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Stability Analysis of Switched Polynomial Systems using Dissipation Inequalities

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***Abstract*— A polynomial approach to deal with stability analysis of polynomial switched systems, i.e., polynomial continuous systems with switching signals, using dissipation inequalities under arbitrary switching is presented. It is shown that the representation of the original switched problem into a continuous polynomial systems allows us to use the dissipation inequality for polynomial systems. With this method and from a theoretical point of view, we provide an alternative way to search for a common Lyapunov function for switched systems.**

***Index Terms*— Dissipation inequalities, Polynomial systems, Switched systems, Stability analysis**

I. INTRODUCTION

We deal with stability analysis of polynomial switched systems, i.e., polynomial continuous systems with switching signals, using dissipation inequalities under arbitrary switching. The most efforts in switched systems research have been typically focused on the analysis of dynamic behaviors, such as global uniform asymptotic stability (GUAS) with respect to switching signals, and several classes of switched systems that has the GUAS property have been idenfied. Several methods have been proposed to find the common Lyapunov function (see [1], [2], and references therein), but most of them are too restrictive from the computational point of view, because it is usually hard to check for a common function over all the subsystems and this could not exist.

In the seminal papers [3], [4] foundations in dissipativity theory of dynamical systems were presented. These foundations are based in terms of an inequality involving a generalized system power input, or, supply rate, and a generalized energy function, or storage function [3]. The interpretation of this storage function establishes the connection between Lyapunov stability and dissipativity. Stability problems can be solved once the dissipativity property is assured, and the storage function becomes a Lyapunov function, which can be used to construct Lyapunov functions for nonlinear dynamical systems. In general, dissipativity theory for switched systems has received just few attention in the last decade [5], [6], [7]. As for a common Lyapunov function a single storage function for all subsystems is usually difficult to find or may not exist (computational problems arise when a common function have to be find) [6]. For these common functions, i.e., common Lyapunov function and common storage function, recent computational tools have emerged to present alternative solutions for these problems. On the other hand, [8] – [10] present results for polynomial differential systems on a manifold, and [8] – [13] present techniques for the class of polynomial control systems. We see that many control problems can be modeled as, transformed into, or approximated by polynomial control systems. In addition, linear control systems can be considered as a special case of polynomial control systems. But this wide generality implies also difficulty to study. However, thanks to the recent computational tools for polynomials analysis, i.e., semidefinite programming and the sum of square decomposition [14], polynomial control systems can be analyzed by reliable and efficient numerical methods [9]. These properties of polynomial systems on a constraint manifold with the dissipasivity theory for stability analysis, give us the elements to establish stability analysis for polynomial switched systems reformulated as a differential polynomial continuous system on a constraint manifold [15].

The main contribution of this paper is twofold. First, we present a reformulation of the switched system as a differential polynomial continuous system on a constraint manifold. This reformulation opens several possibilities of analysis and design of switching systems in a consistent way and also with numerical efficiency [15]. We can profit from some tools developed in the last decade for nonlinear differential-algebraic equations (DAEs) and polynomial system, by numerical methods as analysis as well [8] – [13], [16], [17]. On the other hand, an alternative method to find a common Lyapunov function for switched systems with an efficient numerical method is presented using the simple stability result for stability analysis of polynomial DAEs presented in [8] –[10].

The paper is organized as follows. In Section II we present some definitions and basic concepts. A polynomial approach for the switched system is developed in Section III. Stability analysis for polynomial DAEs is described in Section IV. Finally, in Section V some conclusions are drawn.

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II. DEFINITIONS AND PRELIMINARIES

A. Basic Concepts

A switched system is a system that consists of several continuous-time systems with discrete switching events. A switched system may be obtained from a hybrid system by neglecting the details of the discrete behavior and instead considering all possible switching patterns from a certain class. This represents a significant departure from hybrid systems, especially at the analysis stage [1]. Switched systems have many application examples, such as power electric circuits, automotive controllers, chemical process, etc.

