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On balanced realization and finite-dimensional approximation for infinite-dimensional nonlinear systems

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Abstract— **In this paper, we propose a finite-dimensional approximation for a nonlinear infinite-dimensional system via balanced realization. The proposed method is accomplished by balanced realization and singular value analysis for nonlinear systems. This approach is expected to derive effective approximate models preserving particular input-output behavior. Necessary and sufficient conditions characterizing balanced realization for infinite-dimensional systems are derived. Furthermore, the effectiveness of the proposed method is shown by a numerical simulation.**

I. Introduction

Various model order reduction methods have been proposed [1], balanced truncation method is often used since it conserves input-output behavior. Furthermore, for continuous time systems, the stability of the approximated model is preserved. For nonlinear systems, Scherpen formulated a basic framework for balanced realization [2]. Then a precise solution to this problem and the corresponding balanced truncation method were derived [3], [4].

Glover et al. [5] proposed balanced realization and balanced truncation for infinite-dimensional linear systems. However, there is no result for infinite-dimensional nonlinear systems so far. Therefore we discuss this problem and its finite-dimensional approximation. Conventional finite-dimensional approximation methods, such as mode decomposition, Taylor series expansion and finite element method are accomplished by approximating the whole state equations. On the other hand, balanced truncation pays attention to the input-output behavior of systems and tries to approximate them with respect to input-output relation. In this paper, balanced realization for infinite-dimensional nonlinear systems is formulated and characterized by a pair of Hamilton-Jacobi equations (HJEs). An approximate solution to those equations is proposed based on Galerkin method. Direct application of Galerkin method to HJEs on infinite dimensional signal spaces requires computation of infinite number of integrals. An approximation algorithm with finite number of integral computation is proposed, which works out by applying Galerkin method in two different steps: approximation to the state equations and that to HJEs. As a result, a finite-dimensional approximated system is derived by applying balanced truncation.

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Section II refers to balanced realization and model order reduction for finite-dimensional systems. Section III generalizes them for infinite-dimensional systems. A finite-dimensional approximation method is proposed in Section IV. In Section V, the effectiveness of the proposed method is demonstrated by a numerical simulation.

Notation In this paper, $\|x\| := (x^{\mathrm{T}}x)^{1/2}$ for $x \in \mathbb{R}^n$. The symbol $\langle x, y\rangle := x^{\mathrm{T}}y$ denotes the inner product for $x, y \in \mathbb{R}^n$. Let us define the inner product on L_2 as $\langle x, y\rangle_{L_2} := \int_a^b x(t)^{\mathrm{T}}y(t)\,\mathrm{d}\,t$ and the norm on L_2 as $\|x\|_{L_2} := \sqrt{\langle x, x\rangle_{L_2}}$. Moreover, the class-$K$ and class-KL functions are defined as follows. A continuous function $\alpha : [0,\infty) \to [0,\infty)$ is called class-K if α is monotonically increasing and satisfies $\alpha(0) = 0$. A continuous function $\beta : [0,\infty)\times[0,\infty) \to [0,\infty)$ is called class-KL if $\beta(r,s)$ is a class-K function with respect to r for $\forall s \in [0,\infty)$, if $\beta(r,s)$ is monotonically decreasing with respect to s for $\forall r \in [0,\infty)$ and if $\lim_{s\to\infty}\beta(r,s) = 0$.

II. Preliminaries

This section refers to balanced realization and balanced truncation for finite-dimensional nonlinear systems [3], [4]. Consider an input-affine, time invariant, asymptotically stable nonlinear system

$$\Sigma_f : \begin{cases} \dot{x} &= f(x) + g(x)u \quad x(0) = x^0 \\ y &= h(x) \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^r$. The next assumption is adopted in what follows.

Assumption 1: Hankel singular values of the Jacobian linearization of Σ_f at $x = 0$ are nonzero and distinct.

For this system, the controllability and obserbability functions L_c and L_o are defined as follows.

Definition 1:

$$L_c(x^0) := \min_{\substack{u\in L_2(-\infty,0)\\ x(-\infty)=0,x(0)=x^0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2\,\mathrm{d}\,t \tag{2}$$

$$L_o(x^0) := \tfrac{1}{2}\int_0^\infty \|y(t)\|^2\,\mathrm{d}\,t, \quad x(0) = x^0 \tag{3}$$

By Definition 1, if $\dot{x} = f(x)$ is asymptotically stable, the existence and the positive definiteness of L_c is equivalent to reachability of Σ_f with respect to L_2 input signals, and the existence and the positive definiteness of L_o is equivalent to zero-state observability of Σ_f. The functions L_c and L_o are the solutions of Hamilton-Jacobi equations

$$\frac{\partial L_c(x)}{\partial x}f(x) + \frac{1}{2}\frac{\partial L_c(x)}{\partial x}g(x)g(x)^{\mathrm{T}}\frac{\partial L_c(x)}{\partial x}^{\mathrm{T}} = 0 \tag{4}$$

$$\frac{\partial L_o(x)}{\partial x}f(x) + \frac{1}{2}h(x)^{\mathrm{T}}h(x) = 0 \tag{5}$$

where the origin of $\dot{x} = -f(x) - g(x)g(x)^{\mathrm{T}}(\partial L_c(x)/\partial x)^{\mathrm{T}}$ is asymptotically stable. Related to L_c and L_o, the following theorem is characterized by the equation of singular value analysis for the corresponding nonlinear Hankel operators.

