end{bmatrix} \Gamma \begin{bmatrix} e_x(k) \\ \widetilde{u}(k) \\ w(k) \end{bmatrix}$$

$$\begin{aligned} \lambda_2(k) \;\; = \;\; & -tr\{(\eta(k) - \Delta K^*(k))(\eta(k) - \Delta K^*(k))^T\} \\ & +2tr\{\delta(k)(\Delta K^*(k) + \beta K(k-1))^T\} \end{aligned}$$

and where

$$\Gamma \triangleq \begin{bmatrix} E_{11} & E_{12} & \widetilde{A}^T P \widetilde{B_1} + \widetilde{C}_1^T \widetilde{D}_{11} \\ * & E_{22} & \widetilde{D}_{21} + \widetilde{D}_{12}^T \widetilde{D}_{11} + \widetilde{B}_2^T P \widetilde{B}_1 \\ * & * & -\gamma^2 + \widetilde{B}_1^T P \widetilde{B}_1 + \widetilde{D}_{11}^T \widetilde{D}_{11} \end{bmatrix} < 0,$$

$$\begin{aligned} & E_{11} = \widetilde{A}^T P \widetilde{A} - P + \widetilde{C}_1^T \widetilde{C}_1, \\ & E_{12} = \widetilde{A}^T P \widetilde{B_2} - \widetilde{C_2}^T + \widetilde{C}_1^T \widetilde{D}_{12}, \\ & E_{22} = -2\widetilde{D_{22}} + \widetilde{B}_2^T P \widetilde{B}_2 + \widetilde{D}_{12}^T \widetilde{D}_{12}. \end{aligned} \qquad (37)$$

We shall show that by requiring (33), $\lambda_1(k)$ will be negative. Using Schur complements [10], it is readily found that $\lambda_1(k) < 0$ for all $col\{e_x(k), w(k), \widetilde{u}(k)\} \neq 0$ if (33) is satisfied. Note that using (23) we find that

$$\begin{aligned} \lambda_2(k) \;\; = \;\; & -tr\{(\delta(k) - \delta(k-1))(\delta(k) - \delta(k-1))^T\} \\ & +2tr\{\delta(k)(\Delta K^*(k) + \beta K(k-1))^T\} \end{aligned}$$

is non-definite. But, using (6) and (7) and the fact that the first term of $\delta(k)$ is zero, we obtain that

$$\begin{aligned} \lambda_2(k) \;\; = \;\; & -tr\{(\delta(k) - \delta(k-1))(\delta(k) - \delta(k-1))^T\} \\ & -2\beta tr\{\delta(k)\delta(k-1^T)\} \end{aligned}$$

which for $\beta \in (0,1)$ is negative, hence $\Delta V(k)$ becomes negative. QED

We note that Theorem 1, deals with the general Multi Input Multi Output (MIMO) case, but it does not suggest an algorithm for finding the set $\mathcal{K}$. Therefore, we next focus on the Single Input Single Output (SISO) case where $\widehat{K}_e(k)$ is scalar valued. In such a case, we readily obtain the following simplified version of Theorem 1 which allows a double line-search for the gains range:

Theorem 2: For an ASPR plant, the adaptive scheme consists that plant (10), the control law (18) and the gain adaptation formula (20), creates for any input command, bounded gains and states and achieves a disturbance attenuation level γ, for any $\beta \in (0,1)$ and any $w(k) \in \ell_2$ if there exist a positive definite matrix $P \in \mathcal{R}^{n\times n}$ and

$0 < \widehat{K}_{emin} < \widehat{K}_{emax} < D_{22}^{-1}$ so that the BMI of (33) is feasible for $\widehat{K}_e(k) \in \{\widehat{K}_{emin}, \widehat{K}_{emax}\}$. In such a case, the controller that achieves the disturbance attenuation of γ is given by (18) where the gains are limited by $\widehat{K}_{emin}$ and $\widehat{K}_{emax}$.

V. Robust SAC with Disturbance

We extend the results of Theorem 1 to the case where the matrices A , B_1 and B_2 of the system (10) are not exactly known. Denoting

$$\Omega = \left\{ \begin{array}{ccc} A & B_1 & B_2 \end{array} \right\} \tag{38}$$

where $\Omega \in Co\{\Omega_i,\ i = 1, ...N\}$, namely,

$$\Omega = \sum_{i=1}^{N} f_i \Omega_i \quad \text{for some} \quad 0 \le f_i \le 1,\ \sum_{i=1}^{N} f_i = 1 \tag{39}$$

where the vertices of the polytope are described by

$$\Omega_i = \left\{ \begin{array}{ccc} A^{(i)} & B_1^{(i)} & B_2^{(i)} \end{array} \right\}, \quad i = 1, 2..., N. \tag{40}$$

The next theorem describes conditions that not only guaranty that the closed-loop system (31) is stable, but also that it has a H_∞ disturbance attenuation level γ over $Co\{\Omega_i\}$.

