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Global Finite-time Stabilization of a Nonlinear System using Dynamic Exponent Scaling

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***Abstract*— In this paper we consider the problem of global finite-time stabilization for a class of triangular nonlinear systems. The proposed design method is based on backstepping and dynamic exponent scaling using an augmented dynamics, from which, a dynamic *smooth* feedback controller is derived. The finite-time stability of the closed-loop system and the boundedness of the controller are proved by the finite-time Lyapunov stability theory and a new notion 'degree indicator'.**

I. INTRODUCTION

After the concept of finite-time stability was introduced in the 1950s [12], many researchers have made an effort to solve this problem because of fast convergence and good performances on robustness and disturbance rejection. Since the bang-bang time optimal feedback control was applied to the double integrator [1], many results have been presented in the literature for various systems [2]–[5], [7]–[11], [14]–[17].

Most of them are concerned with the *continuous* state feedback and output feedback. In particular, the authors of [4] introduced the Lyapunov theory for finite-time stability and suggested the continuous state feedback which achieves finite-time stability of the double integrator system. After then, the paper [5] gave the Lyapunov theorem for finite-time stability of continuous autonomous systems. Results based on the concept of homogeneity appear in [8]–[10]. In [8], an output feedback finite-time stabilization problem for the double integrator system was handled and, in [9], a continuous finite-time stabilizer for a class of controllable systems, especially a chain of power-integrators, was proposed. Moreover, the problem of finite-time output feedback was studied in [10], which solved the problem using finite-time state feedback and finite-time observer for the same system. On the other hand, by backstepping and domination approach, the consequence of [11] constructed Hölder continuous state feedback for a lower-triangular systems with uncertainty and its notions are extended to output feedback problem for various systems [14]–[17]. These techniques were further extended in [2].

Our objective is to design a global finite-time stabilizer for a class of triangular nonlinear systems using an augmented dynamics. Our tools are backstepping and 'dynamic exponent scaling' in conjunction with a specially designed augmented dynamics, from which a smooth ($\mathcal{C}^\infty$) state feedback is obtained. One benefit of smooth ($\mathcal{C}^\infty$) feedback over the continuous ($\mathcal{C}^0$) feedback is that the uniqueness of the solution is directly guaranteed because the closed-loop system becomes smooth. In contrast, most of previous results such as [2], [11], [14]–[17] just guaranteed global 'strong stability[1]', or some authors of, e.g., [3]–[5], [8]–[10] presumed uniqueness of the solution in forward time, which is hard to verify. Another benefit of the proposed design is that it gives relatively less hardened[2] feedback, compared to, e.g., [11]. This is because the domination method used in [11] intrinsically yields somewhat hardened control, while the proposed method need not use the domination method. Finally, the proposed design is relatively simple compared to [11]. This is again the benefit from the smoothness. In fact, since the virtual control at each step is also smooth, the domination method need not be used which makes the design relatively simple. The only cost to pay for the proposed design is that the proof should guarantee that the proposed controller is bounded until the solution gets into the origin in finite-time because the proposed controller has some state in its denominator (that will become zero). In this paper, we provide the proof using a new notion 'degree indicator'.

To introduce our idea, while avoiding unnecessary complexity, we limit ourselves in this paper to the 3rd-order triangular systems of the form

$$
\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1),\\
\dot{x}_2 &= x_3 + f_2(x_1, x_2),\\
\dot{x}_3 &= u + f_3(x_1, x_2, x_3),
\end{aligned} \tag{1}
$$

where $[x_1, x_2, x_3]^T \in \mathbb{R}^3$ is the system state, $u \in \mathbb{R}^1$ is the system input, and $f_i(\cdot)$, $i = 1, 2, 3$, are smooth functions with $f_i(0) = 0$. For (1), a dynamic controller of the form

$$
\dot{x}_0 = f_0(x_0, x_1, x_2, x_3), \qquad x_0(0) > 0, \tag{2}
$$

$$
u = u(x_0, x_1, x_2, x_3), \tag{3}
$$

will be constructed. The dynamics (2) is called as an augmented system, whose intial condition is always set to be any positive number.

The paper is organized as follows. In Section II, we present a motivational example where uniqueness of solution, finite-time stability of the closed-loop system, and boundedness of the controller are studied in a simplified setting. Section III is devoted to the main theorem and the proof, which consists of two parts: an algebraic design of the control law and a consideration of the dynamics to prove boundedness of the controller. Concluding remarks are given in Section IV.

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[1] That is, there may be many solutions but they are all stable.

[2] The 'hardened' control exhibits unnecessarily high local gains in some regions of the state space, which might cause excessive control effort such as high-magnitude chattering in the control signal [6].