Theorem 1: [6], [3] Assume that $\lambda \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ satisfy

$$\nabla L_o(\xi) = \lambda \nabla L_c(\xi). \tag{6}$$

Then $\sigma = \sqrt{L_o(\xi)/L_c(\xi)}$ is Hankel singular value, and the solution ξ to (6) is on the coordinate axes of balanced realization.

According to Theorem 1, we can obtain Hankel singular values by solving (6). Balanced realization of Σ_f can be derived by transforming the solution curves of ξ into the coordinate axes. Furthermore, the following theorem indicates nonlinear balanced realization.

Theorem 2: [3], [4] Under Assumption 1, there exists a coordinate transformation $x = \Phi(z)$ on a neighborhood U of the origin satisfying

$$L_c(\Phi(z)) = \tfrac{1}{2}\textstyle\sum_{i=1}^{n} \frac{z_i^2}{\sigma_i(z_i)},\ L_o(\Phi(z)) = \tfrac{1}{2}\sum_{i=1}^{n} z_i^2\sigma_i(z_i).$$

Balanced truncation for nonlinear systems is implemented as follows. Suppose that Σ_f is in a balanced realization Σ_{br} whose state is denoted z. Assume that the Hankel singular values satisfy $\sigma_1 \geq \cdots \geq \sigma_k \gg \sigma_{k+1} \geq \cdots \geq \sigma_n \geq 0$ for a certain k, $(1 \leq k \leq n)$. This means that $z^a := (z_1, \cdots, z_k) \in \mathbb{R}^k$ has much more influence on the input-output behavior of Σ_{br} than $z^b := (z_{k+1}, \cdots, z_n) \in \mathbb{R}^{n-k}$. According to the division $z = (z^a, z^b)$, let us divide $f(z) = (f^a(z), f^b(z))$ and $g(z) = (g^a(z)^{\mathrm{T}}, g^b(z)^{\mathrm{T}})^{\mathrm{T}}$ correspondingly. Then, the nonlinear reduced model is obtained by substituting $z^b \equiv 0$ as follows.

$$\Sigma_{br}: \begin{cases} \dot{z^a} &= f^a(z^a, 0) + g^a(z^a, 0)u \quad z(0) = z^0 \\ y &= h(z^a, 0) \end{cases}$$

III. Balanced Realization for Infinite-Dimensional Nonlinear Systems

It is shown in this section that L_c and L_o for infinite-dimensional nonlinear systems are also characterized by HJEs. Consider an input-affine, time invariant, asymptotically stable nonlinear infinite-dimensional system

$$\Sigma: \begin{cases} \frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t} &= F(x(t)) + \sum_{i=1}^{m} u_i(t)G_i(x(t)) \\ y(t) &= H(x(t)) \end{cases} \tag{7}$$

with $x(t)$ in Sobolev spaces $\mathcal{H}^1(\Omega)$ with $\Omega \subset \mathbb{R}^n, u(t) = (u_1(t), \cdots, u_m(t))^{\mathrm{T}} \in \mathbb{R}^m, y(t) \in \mathbb{R}^r,\ F, G_i : \mathcal{H}^1(\Omega) \to L_2^n (i = 1, \cdots, m)$ and $H : \mathcal{H}^1(\Omega) \to \mathbb{R}^r$. The origin $x = 0$ is the equilibrium, that is, $F(0) = 0$ and $H(0) = 0$ without loss of generality. Let Σ be asymptotically stable in the following sense.

Definition 2: The origin $x = 0$ of Σ is said to be asymptotically stable if there exists a class-KL function β satisfying $\|x(t)\|_{L_2} \leq \beta(\|x(0)\|_{L_2}, t)$.

The definition of L_c and L_o for infinite-dimensional systems are similar to (2) and (3), respectively.

$$L_c(x^0) := \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0, x(0)=x^0}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 \,\mathrm{d}\,t$$
$$L_o(x^0) := \tfrac{1}{2}\textstyle\int_0^\infty \|y(t)\|^2 \,\mathrm{d}\,t, \quad x(0) = x^0$$

Next theorem characterizes L_c and L_o for infinite-dimensional systems as natural generalization for the finite-dimensional ones.

Theorem 3: Consider the system (7). Suppose that the origin of $\dot{x} = F(x)$ is asymptotically stable on a neighborhood W of 0. Then the following statements hold.

(i) Suppose that

$$\langle \nabla \check{L}_o(x), F(x) \rangle_{L_2} + \tfrac{1}{2}H(x)^{\mathrm{T}}H(x) = 0, \quad \check{L}_o(0) = 0 \tag{8}$$

has a smooth solution $\check{L}_o$ on $W_o \subset W$ satisfying $0 \leq \check{L}_o(x) \leq \alpha_o(\|x\|_{L_2})$ with a class-K function α_o. Then the observability function L_o exists and coincides with the unique solution $\check{L}_o$ of (8) on W_o.

(ii) Suppose that

$$\langle \nabla \check{L}_c(x), F(x) \rangle_{L_2} + \tfrac{1}{2}\textstyle\sum_{i=1}^{m} \langle \nabla \check{L}_c(x), G_i(x) \rangle_{L_2}^2 = 0,$$
$$\check{L}_c(0) = 0 \tag{9}$$

has a smooth solution $\check{L}_c$ on a neighborhood $W_c \subset W$ such that the origin of

$$\dot{x} = -F(x) - \textstyle\sum_{i=1}^{m} G_i(x) \langle \nabla \check{L}_c(x), G_i(x) \rangle_{L_2} \tag{10}$$

is asymptotically stable on W_c and $\check{L}_c$ satisfies $0 \leq \check{L}_c(x) \leq \alpha_c(\|x\|_{L_2})$ with a class-K function α_c. Then the controllability function L_c exists on W_c and coincides with the solution of (9) on W_c such that the origin of the system (10) is asymptotically stable.