The mathematical model can be described by

$$\dot{x}(t) = f_{\sigma(t)}(x, u, t), \tag{1}$$

where $f_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}^n$ are vector fields, the exogenous input $u \in \mathbb{R}^m$ and $\sigma : [0, t_f] \to \mathcal{Q} \in \{0, 1, 2, ..., q\}$ is a *piecewise constant* function of time. Every mode of operation corresponds to a specific subsystem $\dot{x}(t) = f_i(x, u, t)$, for some $i \in \mathcal{Q}$, and the *switching signal* σ determines which subsystem is followed at each point in time, into the interval of time $[0, t_f]$, with t_f as the final time. The control inputs, σ and u, are both measurable functions. No assumptions about the number of switches nor about the mode sequence are made. In addition, we consider non Zeno behavior, i.e., we exclude an infinite switching accumulation points in time. The state does not have jump discontinuities.

B. Dissipativity, Passivity and Stability

Intuitively, from the concept of passivity it could be inferred that passive systems do not have the possibility to produce energy by themselves. It is possible to demonstrate that if the system is expressed as purely passive system the origin is an asymptotically unfluctuating equilibrium point, and the storage function V turns into a Lyapunov function. The functionality for stability analysis of passivity, dissipativity is that this characteristic is preserved under interconnection [18], [6]. Then, passivity is considered to be an efficient tool for the analysis and design of large-scale systems. In the same way we know that storage functions induced by dissipativity is a candidate Lyapunov function for stability analysis, this shows that stability and stabilization problems can be solved once the dissipativity property is assured [6].

Even if the definition of passivity and the energy speculations that guide to stability are intuitive, the definition could be misguiding when trading with hybrid systems. It would be well-founded to conclude that if the system can switch between two sets of state equations and if each set of them defines a passive system, the hybrid system produced must also be passive. But it has been proved that this conclusion would be incorrect [18].

Switched systems have an unusual behavior. A storage function of an inactive subsystem continues changing or even grows on the time interval when it is inactivated. In fact, this happens because all subsystems share the same state variables [6]. We are dealing with analysis of stability in switched systems under arbitrary switching, which is the property that the switched system state goes to zero asymptotically for any switching sequence. If this property holds for any initial conditions, we have global uniform asymptotic stability (GUAS) [1], [2].

III. POLYNOMIAL APPROACH

In this section, we show how the system (1) can be reformulated into a polynomial expression that mimics the behavior of the original system [15]. The approach followed here has had in spirit some counterpart for 0-1 programs, see for instance [19].

A. Polynomial Expression

The polynomial expression that is able to mimic the behavior of the switched system is developed using a new variable s, which works as a control variable. The starting point is to rewrite (1) as a continuous non-switched control system in its more general case.

First, we define a drift vector field $\mathbf{F}(x, u) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$

$$\mathbf{F}(x, u) = [f_0(x, u), f_1(x, u), ..., f_q(x, u)] \tag{2}$$

where $f_i(x, u)$, $i \in \mathcal{Q}$ is the function for each subsystem of the switched systems given in (1). Now, in order to find the polynomial expression we need for each $i \in \mathcal{Q} = \{0, 1, ..., q\}$, a quotient l_k with the property that $l_k(i) = 0$ when $i \neq k$, and $l_k(k) = 1$.

Let $\mathbf{L}$ be the vector of Lagrange polynomial interpolation quotients ([20]) defined with the new variable s, i.e.,

$$\mathbf{L}(s) = [l_0(s), l_1(s), ..., l_q(s)]^T \tag{3}$$

where

$$l_k(s) = \prod_{\substack{i=0 \\ i \neq k}}^{q} \frac{(s - i)}{(k - i)} \tag{4}$$

Secondly, using these Lagrange polynomial interpolation quotients, we see that there exists an unique polynomial P of order $q + 1$ with the property of

$$f_i(x, u) = P(x, u, i) \quad \text{for each } i \in \mathcal{Q}$$

This polynomial is given by

$$\begin{aligned} P(x, u, s) &= \mathbf{F}(x, u)\mathbf{L}(s) \\ &= \textstyle\sum_{k=0}^{q} f_k(x, u) l_k(s) \end{aligned} \tag{5}$$

where $s \in \mathbb{R}$, and $l_k(s)$ is given by (6).