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II. MOTIVATIONAL EXAMPLE

To see the basic idea effectively, we begin by

$$\dot{x} = u. \tag{4}$$

For this system, consider a dynamic controller[3]

$$\dot{x}_0 = -k_0 x_0^d + \frac{x^2}{x_0^{2-d}} =: f_0(x_0, x) \tag{5a}$$

$$u = -\left(1 + \frac{1}{b}\right)\frac{x}{x_0^{1-d}} =: u(x_0, x) \tag{5b}$$

where d is a fraction such that $0 < d < 1$ whose numerator and denominator are odd integers, $k_0 = 1 + 1/a$ where $a = 2/(1-d)$, and $b = 2/(1+d)$. Note that the controller is well-defined and smooth in the set $\mathbb{R}^{(+,1)} := \{(x_0, x) : x_0 > 0\}$. We set the initial condition $x_0(0)$ of (5a) to be any positive number (i.e., $x_0(0) > 0$), and it will be seen that the solution $x_0(t)$ remains positive before the solution $(x_0(t), x(t))$ gets to the origin in finite time. We now claim that the controller (5) plays the role of finite-time stabilizer by the following arguments.

(1) For any initial condition $x(0)$ and any $x_0(0) > 0$, the unique solution $(x_0(t), x(t))$ of the closed-loop system (4) *and* (5) *exists as long as $(x_0(t), x(t)) \in \mathbb{R}^{(+,1)}$.*

This is because the closed-loop system is smooth in the open set $\mathbb{R}^{(+,1)}$ (see [13]).

(2) The solution $(x_0(t), x(t))$ becomes $(0, 0)$ at a finite time $T > 0$, and $x_0(t) > 0$ for $0 \le t < T$.

Basically, it is enough to show that the solution escapes from the set $\mathbb{R}^{(+,1)}$ in finite time through the origin. To see this, let the Lyapunov function $V = (x_0^2 + x^2)/2$. Then, we have

$$\dot{V} = -k_0 x_0^{1+d} + \frac{x^2}{x_0^{1-d}} + xu + (-x^{1+d} + x^{1+d}),$$

in which the term x^{1+d} is added and subtracted. Here, to make the term x^{1+d} to be x^2, we use the following inequality[4]:

$$x^{1+d} \times \frac{x_0^r}{x_0^r} = x_0^r \times \frac{x^{1+d}}{x_0^r} \le \frac{x_0^{1+d}}{a} + \frac{1}{b}\frac{x^2}{x_0^{1-d}}, \tag{6}$$

with $a = 2/(1-d)$, $b = 2/(1+d)$, and $r = (1+d)/a$. We name the above inequality 'dynamic exponent scaling' since the augmented state x_0 is used to increase the degree of x.

Using the inequality (6), we arrive at

$$\dot{V} \le -\left(k_0 - \frac{1}{a}\right)x_0^{1+d} - x^{1+d} + xu + \left(1 + \frac{1}{b}\right)\frac{x^2}{x_0^{1-d}}.$$

Therefore, the control (5) with $k_0 = 1 + 1/a$ yields that

$$\dot{V} \le -x_0^{1+d} - x^{1+d}.$$

Now we use the fact that if there exists a $\mathcal{C}^1$ positive definite radially unbounded Lyapunov function V such that $\dot{V} + kV^\alpha \le 0$ along the solution of the system, with $k > 0$ and $0 < \alpha < 1$, then the origin is globally finite-time stable [4]. For our case, with $\alpha = (1+d)/2$, it follows that

$$\begin{aligned}\dot{V} + kV^\alpha &\le -(x_0^{1+d} + x^{1+d}) + k\left(\frac{x_0^2 + x^2}{2}\right)^\alpha \\ &\le -(x_0^{1+d} + x^{1+d}) + \frac{k}{2^\alpha}\left(x_0^{1+d} + x^{1+d}\right) \\ &= -\left(1 - \frac{k}{2^\alpha}\right)\left(x_0^{1+d} + x^{1+d}\right) \le 0,\end{aligned} \tag{7}$$

in which we choose k such that $0 < k \le 2^\alpha$.