Theorem 3: For an ASPR plant, the addressed SAC scheme, creates bounded gains and states and achieves a disturbance attenuation level γ for any $\beta \in (0, 1)$ over $Co\{\Omega_i\}$, for any input command and any $w(k) \in \ell_2$ if there exist a positive definite matrix $P \in \mathcal{R}^{n\times n}$ and a compact set $\mathcal{K}$ so that the following BMIs are satisfied for all $\widehat{K}_e(k) \in \mathcal{K}$:

$$\Phi \triangleq \begin{bmatrix} -P & -\widetilde{C}_2^T & 0 & \widetilde{C}_1^T & \widetilde{A}^{(i)T} P \\ * & -2\widetilde{D}_{22} & \widetilde{D}_{21} & \widetilde{D}_{12}^T & \widetilde{B}_2^{(i)T} P \\ * & * & -\gamma^2 I & \widetilde{D}_{11}^T & \widetilde{B}_1^{(i)T} P \\ * & * & * & -I & 0 \\ * & * & * & 0 & -P \end{bmatrix} < 0, \quad i = 1, 2..., N. \tag{41}$$

Proof: The latter is affine in $A^{(i)}$ and $B_1^{(i)}$ and $B_2^{(i)}$. We thus readily obtain that $\Phi < 0$ is satisfied over Ω by multiplying (41) by f_i and summing over $i = 1, 2, ..., N$.

VI. Numerical Example

In this section we present a numerical example to demonstrate the application of the theory developed above.

Consider a modified version of the angle of attack/pitch-rate dynamics example of [11]. This example describes the short period dynamics of a missile and was used in [11] to study gain scheduled control. A servo model with time constant of $1/\tau = 30[rad/sec]$ was applied. The state-vector is $x = [\theta, \alpha, q, \delta_e]^T$ where,$\theta[rad]$ is the pitch angle, $\alpha[rad]$ is the angle of attack, $q[rad/sec]$ is the pitch rate angle and $\delta_e[rad]$ is the elevator angle. The plant input is the elevator angle command $\delta_{com}[rad]$, and the plant output is the pitch-angle plus 0.1 of pitch-rate plus $0.01\delta_{com}$ where the latter terms were added in order to respectively improve the effective damping of the missile short period mode and to ensure the ASPR property of the open-loop discrete-time system. It should be also noted that the nonzero but small $D_{22} = 0.01$ has no particular physical significance. The plant is described by continuous time state-space model for $N = 4$, where:

$$A = \begin{bmatrix} -0.001 & 0 & 1 & 0 \\ 0 & -Z_{\alpha_j} & 1 & 0 \\ 0 & -M_{\alpha_j} & 0 & 1 \\ 0 & 0 & 0 & -1/\tau \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/\tau \end{bmatrix} \tag{42}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0.1 & 0 \end{bmatrix} \ \text{and}\ D_{22} = 0.01$$

$$C_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \ \text{and}\ D_{21} = 0.1, D_{12} = 0.01$$

and where the parameters of the four vertices (Mach-Altitude Pairs of $(0.5, 0), (0.5, 18km), (4, 0), (4, 18km)$ are $Z_\alpha \in \{0.5, 0.5, 4, 4\}$ and $M_\alpha \in \{6, 106, 6, 106\}$. As can be seen the uncertainty appears only in A. The discrete-time version of the above continuous-time plant is first derived, assuming a zero-order hold at the plant input and taking a sampling-time of $T_s = 1/64[sec]$. The matrices of the reference system model are

$$A = \begin{bmatrix} -3 & -10 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} 0 & 10 \end{bmatrix}. \tag{43}$$

Our aim is to track the reference model outputs with, say,

$$w(k) = 30\sin(50k)e^{-0.01k}.$$

Note that since $D_{22} = 0.01$, it turns out (using (29)) that

$$0 < \widehat{K}_e(k) < 100.$$

For any given value of $\widehat{K}_e(k)$, the BMI (33) can be solved numerically to obtain the corresponding minimum disturbance attenuation level γ and P that depends on $\widehat{K}_e(k)$ at all the four vertices. Fig. 1, obtained by performing this calculation over $\widehat{K}_e(k)$ within a grid in the latter range, describes the minimum achievable disturbance attenuation level γ as a function of the (scalar, in our example) gain $\widehat{K}_e(k)$. It can be seen from Fig. 1 that γ sharply increases at low adaptive gain ($\widehat{K}_e(k) < 50$) and at high adaptive gain. In fact, γ tends to infinity when $\widehat{K}_e(k)$ tends to D_{22}^{-1}. The smallest γ is 1.72, and it is achieved when $\widehat{K}_e(k) = 68$. We next seek single P and $\widehat{K}_{emin}, \widehat{K}_{emax}$ that satisfy the requirements of Theorem 3 for $\gamma = 2$. Applying a double line-search on the latter gain limits, we find that for

$$P = \begin{bmatrix} 16039 & -11445 & 112.04 & 5.3809 \\ -11445 & 13873 & 32.718 & 1.4446 \\ 112.04 & 32.718 & 23.073 & 0.29878 \\ 5.3809 & 1.4446 & 0.29878 & 0.027286 \end{bmatrix},$$