On the other hand, we can use a complementary polynomial, which is used to constrain s to take only integer values. Let $Q(s)$ be the constraint polynomial so that

$$Q(s) = \prod_{k=0}^{q} (s - k) \tag{6}$$

A related continuous polynomial DAE system of the switched system (1) is constructed in the following proposition.

Proposition 1: Consider a switched system of the form given in (1) with a drift vector field which are in the form given in (3). There exists an unique polynomial differential-algebraic equation with the polynomial state equation $P(x,u,s)$ given in (5) of order $q+1$, and the constraint algebraic polynomial $Q(s)$ given in (6) as follows

$$\begin{aligned} \dot{x} &= P(x,u,s) \\ 0 &= Q(s) \end{aligned} \tag{7}$$

This polynomial DAE is an equivalent representation of the switched system (1)

Proof. The system we are looking for has to be a polynomial system consisting of a polynomial state equation $P(x,u,s)$ of degree $q+1$, and a constraint polynomial $Q(s)$, also of degree $q+1$. We know that the Lagrange polynomial is a solution to the interpolation problem, and $l_j(s)$ is a polynomial and has degree $q+1$. So that from the definition of the Lagrange polynomial interpolation ([20]), $\mathbf{L}(s)$ is the unique polynomial interpolating the given data. Now, using equation (7), we have the equivalent representation

$$\begin{aligned} \dot{x} = P(x,u,s) &= \textstyle\sum_{k=0}^{q} f_k(x,u) l_k(s) \\ 0 = Q(s) &= \prod_{k=0}^{q} (s-k) \end{aligned}$$

we see that the solutions of the algebraic equation $Q(s)=0$ constrain the values of the variable s to be in the set of values $\mathcal{Q}$. Let q be the finite number of subsystems of the switched system (1), i.e., $f_0, \cdots, f_q$. Then, the polynomial state equation $P(x,u,s)$ is unique due to the quotients of the Lagrange polynomial interpolation $l_0, ..., l_q$ are unique. From the numerator of the above definition, we see that $l_k(s)$ is an order $n+1$ polynomial having zeros at all of the subsystems except the k-th. The denominator is simply the constant which normalizes its value to 1 when f_k is activated. Thus, we have

$$l_k(s) = \delta_{k_s} \equiv \begin{cases} 1, & s = k \\ 0, & s \neq k \end{cases}$$

■

For instance, we show the most simply case when $q=1$, the system (7) has the same form of the convex combination of two subsystems. Note that the trajectories of the original switched system (1) correspond to piecewise constant controls taking values in the set $\sigma \in \{0,1,...,q\}$.

The above proposition allows us to represent the switched system as a differential algebraic equation (DAE). The notion of DAE represents the fact that (7) consist of differential equations coupled with algebraic equations. Depending of the problem modeling, DAEs are also called, singular systems, descriptor systems, semistate equation, implicit systems, and differential equations on manifolds. For this approach, DAEs are regarded as explicit ordinary differential equations on manifolds. While DAEs provide a convenient modeling concept, many numerical difficulties arise due to the fact that the dynamics are constrained to a manifold [17]. These difficulties are characterized by one of many index concepts that exist for DAEs, i.e., global index, tractability index, geometrical index, perturbation index, differential index, among others (see [21], [17], and the reference therein). The differential index is the notion used in this approach. In general, we define the DAE as follows:

$$G(x,\dot{x},s) = 0 \tag{8}$$

We note that variable s may be regarded as another state variable for analysis purpose. The function $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ is assumed to be sufficiently smooth with $G(0,0,0) = 0$. The definition of index is described as follows.

Definition 2: [22] The index of the DAE in Equation (8) is the minimum number of times that all or part of a nonlinear DAE must be differentiated with respect to t in order to determine $\dot{x}$.