We suppose that $x_0(t) > 0$ for $0 \le t < T_{x_0}$ and $x_0(T_{x_0}) = 0$, with the possibility that $T_{x_0} = \infty$. Then, during $0 \le t < T_{x_0}$, the solution $(x_0(t), x(t))$ is in the set $\mathbb{R}^{(+,1)}$, and thus, the inequality (7) is valid for that period. This in turn implies that the function V becomes zero at a time $T_V > 0$ (noting that $V > 0$ at $t = 0$). Because $V = 0$ implies that $x_0 = 0$, it is not possible that $T_{x_0} = \infty$ or $T_{x_0} > T_V$. On the other hand, $T_{x_0} < T_V$ is not possible either. In fact, if $T_{x_0} < T_V$, then $x_0(T_{x_0}) = 0$ and $x(T_{x_0}) \neq 0$. This implies from (5a) that $\dot{x}_0 = -k_0 x_0(t)^d + \frac{x(t)^2}{x_0(t)^{2-d}} > 0$ for a short time period just before T_{x_0}, say $t \in [T_{x_0} - \epsilon, T_{x_0})$, because x_0 is very small but positive while $|x(t)|$ is strictly greater than zero. This implies that $x_0(t)$ does not decrease, which is a contradiction. Therefore, it follows that $T_V = T_{x_0}$, and proves the claim with $T = T_V$.

(3) The right-hand sides of the controller (5a) *and* (5b) *(i.e., $f_0(x_0, x)$ and $u(x_0, x)$) remain bounded for $0 \le t < T$.*

Define $\mathcal{P} := \{(x_0, x) : k_0 x_0^2 \ge x^2, x_0 > 0\}$, and $\mathcal{P}_R := \mathcal{P} \cap B_R$ where B_R is a ball of a positive radius R centered at the origin. The state $x_0(t)$ does not increase in the set $\mathcal{P}$ because of (5a), while $x_0(t)$ increases in $\mathbb{R}^{(+,1)} \backslash \mathcal{P}$. There exist $R > 0$ and $T_R \ge 0$ such that the solution $(x_0(t), x(t))$ remains in P_R for $t \in [T_R, T)$. This is because $x_0(t)$ should decrease just before it becomes zero. Obviously, for $t \in [0, T_R]$, the singular terms x^2/x_0^{2-d} and x/x_0^{1-d} in (5) are bounded because they are continuous on a compact time interval. Hence, it is left to show that they are still bounded for the period $[T_R, T)$. In fact, it will be shown that two functions f_0 and u are bounded in the set $\mathcal{P}_R$.

By noting that the singularity happens only when $x_0 = 0$, we need to prove that

$$\lim_{x_0 \to 0^+} \max_{\sqrt{x^2/k_0} \le x_0} |g(x_0, x)| < \infty$$

where g represents f_0 and u, respectively. To facilitate it, we define 'degree indicator' as

$$\begin{aligned} D(g(x_0, x)) &:= \inf \beta \quad \text{subject to} \\ &\limsup_{x_0 \to 0^+} \bar{g}(x_0) x_0^\beta < \infty \end{aligned} \tag{8}$$

where $\bar{g}(x_0) := \max_{\sqrt{x^2/k_0} \le x_0} |g(x_0, x)|$. Then, the function $g(x_0, x)$ is unbounded in $\mathcal{P}_R$ if and only if $D(g(x_0, x)) > 0$. Finally, since $D(f_0(x_0, x)) = -d$ and $D(u(x_0, x)) = -d$, it is ensured that they are bounded in $\mathcal{P}_R$.[5]

[3]Precisely speaking, the controller (5) should be

$$\dot{x}_0 = \begin{cases} (5a), & x_0 \neq 0 \\ 0, & x_0 = 0 \end{cases} \quad \text{and} \quad u = \begin{cases} (5b), & x_0 \neq 0 \\ 0, & x_0 = 0 \end{cases}$$

since the controller (5) is not defined when $x_0 = 0$.

[4]It is based on Young's inequality:

$$|x||y| \le \frac{|x|^a}{a} + \frac{|y|^b}{b}$$

where $1/a + 1/b = 1$.

[5]In fact, $D(f_0(x_0, x)) = -d$ if $k_0 \neq 1$ or $D(f_0(x_0, x)) = -\infty$ if $k_0 = 1$, for example. (But, note that $k_0 > 1$ in the example.)

Fig. 1 shows the phase portrait of the closed-loop system (4) and (5) for various initial conditions, $d = 1/3$ and $k_0 = 1 + 1/a = 4/3$.

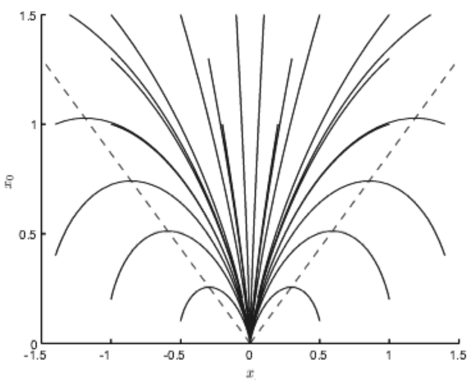

Fig. 1. Phase portrait of the closed-loop system (4) and (5).