$\widehat{K}_{emin} = 55$ and $\widehat{K}_{emax} = 83$ satisfy (41). Since we have

$$K_e(k) = (I - \widehat{K}_e(k)D_{22})^{-1}\widehat{K}_e(k), \tag{44}$$

each $\widehat{K}_e(k)$ corresponds to some $K_e(k)$ in the range $[0 \quad \infty)$. For $\widehat{K}_e(k) = \widehat{K}_{emin}$ the lower bound of $K_e(k)$ is 122.5 and for $\widehat{K}_e(k) = \widehat{K}_{emax}$ the upper bound of $K_e(k)$ is 488.3. According to Theorem 2, H_∞ performance that corresponding to $\gamma = 2$ will be obtained if one adopts the practice of initializing the gain adaptation by $K_e(k) \approx 123$ and by bounding $K_e(k)$ to 488. Simulation results are given in Fig. 2 for $\beta = 0.01$ and $K_{e0} = 130$. Fig. 2 depicts the input and output of the reference model and the output of the plant at all the four operating points. Evidently, the plant output successfully tracks the reference model outputs by the proposed control law (18) and the gain adaptation formula (20). Fig. 3 describes the state θ, and the elevator angle command (δ_{com}) for each time step of the all 4 operation points. Apparently, pitch angle tracks the output of the reference model and all the other states are regulated to zero. Somewhat a more sluggish tracking is observed for the second operating point (dashed line) which corresponds to the lowest dynamic pressure requiring the largest δ_{com} which causes the largest $|\theta - y|$.

VII. Conclusions

In this paper the theory of Simplified Adaptive Control model following has been generalized to discrete-time systems with parameter uncertainties and H_∞ disturbance attenuation requirements. The results guaranty closed-loop stability and prescribed disturbance attenuation level, under the requirement of Almost-Strictly-Positive-Realness of the system (or, equivalently, minimum phase requirement).

These results are illustrated via an example from the field of flight control. The results encourage further research in this area, such as simplified adaptive control with exogenous disturbance and measurement noise for non minimum-phase systems.

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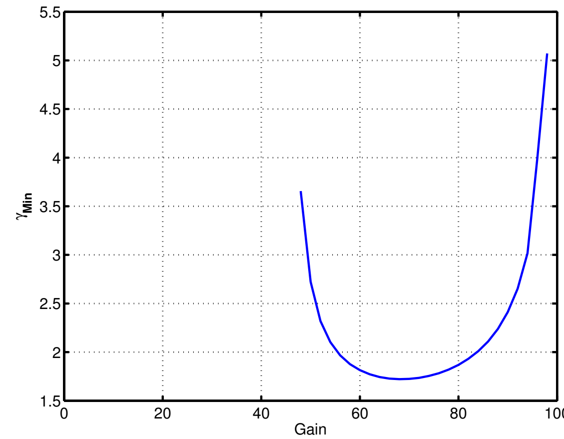

Fig. 1. Minimum disturbance attenuation level γ as function of the gain $\widehat{K}_e$

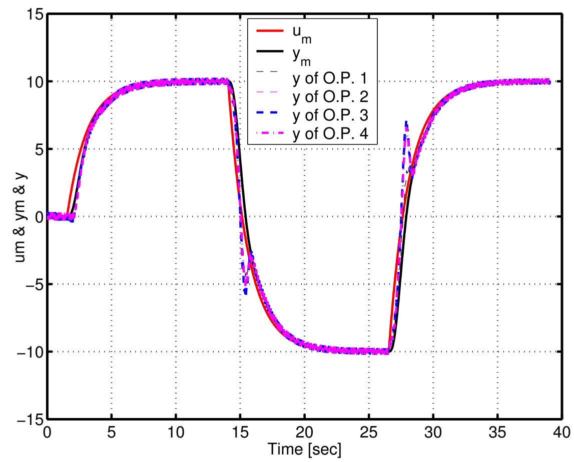

Fig. 2. Simulation results at the 4 operating points - The input and output of the reference model and the output of the plant

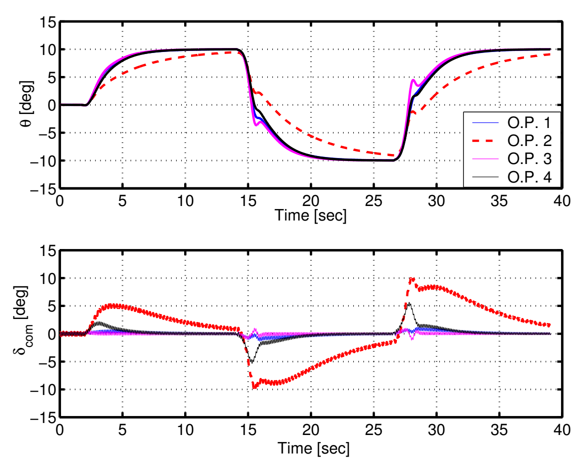

Fig. 3. Simulation results at the 4 operating points - the state θ, and the elevator angle command (δ_{com})