According to the definition, an implicit ODE has index zero. In [21], the constrained manifold is used to define the differentiation index and in [22], a proposition is presented for a semi-explicit DAE that it can be extended to (7). Then, it can be seen that the semi-explicit DAE (7) has index one if and only if $\partial Q(s)/\partial s$ is nonsingular. Now, from the semi-explicit DAE (7), we note that all solutions of the DAE lie in the manifold

$$\Gamma = \{(x,s)^T : Q(s) = 0\}$$

This clearly implies that we cannot find a solution if the starting point does not belong to this manifold. Finally, the solution of (7) may be interpreted as an explicit ODE on the manifold Γ.

1) Numerical Example of Switched Nonlinear System: In this section we present an illustrative example of application of a switched nonlinear system reformulated by Proposition 1 as a differential equation on a manifold. Simulations are done with MATLAB using the function ode15s, which is used to solve DAE. With this example we illustrate an efficient computational treatment to simulate switched systems reformulated as a polynomial expression.

Consider the set of systems described by the drift vector field

$$\mathbf{F}(x) = [f_0(x), f_1(x)]$$

with

$$f_0(x) = \begin{bmatrix} -x_1^2 + x_2^2 \\ -2x_1x_2 \end{bmatrix}$$

and $f_1 = -f_0$. This system is the so-called Artstein's circle, which is asymptotically controllable and asymptotically stabilizable [23], [24]. This is presented as a switched nonlinear system for stabilizing analysis. In Fig. 1 we can see the phase plane for two different initial conditions. The system shows a stable behavior for both cases.

B. Stability Problem

The switched system expressed as a polynomial differential-algebraic system allows us to establish an alternative approach for stability analysis. Instead of searching for a common Lyapunov function, or multiple Lyapunov functions in order to provide stability under arbitrary switching law (which are usually very restrictivetechniques based on

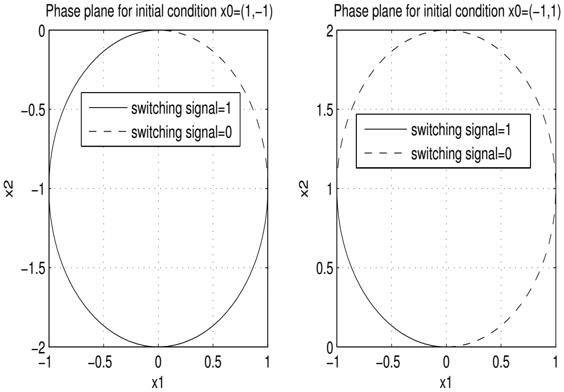

Fig. 1. Phase plane for two different initial conditions

exhaustive algorithms [1]), we can look for a Lyapunov function using techniques developed for polynomial continuous systems. It means that we can find a common Lyapunov function using dissipativity inequalities as in [9]. With this reformulation we are dealing with a polynomial differential system on a manifold as mentioned before.

Basically, the stability problem of differential-algebraic systems is related to the problem of stability on manifolds, which are defined by the constraints in the system description. The main components for a stability analysis of DAEs are the constraints as well as penalization arguments from optimization theory. Using these concepts in a more general setting, a simple yet general stability theorem in terms of a dissipation inequality for differential-algebraic systems of the form (8) is obtained in [8], [9], [10] as outlined in the next section.

IV. RESULTS IN STABILITY ANALYSIS

In the previous section the switched systems is expressed as a polynomial differential-algebraic system or a constrained control system. With this reformulation, we can apply the stability results for constrained control systems using dissipation inequalities and sum of squares as in [8], [9], [10].

A. Stability Result for Polynomial Differential-Algebraic Systems

The main idea behind the proposed approach is simple and it is presented as follows. First, define the following stacked vector of hidden constraints containing the first μ derivatives of the vector field G along trajectories, i.e.,

$$\mathbf{G}_\mu(\xi,\varsigma) = \left[G(x,\dot{x},s), \frac{d}{dt}G(x,\dot{x},s), \cdots \frac{d^\mu}{dt^\mu}G(x,\dot{x},s)\right]^T \tag{9}$$

with $\xi = (x, \dot{x}, ..., x^{\mu+1})$, $\varsigma = (s, \dot{s}, ..., s^{\mu+1})$, and $(d/dt)G(x,\dot{x},s) = (\partial/\partial x)G(x,\dot{x})\dot{x} + (\partial/\partial \dot{x})G(x,\dot{x})\ddot{x} + (\partial/\partial s)G(x,\dot{x},s)\dot{s}$, and so on. The following theorem gives a simple stability result for the general differential-algebraic systems of type (7).