III. Finite-time Stabilizer for Triangular Systems

In this section we present the main theorem with its proof. For this, let $x := [x_1, x_2, \cdots, x_n]^T$ and $\mathbb{R}^{(+,n)} := \{(x_0, x) \in \mathbb{R}^{n+1} : x_0 > 0\}$. (However, we consider only when $n = 3$ in this paper.)

Theorem 1: Let d be a fraction whose numerator and denominator are odd integers satisfying

$$\frac{2^{n-1} - 1}{2^{n-1}} < d < 1. \tag{9}$$

Then, for the system (1), there exists a dynamic controller

$$\begin{aligned} \dot{x}_0 &= f_0(x_0, x) \\ u &= u(x_0, x) \end{aligned} \tag{10}$$

with any $x_0(0) > 0$, where f_0 and u are smooth functions in $\mathbb{R}^{(+,3)}$, and the controller (10) renders the origin of the closed-loop system globally finite-time stable (in the sense that the origin is stable and, for each $(x_0(0), x(0)) \in \mathbb{R}^{(+,3)}$, there exists $T > 0$ such that $\lim_{t\to T}(x_0(t), x(t)) = (0, 0)$). In addition, the right-hand sides of (10) (i.e., $f_0(x_0(t), x(t))$ and $u(x_0(t), x(t))$) are bounded while the solution reaches the origin.

In order to prove Theorem 1, we first present an algebraic construction of the smooth controller (10) based on the dynamic exponent scaling technique, which yields the inequality $\dot{V} \leq -kV^\alpha$ with some $k > 0$ and $0 < \alpha < 1$. As a second step, we then provide the proof that the right-hand sides of the controller are bounded throughout the control horizon, with the help of degree indicator.

A. Algebraic Construction of the Smooth Dynamic Controller

In this subsection, we construct the controller (10) with a Lyapunov function, which will show the global finite-time stability of the origin of the closed-loop system.

For the system (1), the augmented system can be designed as

$$\dot{x}_0 = -k_0 x_0^d + \frac{\sum_{i=1}^3 \gamma_i \bar{x}_i^2}{x_0^{2-d}} =: f_0(x_0, x), \tag{11}$$

where

$$\bar{x}_i = x_i - x_i^*,$$

in which, x_i^*'s $(i = 1, 2, 3)$ are virtual controls to be designed, and k_0 and γ_i's are tuning gains also to be determined. The reason to design x_0-dynamics such as (11) is to ensure that $x_0(t)$ never reaches zero before any system state does.

Step 1: We define $x_1^* \equiv 0$ and design the virtual control x_2^* in this step. Choosing the Lyapunov function $V_1 = (x_0^2 + \bar{x}_1^2)/2$ yields

$$\dot{V}_1 = -k_0 x_0^{1+d} + \frac{\sum_{i=1}^3 \gamma_i \bar{x}_i^2}{x_0^{1-d}} + \bar{x}_1(x_2 + f_1(x_1)) \pm k_1 \bar{x}_1^{1+d}$$

where k_1 is any positive number of the designer's choice.

Using the dynamic exponent scaling

$$|k_1 \bar{x}_1^{1+d}| \times \frac{x_0^r}{x_0^r} \leq \frac{x_0^{1+d}}{a} + \frac{\bar{x}_1^2 \sigma_1}{x_0^{1-d}}, \tag{12}$$

where a, b, and r are the same as in Section II, and $\sigma_1 = k_1^b/b$, we have

$$\begin{aligned} \dot{V}_1 \leq &-\left(k_0 - \frac{1}{a}\right) x_0^{1+d} - k_1 \bar{x}_1^{1+d} + \frac{\sum_{i=1}^3 \gamma_i \bar{x}_i^2}{x_0^{1-d}} \\ &+ \bar{x}_1(x_2 + f_1(x_1)) + \frac{\bar{x}_1^2 \sigma_1}{x_0^{1-d}}. \end{aligned}$$

Our strategy is that each term $\gamma_i \bar{x}_i^2$ in $\sum_{i=1}^3 \gamma_i \bar{x}_i^2$ will be removed in the step i. With this in mind, we pull $\frac{\gamma_1 \bar{x}_1^2}{x_0^{1-d}}$ out of $\frac{\sum_{i=2}^3 \gamma_i \bar{x}_i^2}{x_0^{1-d}}$ as

$$\begin{aligned} \dot{V}_1 \leq &-\left(k_0 - \frac{1}{a}\right) x_0^{1+d} - k_1 \bar{x}_1^{1+d} + \frac{\sum_{i=2}^3 \gamma_i \bar{x}_i^2}{x_0^{1-d}} \\ &+ \bar{x}_1(x_2 + f_1(x_1)) + \frac{\bar{x}_1^2(\sigma_1 + \gamma_1)}{x_0^{1-d}}. \end{aligned}$$