Proof. See Appendix.

Theorem 3 proves the sufficiency of the Hamilton-Jacobi equations (8) and (9) for the existence of L_c and L_o. The following theorem proves the necessity.

Theorem 4: Consider the system (7). Suppose that the origin of $\dot{x} = F(x)$ is asymptotically stable on a neighborhood W of 0. Then the following statements hold.

(i) Suppose that the observability function L_o exists on $W_o \subset W$ and is smooth. Then

$$\langle \nabla \check{L}_o(x), F(x) \rangle_{L_2} + \tfrac{1}{2}H(x)^{\mathrm{T}}H(x) = 0, \quad \check{L}_o(0) = 0 \tag{11}$$

has a solution $\check{L}_o = L_o$ on W_o.

(ii) Suppose that a controllability function L_c exists on $W_c \subset W$ and is smooth. Then

$$\langle \nabla \check{L}_c(x), F(x) \rangle_{L_2} + \tfrac{1}{2}\textstyle\sum_{i=1}^{m} \langle \nabla \check{L}_c(x), G_i(x) \rangle_{L_2}^2 = 0,$$
$$\check{L}_c(0) = 0 \tag{12}$$

has a solution $\check{L}_c = L_c$ such that the origin of (10) is asymptotically stable on W_c.

Proof. See Appendix.

IV. Approximate solutions to Hamilton-Jacobi equations for infinite-dimensional systems

This section proposes an approximate solution to (8) and (9) based on Galerkin method. Approximate solutions to (4) and (5) have been already proposed by many authors, see e.g. [7] and [8]. The method based on [8] is adopted here. An approximate solution given by Galerkin method comes closer and closer to the exact one as the approximate accuracy becomes higher and higher [9], [10].

A. Galerkin method

Let X be a real Hilbert space and X' be the dual space of X. Let $a : X \times X \to \mathbb{R}$ be a continuous bilinear operator. Consider a problem to find $x \in X$ satisfying

$$a(x, y) = f(y), \forall y \in X \tag{13}$$

for $f \in X'$. If the operator a satisfies a certain condition, the problem (13) has the unique solution [10]. Suppose that a satisfies this condition. Then try to find an approximate solution $x \in X$ on a finite-dimensional subspace $X_h := \text{span}\{x_1, \cdots, x_N\} \subset X$. That is, consider the problem $a(x_h, y_h) = f(y_h), \quad \forall y_h \in X_h$ instead of (13) and find an approximate solution x_h in X_h. Here the subscript h indicates the approximate accuracy where $N \to \infty$ as $h \to 0$.

B. Proposed algorithm

Utilizing Galerkin method to solve (8) and (9), approximate solutions to L_c and L_o can be found. These solutions converge to the exact ones as $h \to 0$. However, direct application of Galerkin method to (8) and (9) requires one to compute infinite number of integrals. Therefore we cannot calculate the approximate solution in practice. Here, let us derive finite-dimensional HJEs first by applying Galerkin method to (7). Then obtain approximate solutions to L_c and L_o by solving the approximated HJEs by applying Galerkin method once again. Since it is easier to solve (8) than to solve (9), we discuss how to solve (8) in detail below.

Consider an input-affine system

$$\Sigma : \begin{cases} \frac{\partial x(s,t)}{\partial t} &= F(x(\cdot, t), s) + \sum_{i=1}^{m} u_i(t) G_i(x(t)) \\ y(t) &= H(x(\cdot, t)) \end{cases} \tag{14}$$

with $x(\cdot, t) \in \mathcal{H}^1(\Omega), u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^r$. The origin is an equilibrium. The variable $s \in \mathbb{R}$ is the spatial axis. The symbols F, G_i and H are operators including calculation such as integrals with respect to s. Suppose that Σ is asymptotically stable. Let $x(s,t)$ be described by a series expansion as $x(s,t) = \sum_{j=1}^{\infty} \langle x(s,t), \phi_j(s) \rangle_{L_2}, \phi_j(s) = \sum_{j=1}^{\infty} \xi_j(t) \phi_j(s)$ with a set of complete orthogonal functions $\{\phi_j\}, (j = 1, 2, \cdots)$ in the spatial space. Approximation of x is represented in a subspace spanned by $\{\phi_j\}, (j = 1, \cdots, N_s)$ as follows.

$$x(s,t) = \sum_{j=1}^{N_s} \xi_j(t) \phi_j(s) \tag{15}$$

The inner product calculation is carried out by substituting (15) for the state equation in (14) as

$$\begin{aligned} &\left\langle \tfrac{\partial}{\partial t} (\textstyle\sum_{j=1}^{N_s} \xi_j(t) \phi_j(s)) - F(x(\cdot, t), s) \right. \\ &\left. - G(x(\cdot, t), s) u(t), \phi_i(s) \right\rangle_{L_2} = 0, i = 1, \cdots, N_s. \end{aligned} \tag{16}$$