Theorem 3: ([9], [10]) The equilibrium point $x = 0$ of the differential-algebraic system (7) is stable for any admissible input $s = s(t)$, if there exists a Lyapunov function candidate $V : \mathbb{R}^n \mapsto \mathbb{R}$, a function ρ:$\mathbb{R}^{(\mu+2)(n+q)} \mapsto \mathbb{R}$, and an integer number μ such that the dissipation inequality

$$\nabla V(x)\dot{x} \leq \left\|\mathbf{G}_\mu(\xi,\varsigma)\right\|^2 \rho(\xi,\varsigma) \tag{10}$$

is satisfied for some x-neighborhood $\Omega_x((\xi,\varsigma)=0)$.
The main idea behind the inequality (10) is just to check negative semidefiniteness of $\dot{V}$ with respect to the constraint set. Otherwise, if $\mathbf{G}_\mu(\xi,\varsigma) \neq 0$, one can always find a function ρ such that inequality above is satisfied, by just making ρ big enough. It can be interpreted ρ as Lagrange multiplier. On the other hand, in general, according to [9], [10], there is no universal rule how to choose μ, but in certain cases the choice is clear. The implicit stability analysis given by the dissipation inequality (10) corresponds exactly to an explicit stability analysis of the vector field defined on this constrained manifold. In this case, the differentiation index of the differential-algebraic system is known, therefore it is reasonable to choose μ to be equal to the differentiation index (i.e., $\mu = 1$).

B. Stability Analysis of Polynomial Switched Systems

In general, it is very difficult to search for a Lyapunov function V and a function ρ for practical problems. However, recently established methods based on semidefinite programming and the sum of squares decomposition allow us to verify Lyapunov inequalities of the form (10) very efficiently in the case where G, V, ρ are assumed to be polynomials [8]. Certainly, in our case functions G, V, and ρ are all of polynomial nature. Hence, we have to establish an algorithm in order to find a Lyapunov function V and a function ρ.

The algorithm can be summarized as follows. Given functions G, G_μ, and μ, find a radially unbounded differentiable positive definite function V and a function ρ such that the dissipation inequality (10) is satisfied. It is impossible to search over all functions V, ρ. Here, it is assumed that V and ρ are polynomials up to certain degrees. Now, we can define the dissipation inequalities for the polynomial representation of the switched system. Since we are dealing with the study of global uniform asymptotically stable (GUAS) systems, it means that we are searching for a common Lyapunov function regardless of what a switching sequence is. On the other hand, we know from Definition 2 that the differential-algebraic system (7) is of index one. Therefore, we may choose $\mu = 1$, and if we try to prove global stability of the system (5), the following polynomial inequalities must be satisfied,

$$\begin{gathered} V(x) > 0 \\ \nabla V(x)P(x,s) \leq \|Q(s)\|^2 \rho(x,s) \end{gathered}$$

which implies that

$$\begin{gathered} V(x) > 0 \\ \nabla V(x)\left(\textstyle\sum_{k=0}^q f_k(x,u)l_k(s)\right) \leq \left\|\prod_{k=0}^q (s-k)\right\|^2 \rho(x,s) \end{gathered} \tag{11}$$

for all $\xi = (x, \dot{x}, s) \in \mathbb{R}^{2n}$. Note that if V is polynomial and positive definite, it implies that V is radially unbounded. To verify such polynomial inequalities is an NP-hard computational problem [8]. However, with the help of the sum of squares decomposition, it is possible to verify such polynomial inequalities very efficiently. On the other hand, this problem coincides with the problem of searching for a common Lyapunov function for the vector field $\mathbf{F}(x,u) = [f_0(x,u), f_1(x,u), ..., f_q(x,u)]$.