Now the virtual control x_2^* is constructed as

$$x_2^* = -f_1(x_1) - \frac{\bar{x}_1(\sigma_1 + \gamma_1)}{x_0^{1-d}}, \tag{13}$$

which yields that

$$\dot{V}_1 \leq -\left(k_0 - \frac{1}{a}\right) x_0^{1+d} - k_1 \bar{x}_1^{1+d} + \frac{\sum_{i=2}^3 \gamma_i \bar{x}_i^2}{x_0^{1-d}} + \bar{x}_1 \bar{x}_2.$$

Step 2: With the Lyapunov function $V_2 = V_1 + \bar{x}_2^2/2$, we have

$$\dot{V}_2 = \dot{V}_1 + \bar{x}_2(x_3 + f_2(\cdot)) - \bar{x}_2 \sum_{i=0}^1 \frac{\partial x_2^*}{\partial x_i} \dot{x}_i.$$

By adding and subtracting $k_2 \bar{x}_2^{1+d}$, with any $k_2 > 0$, and using the dynamic exponent scaling similar to (12), we obtain that

$$\begin{aligned} \dot{V}_2 \leq &-\left(k_0 - \frac{2}{a}\right) x_0^{1+d} - \sum_{i=1}^2 k_i \bar{x}_i^{1+d} + \frac{\sum_{i=2}^3 \gamma_i \bar{x}_i^2}{x_0^{1-d}} \\ &+ \bar{x}_1 \bar{x}_2 + \bar{x}_2(x_3 + f_2(\cdot)) - \bar{x}_2 \sum_{i=0}^1 \frac{\partial x_2^*}{\partial x_i} \dot{x}_i + \frac{\bar{x}_2^2 \sigma_2}{x_0^{1-d}}, \end{aligned} \tag{14}$$

where $\sigma_2 = k_2^b/b$. Noting that $\dot{x}_0$ contains $\sum_{i=1}^{3} \gamma_i \bar{x}_i^2 / x_0^{2-d}$, we design x_3^* only to cancel $\sum_{i=1}^{2} \gamma_i \bar{x}_i^2 / x_0^{2-d}$ leaving $\gamma_3 \bar{x}_3^2 / x_0^{2-d}$ to be handled in the next step. As the same manner, only the term $\gamma_2 \bar{x}_2^2 / x_0^{1-d}$ will be handled in the third term of (14). As a result, the virtual control x_i^* is kept as a function of variables $x_1, \ldots, x_{i-1}$, (or $\bar{x}_1, \ldots, \bar{x}_{i-1}$) only. Therefore,

$$\begin{aligned}
\dot{V}_2 \leq &- \left(k_0 - \frac{2}{a}\right) x_0^{1+d} - \sum_{i=1}^{2} k_i \bar{x}_i^{1+d} + \frac{\gamma_3 \bar{x}_3^2}{x_0^{1-d}} + \bar{x}_1 \bar{x}_2 \\
&+ \bar{x}_2 (x_3 + f_2(\cdot)) - \bar{x}_2 \frac{\partial x_2^*}{\partial x_0} \left(-k_0 x_0^d + \frac{\sum_{i=1}^{2} \gamma_i \bar{x}_i^2}{x_0^{2-d}} \right) \\
&- \bar{x}_2 \frac{\partial x_2^*}{\partial x_0} \frac{\gamma_3 \bar{x}_3^2}{x_0^{2-d}} - \bar{x}_2 \frac{\partial x_2^*}{\partial x_1} \dot{x}_1 + \frac{\bar{x}_2^2 (\sigma_2 + \gamma_2)}{x_0^{1-d}} \\
\leq &- \left(k_0 - \frac{2}{a}\right) x_0^{1+d} - \sum_{i=1}^{2} k_i \bar{x}_i^{1+d} + \frac{\gamma_3 \bar{x}_3^2}{x_0^{1-d}} + \bar{x}_2 \bar{x}_3 \\
&- \bar{x}_2 \frac{\partial x_2^*}{\partial x_0} \frac{\gamma_3 \bar{x}_3^2}{x_0^{2-d}}.
\end{aligned}$$

with the virtual control

$$\begin{aligned}
x_3^* = &-\bar{x}_1 - f_2(\cdot) + \frac{\partial x_2^*}{\partial x_0} \left(-k_0 x_0^d + \frac{\sum_{i=1}^{2} \gamma_i \bar{x}_i^2}{x_0^{2-d}} \right) \\
&+ \frac{\partial x_2^*}{\partial x_1} \dot{x}_1 - \frac{\bar{x}_2 (\sigma_2 + \gamma_2)}{x_0^{1-d}}.
\end{aligned} \tag{15}$$