A finite-dimensional approximate state equation is derived by solving (16). Also an approximate solution to the output equation in (14) is calculated on the same subspace. As a consequence, a finite-dimensional approximate state space equation is derived as follows.

$$\Sigma_\xi : \begin{cases} \dot{\xi} &= f(\xi) + g(\xi) u \\ y &= h(\xi) \end{cases} \tag{17}$$

Next, let L_o be described by a series expansion with a set of complete orthogonal functions $\{\psi_j\}, (j = 1, 2, \cdots)$ with respect to ξ, $L_o(\xi) = \sum_{j=1}^{\infty} q_j \psi_j(\xi)$, $q_j \in \mathbb{R}$ $(j = 1, 2, \cdots)$. Then solve (8) related to L_o by Galerkin method. On a subspace X_{N_ξ} spanned by $\{\psi_j\}, (j = 1, \cdots, N_\xi)$, the approximation to L_o is described as follows.

$$L_o(\xi) = \sum_{j=1}^{N_\xi} q_j \psi_j(\xi) \tag{18}$$

Substitute (17) and (18) for (8) and calculate

$$\left\langle \langle \nabla \textstyle\sum_{j=1}^{N_\xi} q_j \psi_j(\xi), f(\xi) \rangle + \frac{1}{2} h(\xi)^{\mathrm{T}} h(\xi), \psi_i(\xi) \right\rangle = 0 \tag{19}$$

to find $q_j (j = 1, \cdots, N_\xi)$ where $i = 1, \cdots, N_\xi$. Then an approximate solution to L_o is obtained. Since this solution depends on the choice of $\{\phi_i\}$ and $\{\psi_i\}$, it is important to choose them to preserve the feature of the original system. For example, the eigenfunctions of the system were chosen in the numerical example described in Section V. Here the convergence of the approximate solution to the exact one is an important issue. The convergence of Galerkin solutions is proven stated in the beginning of Section IV. Here let $N_s \to \infty$ first, next let $N_\xi \to \infty$, then the approximate solution to L_o will converge to the exact one because of (19). The detailed treatment on the convergence of the approximate solutions of finite-dimensional Hamilton-Jacobi equations based on Galerkin method is discussed in [11].

Equation (9) can also be solved by the above method. For Σ_ξ, the function L_c is described by $L_c = \sum_{i=1}^{N_\xi} p_j \psi_j(\xi)$ on X_{N_ξ} where $p_j \in \mathbb{R}$ $(j = 1, .., N_\xi)$. However, this is a quadratic equation with respect to the variable p_j as

$$\begin{aligned} &\left\langle \langle \nabla \textstyle\sum_{j=1}^{N_\xi} p_j \psi_j(\xi), f(\xi) \rangle + \frac{1}{2} \sum_{i=1}^{m} \langle \nabla \sum_{j=1}^{N_\xi} p_j \psi_j(\xi), \right. \\ &\left. g_i(\xi)^{\mathrm{T}} \rangle^2, \psi_k(\xi) \right\rangle = 0, \quad k = 1, \cdots, N_\xi. \end{aligned} \tag{20}$$

This equation is solved by the asymptotic approximation [12] to solve (4) the Hamilton-Jacobi equation in the finite dimensional case. This method is to solve a linear recurrence equation (21) with respect to p_{kj} instead of the quadratic equation (20).

$$\begin{aligned} &\left\langle \langle \nabla \textstyle\sum_{j=1}^{N_\xi} p_{kj} \psi_j(\xi), f(\xi) \rangle + \frac{1}{2} \sum_{i=1}^{m} \langle \nabla \sum_{j=1}^{N_\xi} p_{kj} \psi_j(\xi), \right. \\ &\left. g_i(\xi)^{\mathrm{T}} \rangle \langle \nabla \textstyle\sum_{j=1}^{N_\xi} p_{(k-1)j} \psi_j(\xi), g_i(\xi)^{\mathrm{T}} \rangle, \psi_k(\xi) \right\rangle = 0, \end{aligned} \tag{21}$$

$k = 1, \cdots, N_\xi$. It is proven in [12] that if (5) is solved by asymptotic approximation, then the solution converges to the exact one. Therefore, similarly to the L_o case, let $N_s \to \infty$, then let $N_\xi \to \infty$, and employ another iteration for the asymptotic approximation, then the approximation to L_c will converge to the exact one.

It is expected that the accuracy of the solutions of (6) can be improved by improving that of the approximate solutions to L_c and L_o obtained by the above procedure since (6) is characterized by the gradients of L_c and L_o. Therefore, the accuracy of balanced realization and Hankel singular values are also improved. The computational algorithm is summarized as follows.

The computational algorithm

1) Apply Galerkin method to the state equation of the system (7).
2) Derive (8) and (9) from finite-dimensional approximation to the state equation, and solve them by Galerkin method. As a result, approximate solutions to L_c and L_o are obtained.
3) Solve (6) to find the Hankel singular values and the coordinate axes which yield a balanced realization. The detailed way of calculation is shown in [13].
4) Apply balanced truncation to obtain a reduced order finite-dimensional model.