For illustration and clarity of exposition, consider the case when $q = 1$. The dissipation inequality is of the form

$$\nabla V(x)\left(f_0(x)(1-s) + f_1(x)s\right) \leq \|s(1-s)\|^2 \rho(x,s) \tag{12}$$

Before we state an analytical solution of (12), we need to introduce some basic concepts of sum of squares decomposition. A more detailed description can be found in [14], [13], and the references therein.

The semidefinite programming method for computing the sum of squares decomposition is based on the Gram matrix method (see [25] for more details).

Definition 4: , [13], [14] For $x \in \mathbb{R}^n$ a multivariate polynomial $p(x)$ is sum of squares (SOS) if there exist some polynomials $r_i(x)$, $i = 1, ..., M$ such that

$$p(x) = \sum_{i=1}^{M} r_i^2(x) \tag{13}$$

It is clear that $p(x)$ being an SOS naturally implies $p(x) \geq 0$ for all $x \in \mathbb{R}^n$. An equivalent characterization of SOS polynomials is given in the following proposition.

Proposition 5: [14] A polynomial $p(x)$ of degree $2d$ is a SOS if and only if there exists a positive semidefinite matrix Q and a vector of monomials $Z(x)$ containing monomials in x of degree $\leq d$ such that

$$p(x) = Z(x)^T Q Z(x)$$

Since, we have that $P(x,s)$ is a polynomial vector field, and that we are searching for $V(x)$ that is also a polynomial in x. To solve the testing conditions inequality (12), we can restrict our attention to cases in which the conditions admit SOS decompositions. The only apparent difficulty is the restriction of $V(x)$ to be positive definite, not just positive semidefinite. To deal with this problem we can use the following proposition.

Proposition 6: [14] Given a polynomial $V(x)$ of degree $2d$, let $\varphi(x) = \sum_{i=1}^{n}\sum_{j=1}^{d} \epsilon_{i,j} x_i^{2j}$ such that,

$$\sum_{j=1}^{d} \epsilon_{i,j} > \gamma \quad \forall i = 1, ..., n, \tag{14}$$

with γ a positive number, and $\epsilon_{i,j} \geq 0$ for all i and j. Then the condition

$$V(x) - \varphi(x) \quad \text{is a SOS} \tag{15}$$

guarantees the positive definiteness of $V(x)$.

Now we can pose the inequality (12) as follows. Find a polynomial $V(x)$ such that

$$\begin{array}{c} V(x) - \varphi(x) \geq 0 \text{ is SOS} \\ -\nabla V(x)\left(\sum_{k=0}^{q} f_k(x,u) l_k(s)\right) + \cdots \\ \left\| \prod_{k=0}^{q} (s-k) \right\|^2 \rho(x,s) \geq 0 \text{ is SOS} \end{array} \tag{16}$$

the polynomials $V(x)$, $\rho(x,s)$, and the positive definite function $\varphi(x)$ can be computed using SOSTOOLS [11].

C. Numerical Example 1: Two-switched linear case

Consider for simplicity the polynomial (7) when $q = 1$. First, we deal with the optimal control problem, when all the vector fields are linear, i.e., $f_i(x,u,t) = A_i x$, where $A_i \in \mathbb{R}^{n\times n}$. This yields linear autonomous switched system, i.e.,

$$\dot{x}(t) = A_{\sigma(t)} x(t) \tag{17}$$

Using the polynomial transformation (7) we have,

$$\begin{aligned} \dot{x}(t) = P(x,s) &= \left(\sum_{k=0}^{1} A_k L_k(s)\right) x(t) \\ &= A_0 L_0 x(t) + A_1 L_1 x(t) \end{aligned} \tag{18}$$

with $L_0 = (1-s)$, $L_1 = s$. Combining (17) and (18), we obtain the dynamics given by

$$\begin{aligned} \dot{x}(t) &= \left(A_0(1-s) + A_1 s\right) x(t) \\ Q(s) &= s(s-1) = 0 \end{aligned} \tag{19}$$