Final Step: Let the Lyapunov function $V = V_2 + \bar{x}_3/2$. Then, we have

$$\dot{V} = \dot{V}_2 + \bar{x}_3 (u + f_3(\cdot)) - \bar{x}_3 \sum_{i=0}^{2} \frac{\partial x_3^*}{\partial x_i} \dot{x}_i + (k_3 \bar{x}_3^{1+d} - k_3 \bar{x}_3^{1+d}).$$

After the scaling for $k_3 \bar{x}_3^{1+d}$, we arrive at

$$\begin{aligned}
\dot{V} \leq &- \left(k_0 - \frac{3}{a}\right) x_0^{1+d} - \sum_{i=1}^{3} k_i \bar{x}_i^{1+d} + \frac{\gamma_3 \bar{x}_3^2}{x_0^{1-d}} + \bar{x}_2 \bar{x}_3 \\
&- \bar{x}_2 \frac{\partial x_2^*}{\partial x_0} \frac{\gamma_3 \bar{x}_3^2}{x_0^{2-d}} + \bar{x}_3 (u + f_3(\cdot)) - \bar{x}_3 \sum_{i=0}^{2} \frac{\partial x_3^*}{\partial x_i} \dot{x}_i \\
&+ \frac{\bar{x}_3^2 \sigma_3}{x_0^{1-d}}
\end{aligned}$$

where $\sigma_3 = k_3^b/b$. Therefore, the final control u would be

$$u = -\bar{x}_2 - f_3(\cdot) + \sum_{i=0}^{2} \frac{\partial x_3^*}{\partial x_i} \dot{x}_i + \frac{\partial x_2^*}{\partial x_0} \frac{\gamma_3 \bar{x}_2 \bar{x}_3}{x_0^{2-d}} - \frac{\bar{x}_3 (\gamma_3 + \sigma_3)}{x_0^{1-d}}, \tag{16}$$

so that

$$\dot{V} \leq - \left(k_0 - \frac{3}{a}\right) x_0^{1+d} - \sum_{i=1}^{3} k_i \bar{x}_i^{1+d}.$$

Now choose $k_0 > 3/a$.

Finally, it can be shown, similarly to Section II, that

$$\begin{aligned}
\dot{V} + kV^\alpha \leq &- \left(\min\{k_1, k_2, k_3, k_0 - 3/a\} - \frac{k}{2^\alpha} \right) \\
&\times \left(x_0^{1+d} + \sum_{i=1}^{3} \bar{x}_i^{1+d} \right) \leq 0.
\end{aligned} \tag{17}$$

with k and α such that $\alpha = (1+d)/2$ and $0 < k \leq 2^\alpha \min\{k_i, k_0 - 3/a\}$.

B. Boundedness of the Controller

Boundedness is proved in the $\bar{x}$ coordinate where $\bar{x} := [\bar{x}_1, \bar{x}_2, \bar{x}_3]^T$ since (x_0, x) on $\mathbb{R}^{(+,3)}$ is smoothly transformable into $(x_0, \bar{x})$ on $\mathbb{R}^{(+,3)}$.

Define $\mathcal{P} := \{(x_0, x) \in \mathbb{R}^4 : k_0 x_0^2 \geq \sum_{i=1}^{3} \gamma_i \bar{x}_i^2\}$, and $\mathcal{P}_R := \mathcal{P} \cap B_R$ with some $R > 0$. Now we define the degree indicator as

$$\begin{aligned}
D(g(x_0, \bar{x})) := \inf \beta \quad &\text{subject to} \\
\limsup_{x_0 \to 0^+} \bar{g}(x_0) x_0^\beta &< \infty
\end{aligned} \tag{18}$$

where

$$\bar{g}(x_0) := \max_{\sqrt{\sum_{i=1}^{3} \gamma_i \bar{x}_i^2 / k_0} \leq x_0} |g(x_0, \bar{x})|.$$

By denoting the collection of functions that are smooth on $\mathbb{R}^{(+,3)}$ (i.e., such as $g : \mathbb{R}^{(+,3)} \to \mathbb{R}$) by $\mathcal{G}$, it should be noted that the degree indicator is an operator well-defined on $\mathcal{G}$. In addition, note that the degree indicator measures the degree in the $\bar{x}$-coordinates. Therefore, if we write $D(g(x_0, x))$, it is interpreted as $D\left(g(x_0, x)|_{x=\phi(\bar{x})}\right)$ where ϕ is the diffeomorphism (i.e., coordinate change) between x and $\bar{x}$, which is obtained in the previous subsection. If $D(g(x_0, \bar{x})) \leq 0$, the function $g(\cdot)$ is bounded on $\mathcal{P}_R$. Moreover, the following properties are helpful in the developments to come.