V. Numerical Example

Consider the two-link flexible arm shown in Fig. 1, which rotates in a horizontal plane. See [14] for the detail. Link 1 is a rigid beam. Link 2 is an elastic one. Link 2 has a concentrated mass at the tip of the arm. Each Link i is fixed to a vertical gear shaft driven by a DC Motor i. Let $\tau_i(t)$ be the torque generated by motor i, $\theta_i(t)$ be the rotational angle of Link i, μ_i and k_i be the viscous damping and the elastic coefficient of Link i, respectively. The origin of the inertial coordinate frame O-XY is the rotation center of Motor 1. The origin of the coordinate frame O_i-x_iy_i is fixed to the rotation axis of Motor i. Here $O = O_1$. J_1 is the inertial moment of Link 1, which includes the inertia moment of the rotor of Motor 1. J_2 is the inertial moment of the rotor of Motor 2. L_i is the length of Link i. I is the geometric moment of inertia. E is the Young's modules. ρ is the line density. d is the coefficient of viscous damping generated from the elasticity of Link 2. $w(s,t)$ is the transverse displacement of Link 2 at a spatial point $s \in (0, L_2)$ at time t. Here the spatial variable s is the distance from O_2 to an arbitrary point on the coordinate axis x_2. Since the flexible arm rotates in a horizontal plane, there is no influence of gravity. Therefore, there is neither displacement in vertical plane nor twist of the arm. The vector $\boldsymbol{s} \in \mathbb{R}^2$ denotes the position of an arbitrary point of Link 2 in $X-Y$ coordinates. The vector $\boldsymbol{L} \in \mathbb{R}^2$ denotes the position of the tip of the arm in $X-Y$ coordinates.

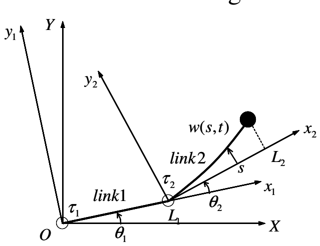

Fig. 1. Two link flexible arm

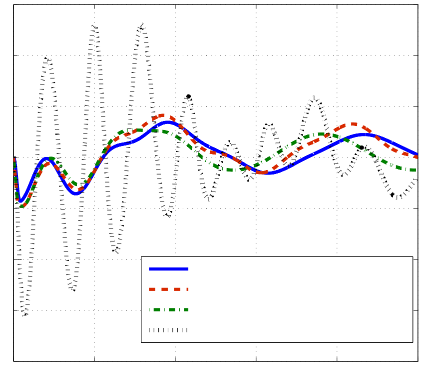

Fig. 2. Time responses of the displacement of the concentrated load

Then, the kinetic energy T, the potential energy V and the external virtual work δW generated by motors are

$$\begin{aligned} T =& \tfrac{1}{2}J_1\dot{\theta}_1^2(t) + \tfrac{1}{2}J_2\left(\dot{\theta}_1(t)+\dot{\theta}_2(t)\right)^2 + \tfrac{1}{2}m\dot{\boldsymbol{L}}^{\mathrm{T}}\dot{\boldsymbol{L}}, \\ &+ \tfrac{1}{2}\textstyle\int_0^{L_2}\dot{\boldsymbol{s}}^{\mathrm{T}}\dot{\boldsymbol{s}}\rho\,\mathrm{d}\,s + \tfrac{1}{2}k_1{\theta_1}^2(t) + \tfrac{1}{2}k_2{\theta_2}^2(t), \\ V =& \tfrac{1}{2}\textstyle\int_0^{L_2} EI(w''(s,t))^2\,\mathrm{d}\,s \\ \delta W =& \textstyle\sum_{i=1}^2 \tau_i(t)\delta\theta_i(t). \end{aligned} \tag{22}$$

By the Hamilton's variational principle, a partial differential equation (PDE) with respect to the elastic displacement, boundary conditions and equations of motion with respect to motors are obtained from (22). The PDE is an infinite-dimensional equation and is represented in the form of the first equation in (7) where the state $x(t) = (\theta_1(t), \theta_2(t), w(s,t), \dot{\theta}_1(t), \dot{\theta}_2(t), \dot{w}(s,t))^{\mathrm{T}}$. Equations of motion with respect to motors are finite-dimensional. As proposed in Section IV-B, in order to solve Hamilton-Jacobi equations by Galerkin method, the PDE is approximated to finite-dimensional one first, then the state x is described by a series of eigenfunctions $\phi_1(s)$ and $\phi_2(s)$ of the arm, $w(s,t) = \sum_{i=1}^2 \xi_i(t)\phi_i(s)$. The eigenfunctions of the system are a set of complete orthogonal functions $\phi_i(s) = \frac{1}{C_i}(\cosh\frac{\gamma_i s}{L_2} - \cos\frac{\gamma_i s}{L_2} - \eta_i(\sinh\frac{\gamma_i s}{L_2} - \sin\frac{\gamma_i s}{L_2}))$, where $\eta_i = \frac{\cosh\gamma_i + \cos\gamma_i}{\sinh\gamma_i + \sin\gamma_i}, i = 1, 2, \cdots$, C_i is an arbitrary constant and γ_i satisfies $1 + \cosh\gamma\cos\gamma + \frac{m}{\rho L_2}(\sinh\gamma\cos\gamma - \cosh\gamma\sin\gamma) = 0$ with $0 < \gamma_1 < \gamma_2 < \cdots$. This derives a finite-dimensional approximate equation with respect to the elastic displacement, from this equation and equations of motion with respect to the motors, a finite-dimensional state equation is derived, which is in the form of (17) with the 8-dimensional state $\xi = (\theta_1, \theta_2, \xi_1, \xi_2, \dot{\theta}_1, \dot{\theta}_2, \dot{\xi}_1, \dot{\xi}_2)^{\mathrm{T}}$, the input torque to each link $u = (\tau_1, \tau_2)^{\mathrm{T}}$ and the output $y(t) = w(L_2, t)$, which is the displacement of the concentrated mass at the tip of the arm. The following physical parameters are adopted in simulations. $L_i = 1$ (m), $J_i = 1$ (kgm^2), $k_i = 1$ (Pa), $\mu_i = 0.1$ (m^2/s), $m = 0.5$ (kg), $\rho = 2$ (kg/m), $E = 10$ (Pa), $I = 1$ (m^4) and $d = 1$ (s).