Note that the trajectories of the original switched system (1) correspond to piecewise constant controls taking values in the set $\{0, 1\}$. In particular, $\dot{x}(t) = A_0 x(t)$ results by setting $s = 0$ in (7), while $\dot{x}(t) = A_1 x(t)$ results by setting $s = 1$. The switching behavior is defined by the constrained polynomial $Q(s)$. For illustration consider the linear two-switched system with

$$A_0 = \begin{bmatrix} -1.1 & 1.4 \\ -0.1 & 0.8 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.3 & 1.1 \\ -0.4 & 0.1 \end{bmatrix}$$

Using these matrices and Equations (18) and (19), we obtain,

$$\begin{array}{c} -\frac{\partial V}{\partial x_1}\left[(-1.1 + 0.8s)\,x_1 + (1.4 - 0.3s)\,x_2\right]\cdots \\ -\frac{\partial V}{\partial x_2}\left[(-0.1 - 0.3s)\,x_1 + (0.8 - 0.7s)\,x_2\right]\cdots \\ +\left(s^4 - 2s^3 + s^2\right)\rho(x,s) \geq 0 \end{array}$$

using the MATLAB toolbox SOSTOOLS [11], a Lyapunov function of degree six is found,

$$\begin{array}{c} V(x) = 3.826x_1^2 - 0.9845x_1^4 + 2.914x_2^4 + 3.14x_2^2 \\ -7.041x_1x_2 - 8.6513x_1x_2^3 + 6.105x_1^2x_2^2 \\ +0.6764x_1^6 - 0.111x_2^6 + 0.1x_1^4x_2^4 \end{array}$$

which proves that the origin is a stable point of the linear switched system (18) reformulated as a polynomial DAE (7).

D. Numerical Example 2: Two-switched nonlinear case - Artstein's circle

A more interesting case is the nonlinear switched system case. We use the example presented in Section III in order to prove stability under arbitrary switching. First, we obtain the polynomial DAE reformulation

$$\dot{x}(t) = \left(f_0(1-s) + f_1 s\right) x(t)$$

$$= \begin{bmatrix} (x_1^2 - x_2^2)(2s-1) \\ (2x_1x_2)(2s-1) \end{bmatrix}$$

$$Q(s) = s(s-1) = 0$$

Again, using these polynomial matrices and equations (19) and (18), we obtain,

$$-\frac{\partial V}{\partial x_1}(x_1^2 - x_2^2)(2s-1) - \frac{\partial V}{\partial x_2}(2x_1x_2)(2s-1)\cdots \\ + \left(s^4 - 2s^3 + s^2\right)\rho(x,s) \geq 0$$

using the MATLAB toolbox SOSTOOLS [11], a Lyapunov function of degree six is found,

$$V(x) = 0.10434x_1^2 - 0.00682x_1^4 - 0.00211x_2^4 + 0.113x_2^2 \\ -0.1x_1x_2^2 + 0.0268x_1^2x_2^2 + 0.1x_1^3x_2^2 + 0.002x_2^6$$

which proves that the Artstein circle system reformulated as a polynomial DAE (7) is stable. We note that for these examples we have tested different values of γ in order to obtain a Lyapunov function with nice coefficients, in this way we have used a value for γ of the order of 10^5.

V. CONCLUSIONS AND FUTURE WORK

In this paper, we have developed a new method for stability analysis of switched systems based on a polynomial approach. First, we transform the original problem into a polynomial system, that is able to mimic the switching behavior but with a continuous differential-algebraic nonlinear representation. From a theoretical point of view, we show that the representation of the original switched problem into a continuous polynomial systems allow us to use the dissipation inequality for polynomial systems. With this method and from a theoretical point of view, we provide an alternative way to search for a common Lyapunov function for switched systems. This work opens several possibilities for the system analysis, as stability analysis by using sum of squares [14], and some other analysis as controllability, observability, sensivity among others. Some results on controllability of switched systems related with non-switched polynomial system have been presented in [26]. It means that with this approach, we have the possibility of analysis and control for nonlinear switched systems in a consistent way.

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