(1) For g, g_1, and g_2 in $\mathcal{G}$ such that $g(x_0, \bar{x}) = g_1(x_0, \bar{x}) + g_2(x_0, \bar{x})$,

$$D(g(\cdot)) \leq \max\{D(g_1(\cdot)), D(g_2(\cdot))\}. \tag{19}$$

(2) For g, g_1, and g_2 in $\mathcal{G}$ such that $g(x_0, \bar{x}) = g_1(x_0, \bar{x}) g_2(x_0, \bar{x})$,

$$D(g(\cdot)) \leq D(g_1(\cdot)) + D(g_2(\cdot)). \tag{20}$$

They can be proved by the fact that $\bar{g}(\cdot) \leq \bar{g}_1(\cdot) + \bar{g}_2(\cdot)$ and $\bar{g}(\cdot) \leq \bar{g}_1(\cdot) \bar{g}_2(\cdot)$. Some examples of the degree indicator are presented here:

- Let $g = 0$ and $g_1 = g_2 = 1$. Then, $D(g) = -\infty$ and $D(g_1) = D(g_2) = 0$, which satisfies the item (1).
- Let $g(x_0, \bar{x}_1, \bar{x}_2) = 1 + \frac{\bar{x}_1}{x_0} + \frac{\bar{x}_1 \bar{x}_2}{x_0^3} + x_0^4$. Then,

$$\begin{aligned}
D(g(\cdot)) &\leq \max\left\{ D(1), D\left(\frac{\bar{x}_1}{x_0}\right), D\left(\frac{\bar{x}_1 \bar{x}_2}{x_0^3}\right), D(x_0^4) \right\} \\
&= \max\{0, 0, 1, -4\} = 1.
\end{aligned}$$

- Let $g(x_0, \bar{x}_1) = \frac{\bar{x}_1^2}{x_0^{1-d}} = \bar{x}_1^2 \times \frac{1}{x_0^{1-d}} =: g_1(\bar{x}_1) \times g_2(x_0)$. Then $D(g(\cdot)) = -1 - d$, $D(g_1(x)) = -2$, and $D(g_2(x_0)) = 1 - d$, with which the item (2) holds.

It is also worthwhile to observe the following.

(3) Suppose that $D(g_i(x_0,\bar{x})) \le -c_i$ with $c_i \ge 0$ for $g_i \in \mathcal{G}$, $i = 1,\cdots,k$ and a smooth function $f : \mathbb{R}^k \to \mathbb{R}$. Let $f_c = f(0,\cdots,0)$. Then it holds that

$$D(f(g_1(x_0,\bar{x}),\cdots,g_k(x_0,\bar{x}))) \begin{cases} \le -\min\{c_1,\cdots,c_k\} & \text{if } f_c = 0 \\ = 0 & \text{if } f_c \ne 0. \end{cases} \quad (21)$$

In fact, $D(g_i(x_0,\bar{x})) \le 0$ implies that, for any $R > 0$, g_i is uniformly bounded on $\mathcal{P}_R$ (i.e., there exists $K \ge 0$ such that $|g_i(x_0,\bar{x})| \le K$ for all $(x_0,\bar{x}) \in \mathcal{P}_R$). Let $f_s(\cdot) = f(\cdot) - f_c$. Then, from the smoothness of $f_s(\cdot)$ and the fact that $f_s(0,\cdots,0) = 0$, it follows that

$$f_s(g_1(x_0,\bar{x}),\cdots,g_k(x_0,\bar{x})) = F_s(g_1(x_0,\bar{x}),\cdots,g_k(x_0,\bar{x})) \begin{bmatrix} g_1(x_0,\bar{x}) \\ \vdots \\ g_k(x_0,\bar{x}) \end{bmatrix},$$

where $F_s : \mathbb{R}^k \to \mathbb{R}^{1\times k}$ is a smooth function [13]. Therefore, $D(f_s(g_1(x_0,\bar{x}),\cdots,g_k(x_0,\bar{x}))) = \inf \beta$ subject to

$$\limsup_{x_0\to 0^+} \max_{\sqrt{\sum_{i=1}^3 \gamma_i \bar{x}_i^2/k_0} \le x_0} \left\| F_s(g_1(x_0,\bar{x}),\cdots,g_k(x_0,\bar{x})) \begin{bmatrix} g_1(x_0,\bar{x}) \\ \vdots \\ g_k(x_0,\bar{x}) \end{bmatrix} \right\| x_0^\beta < \infty.$$