The response of the approximated model is depicted in Fig. 2. For comparison, the figure depicts the responses of

4 different models for wave pulses $\tau_1(t)$ and $\tau_2(t)$ with the initial state $x(0) = 0$. The solid line depicts the response of the 12-dimensional model obtained by mode decomposition, which is supposed to be close to that of the original model. The dotted line depicts that of the 6-dimensional model by mode decomposition. The dashed-dotted line depicts that of the 6-dimensional model derived by applying the proposed method to the Jacobian linearized model of the original. The dashed line depicts that of the 6-dimensional model by the proposed method. Comparing three 6-dimensional models, the response of the model derived by the proposed method is the closest to that of the 12-dimensional model, that is, the proposed method provides the best approximation. Therefore, it shows the effectiveness of the proposed method.

VI. Conclusion

This paper has discussed balanced realization for infinite-dimensional nonlinear systems. A relation between the energy functions and Hamilton-Jacobi equations is clarified. A finite-dimensional approximation method for infinite-dimensional nonlinear systems and its computational algorithm were proposed based on them. Moreover, a numerical example exhibited the effectiveness of the proposed finite-dimensional approximation method. There still exist many open problems: for example, when the approximate solutions obtained by the proposed method will converge to exact ones.

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Appendix

Proof of theorem 3

Proof: Suppose that (8) has a solution $\check{L}_o$ on W_o and that the origin of $\dot{x} = F(x)$ is asymptotically stable. Let the initial state $x(0) = x^0 \in W_o$ and the input of the system (7) equal to 0. Then the following equation is obtained.

$$\begin{cases} \dot{x} = F(x) \\ y = H(x) \end{cases} \tag{23}$$

Consider the solution $x(t)$ of the system (23). Then the definition of the observability function L_o (3) gives

$$\begin{aligned} L_o(x^0) &= \tfrac{1}{2}\int_0^\infty \|y(t)\|^2 \,\mathrm{d}\,t \\ &= \tfrac{1}{2}\int_0^\infty H(x(t))^{\mathrm{T}} H(x(t)) \,\mathrm{d}\,t \\ &= -\int_0^\infty \langle \nabla \check{L}_o(x(t)), F(x(t))\rangle_{L_2} \,\mathrm{d}\,t \\ &= -\int_0^\infty \tfrac{\mathrm{d}}{\mathrm{d}\,t}\check{L}_o(x(t)) \,\mathrm{d}\,t \\ &= -\lim_{t\to\infty} \check{L}_o(x(t)) + \check{L}_o(x^0) \end{aligned} \tag{24}$$

for $x^0 \in W_o$. Since $\check{L}_o(x) \le \alpha_o(\|x\|_{L_2})$, $\check{L}_o(x(t)) \le \alpha_o \circ \beta_o(\|x^0\|_{L_2}, t)$. On the other hand, since $\alpha_o \circ \beta_o \in$ class-KL, which is given by $\alpha_o \in$ class-K and $\beta_o \in$ class-KL, and since $\check{L}_o(x) \ge 0$, $\lim_{t\to\infty} \check{L}_o(x(t)) = 0$, $\forall x^0 \in W_o$. Then (24) implies $L_o(x^0) = \check{L}_o(x^0)$. Therefore, L_o coincides with $\check{L}_o$. The uniqueness of $\check{L}_o$ is proven by reductio ad absurdum. If (8) has another solution $\tilde{L}_o \ne \check{L}_o$ then $\tilde{L}_o = L_o = \check{L}_o$ can be proven immediately, which contradicts $\tilde{L}_o \ne \check{L}_o$. Therefore, (8) has the unique solution $\check{L}_o$. This proves the part (i) of Theorem 3.

For the part (ii) of Theorem 3, suppose that (9) has a solution $\check{L}_c$ on W_c and that the origin of (10) is asymptotically stable. As in the definition (2) of L_c, let $\lim_{t\to-\infty} \|x(t)\|_{L_2} = 0$ and an input u accomplish $x(0) = x^0 \in W_c$. Differentiating the solution $\check{L}_c(x(t))$ of (9) with respect to time t along $x(t)$ satisfying

$$\frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t} = F(x(t)) + \sum_{i=1}^m u_i(t) G_i(x(t)), \tag{25}$$

then

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}\,t}\check{L}_c(x(t)) \\ &= \mathrm{d}\,\check{L}_c(x(t))\left(F(x(t)) + \textstyle\sum_{i=1}^m u_i(t)G_i(x(t))\right) \\ &= \langle \nabla \check{L}_c(x(t)), F(x(t))\rangle_{L_2} \\ &\quad + \mathrm{d}\,\check{L}_c(x(t)) \textstyle\sum_{i=1}^m u_i(t) G_i(x(t)) \\ &= -\tfrac{1}{2}\textstyle\sum_{i=1}^m \langle \nabla \check{L}_c(x(t)), G_i(x(t))\rangle_{L_2}^2 \\ &\quad + \textstyle\sum_{i=1}^m \langle \nabla \check{L}_c(x(t)), G_i(x(t))\rangle_{L_2} u_i(t) \\ &= -\textstyle\sum_{i=1}^m \tfrac{1}{2}(u_i(t) - \langle \nabla \check{L}_c(x(t)), G_i(x(t))\rangle_{L_2})^2 \\ &\quad + \tfrac{1}{2}\|u(t)\|^2 \\ &\le \tfrac{1}{2}\|u(t)\|^2. \end{aligned} \tag{26}$$

On the other hand, since $\check{L}_c$ is positive definite and satisfies $0 \le \check{L}_c(x) \le \alpha_c(\|x\|_{L_2})$, and $\lim_{t\to-\infty} \|x(t)\|_{L_2} = 0$, $\lim_{t\to-\infty} \check{L}_c(x(t)) = 0$. Equation (26) implies

$$\begin{aligned} \check{L}_c(x^0) &= \int_{-\infty}^0 \tfrac{\mathrm{d}}{\mathrm{d}\,t}\check{L}_c(x(t)) \,\mathrm{d}\,t \\ &= \int_{-\infty}^0 \big(-\textstyle\sum_{i=1}^m \tfrac{1}{2}(u_i(t) - \langle \nabla \check{L}_c(x(t)), \\ &\qquad G_i(x(t))\rangle_{L_2})^2 + \tfrac{1}{2}\|u(t)\|^2\big) \,\mathrm{d}\,t \end{aligned}$$

$$\leq \tfrac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 \,\mathrm{d}\, t. \tag{27}$$

Therefore, $\check{L}_c(x^0)$ is a lower bound of $\frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 \,\mathrm{d}\, t$. Since the origin of (10) is asymptotically stable, it is clear that

$$u_i(t) = \langle \nabla \check{L}_c(x(t)), G_i(x(t))\rangle_{L_2},\ i = 1, \cdots, m \tag{28}$$

render $\lim_{t\to -\infty} \|x(t)\|_{L_2} = 0$ for an arbitrary x^0. These input signals give the minimum

$$\check{L}_c(x^0) = \min_{\substack{u\in L_2(-\infty,0)\\ x(-\infty)=0, x(0)=x^0\in W_c}} \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 \,\mathrm{d}\, t.$$

Therefore we have $L_c(x^0) = \check{L}_c(x^0)$. This proves the part (ii) of Theorem 3 and completes the proof. ■

Proof of Theorem 4

Proof: Suppose that there exists the smooth observability function L_o for the system (7) on a neighborhood W_o of 0 and that the origin of $\dot{x} = F(x)$ is asymptotically stable. Let $x(t)$ be the state of the system (23) and $x(0) = x^0 \in W_o$. Since $x(t) \in W_o$, the definition of L_o implies,

$$\begin{aligned} L_o(x(t)) &= \tfrac{1}{2}\int_{t}^{\infty} \|y(\tau)\|^2 \,\mathrm{d}\,\tau \\ &= \tfrac{1}{2}\int_{t}^{\infty} H(x(\tau))^{\mathrm{T}} H(x(\tau)) \,\mathrm{d}\,\tau, \end{aligned}$$

where $u(\tau) \equiv 0,\ t \leq \tau < \infty$. Differentiating the above equation with respect to time t,

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}\,t} L_o(x(t)) = -\tfrac{1}{2} H(x(t))^{\mathrm{T}} H(x(t)) \\ \therefore\quad & \langle \nabla L_o(x), F(x)\rangle_{L_2} + \tfrac{1}{2} H(x)^{\mathrm{T}} H(x) = 0. \end{aligned}$$

Here $L_o(0) = 0$ holds obviously. Therefore L_o is the solution of (11). As in a similar way to the proof to Theorem 3, L_o coincides with the solution of (11). Therefore, the part (i) of Theorem 4 is proven.

For the part (ii) of Theorem 4, suppose that there exists the smooth controllability function L_c of the system (7) on a neighborhood W_c of 0. The definition (2) of L_c derives the following equation for a state $x^t \in W_c$ of the system (7).

$$L_c(x^t) = \min_{\substack{u\in L_2(-\infty,t)\\ x(-\infty)=0, x(t)=x^t}} \frac{1}{2}\int_{-\infty}^{t} \|u(t)\|^2 \,\mathrm{d}\, t \tag{29}$$

Let $u^\star(t)$ denote the optimal input then

$$u^\star(t) = \arg \min_{\substack{u\in L_2(-\infty,0)\\ x(-\infty)=0, x(t)=x^t}} \frac{1}{2}\int_{-\infty}^{t} \|u(t)\|^2 \,\mathrm{d}\, t.$$

Suppose that $x(t)$ is the state of the system (25). Differentiating (29) with respect to time t

$$\begin{aligned} &\tfrac{1}{2}\|u^\star(t)\|^2 \\ &= \tfrac{\mathrm{d}}{\mathrm{d}\,t} L_c(x(t)) \\ &= \langle \nabla L_c(x(t)), F(x(t)) + \textstyle\sum_{i=1}^{m} G_i(x(t)) u_i^\star(t)\rangle_{L_2}. \end{aligned} \tag{30}$$

Now, apply the input $u = u^\star$ and steer the state from $x(t) = x^t$ to $x(t+\varepsilon) = x^{t+\varepsilon}$. Moreover, consider a continuous input u and suppose that u steers the state from $x(t) = x^t$ to $x(t+\varepsilon) = x^{t+\varepsilon}$. Define an input $\hat{u}$ as

$$\hat{u}(\tau) := \begin{cases} u^\star(\tau) & (\tau < t) \\ u(\tau) & (t \leq \tau \leq t+\varepsilon) \end{cases}$$

where $\varepsilon > 0$ is a small constant. Next, consider the following cost function J_ε.

$$\begin{aligned} J_\varepsilon(\hat{u}) &:= \tfrac{1}{2}\int_{-\infty}^{t+\varepsilon} \|\hat{u}(\tau)\|^2 \,\mathrm{d}\,\tau \\ &= \tfrac{1}{2}\int_{-\infty}^{t} \|u^\star(\tau)\|^2 \,\mathrm{d}\,\tau + \tfrac{1}{2}\int_{t}^{t+\varepsilon} \|u(\tau)\|^2 \,\mathrm{d}\,\tau \end{aligned} \tag{31}$$

Since $u^\star$ is optimal, it minimizes J_ε.

$$\begin{aligned} L_c(x^t) &= \min_{\substack{u\in L_2(-\infty,t)\\ x(-\infty)=0, x(t)=x^t}} \frac{1}{2}\int_{-\infty}^{t} \|u(\tau)\|^2 \,\mathrm{d}\,\tau \\ &= \tfrac{1}{2}\int_{-\infty}^{t} \|u^\star(\tau)\|^2 \,\mathrm{d}\,\tau \end{aligned} \tag{32}$$

The continuity of u and the intermediate-value theorem implies the existence of a continuous function a satisfying $0 < a(\varepsilon) < 1$ and

$$\tfrac{1}{2}\int_{t}^{t+\varepsilon} \|u(\tau)\|^2 \,\mathrm{d}\,\tau = \tfrac{1}{2}\varepsilon\|u(t + a(\varepsilon)\varepsilon)\|^2. \tag{33}$$

Substituting (32) and (33) for (31) yields

$$J_\varepsilon(\hat{u}) = L_c(x^t) + \tfrac{1}{2}\varepsilon\|u(t + a(\varepsilon)\varepsilon)\|^2. \tag{34}$$

Moreover, similarly to (32),

$$J_\varepsilon(u^\star) = L_c(x^{t+\varepsilon}) \tag{35}$$

On the other hand, it follows from the intermediate-value theorem and the continuity of L_c with respect to t that there exists a continuous function b satisfying $0 < b(\varepsilon) < 1$ and

$$\begin{aligned} &L_c(x(t+\varepsilon)) \\ &= L_c(x(t)) + \varepsilon \tfrac{\mathrm{d}\, L_c(x(\tau))}{\mathrm{d}\,\tau}\big|_{\tau = t + b(\varepsilon)\varepsilon} \\ &= L_c(x(t)) + \varepsilon\langle \nabla L_c(x(t+b(\varepsilon)\varepsilon), F(x(t+b(\varepsilon)\varepsilon)) \\ &\quad + G(x(t+b(\varepsilon)\varepsilon)) u(t + b(\varepsilon)\varepsilon)\rangle_{L_2}. \end{aligned} \tag{36}$$

Since $u^\star$ is optimal, $J_\varepsilon(\hat{u}) \geq J_\varepsilon(u^\star)$. As a result, the following equation is obtained from (34), (35) and (36).

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}\,\varepsilon}\left(J_\varepsilon(\hat{u}) - J_\varepsilon(u^\star)\right) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}\,\varepsilon}\left(L_c(x^t) + \tfrac{1}{2}\varepsilon\|u(t+a(\varepsilon)\varepsilon)\|^2 - L_c(x^{t+\varepsilon})\right) \\ &= \lim_{\varepsilon\to 0}\Big(\tfrac{1}{2}\|u(t+a(\varepsilon)\varepsilon)\|^2 - \langle \nabla L_c(x(t+b(\varepsilon)\varepsilon), \\ &\quad F(x(t+b(\varepsilon)\varepsilon)) + G(x(t+b(\varepsilon)\varepsilon)) u(t+b(\varepsilon)\varepsilon)\rangle_{L_2}\Big) \\ &= \tfrac{1}{2}\|u(t)\|^2 - \langle \nabla L_c(x(t), F(x(t)) + G(x(t))u(t)\rangle_{L_2} \\ &= -\langle \nabla L_c(x(t)), F(x(t))\rangle_{L_2} \\ &\quad -\tfrac{1}{2}\textstyle\sum_{i=1}^{m}\langle \nabla L_c(x(t)), G_i(x(t))\rangle_{L_2}^2 \\ &\quad +\tfrac{1}{2}\textstyle\sum_{i=1}^{m}(u_i(t) - \langle \nabla L_c(x(t)), G_i(x(t))\rangle_{L_2})^2 \end{aligned} \tag{37}$$

Since the optimal input $u = u^\star = (u_1^\star, \ldots, u_m^\star)$ minimizes the above equation,

$$u_i^\star = \langle \nabla L_c(x), G_i(x)\rangle_{L_2}. \tag{38}$$

By substituting (38) for (30), (12) is obtained. L_c fulfilling (12) is such that (10) is asymptotically stable, the uniqueness of the solution follows from the proof of theorem 3. These prove the part (ii) of Theorem 4 and completes the proof. ■