The left-hand side of the above inequality is less than or equal to

$$\limsup_{x_0\to 0^+} \max_{\sqrt{\sum_{i=1}^3 \gamma_i \bar{x}_i^2/k_0} \le x_0} \|F_s(g_1(x_0,\bar{x}),\cdots,g_k(x_0,\bar{x}))\| \times \limsup_{x_0\to 0^+} \max_{\sqrt{\sum_{i=1}^3 \gamma_i \bar{x}_i^2/k_0} \le x_0} \left\| \begin{bmatrix} |g_1(x_0,\bar{x})| x_0^\beta \\ \vdots \\ |g_k(x_0,\bar{x})| x_0^\beta \end{bmatrix} \right\|.$$

Since $F_s(g_1(x_0,\bar{x}),\cdots,g_k(x_0,\bar{x}))$ is uniformly bounded on $\mathcal{P}_R$ (by the fact that $D(g_i) \le -c_i \le 0$),

$$D(f_s(\cdot)) \le -\min\{c_1,\ldots,c_k\}.$$

Finally, since $D(f_c) = 0$ if $f_c \ne 0$ and $D(0) = -\infty$, the claim easily follows by (19).

From now on, we are going to prove that $D(f_0(x_0,x)) \le 0$ and $D(u(x_0,x)) \le 0$ where f_0 and u are given in (11) and (16), respectively. The former is obvious from (11) since a direct evaluation of $D(f_0(x_0,x))$ leads to the conclusion. To show that $D(u(x_0,x)) \le 0$, we first show that $D(x_i^*) \le 0$ for $i = 2,3$ step by step under the condition of d in (9).

Let us consider x_2^* in (13), which is composed of two terms: the drift term $f_1(\cdot)$ and $-\frac{\bar{x}_1(\sigma_1+\gamma_1)}{x_0^{1-d}}$. The drift term $f_1(\cdot)$ may be 0 or a function of x_1 and the partial derivatives of $f_1(\cdot)$ may satisfy $\frac{\partial^{n_1} f_1}{\partial x_1^{n_1}}(0) = 0$ or $\frac{\partial^{n_1} f_1}{\partial x_1^{n_1}}(0) \ne 0$.

But, since (21) applies to all cases, it is induced that

$$D(f_1(\cdot)) \le -1,\ D\left(\frac{\partial^{n_1} f_1}{\partial x_1^{n_1}}\right) \le 0,\ n_1 \ge 1. \quad (22)$$

The result (22) will be used to compute $D(x_i^*)$, $i \ge 3$ in the following procedure. The fact $D\left(\frac{\partial^{n_1+\cdots+n_i} f_i(\cdot)}{\partial x_1^{n_1}\cdots\partial x_i^{n_i}}\right) \le 0, i \ge 1$, which will be proved, is helpful in estimating $D(x_i^*), i \ge 3$.

Therefore, with (22) and $D\left(-\frac{\bar{x}_1(\sigma_1+\gamma_1)}{x_0^{1-d}}\right) = -d$, we obtain

$$D(x_2^*) = -d,\ D(\dot{x}_1) = D(\bar{x}_2 + x_2^* + f_1(\cdot)) = -d. \quad (23)$$

Next we will get the condition of x_3^* to be bounded. The feature of x_3^* is that it includes the partial derivatives of x_2^*, which may cause $D(x_3^*)$ to be positive. We start with the drift term $f_2(\cdot)$. Since the drift term $f_2(\cdot)$ have $x_i, i = 1,2$ as variables, substituting $\bar{x}_1 = x_1$ and $\bar{x}_2 = x_2 - x_2^*$ into $f_2(x_1,x_2)$ yields $f_2(\bar{x}_1,\bar{x}_2 + x_2^*)$ and, by (13) and (21), we can easily prove that

$$D(f_2(\cdot)) \le -d,\ D\left(\frac{\partial^{n_1+n_2} f_2(\cdot)}{\partial x_1^{n_1}\partial x_2^{n_2}}\right) \le 0,\ n_1+n_2 \ge 1.$$

The third and forth terms of x_3^* include partial derivatives of x_2^* and we have to handle those carefully for reason of $D\left(\frac{\partial x_2^*}{\partial x_i}\right) > 0, i = 0,1$. After simple calculations, the results are summarized as

$$D\left(\frac{\partial^{n_0+n_1} x_2^*}{\partial x_0^{n_0}\partial x_1^{n_1}}\right) \begin{cases} = (n_0+n_1-d), & n_0 \ge 0,\ 0 \le n_1 \le 1 \\ \le 0 & n_0 \ge 0,\ n_1 \ge 2. \end{cases} \quad (24)$$

From (20), (23), (24), and $D\left(-k_0 x_0^d + \frac{\sum_{i=1}^2 \gamma_i \bar{x}_i^2}{x_0^{2-d}}\right) \le -d$, it is clear that

$$D\left(\frac{\partial x_2^*}{\partial x_0}\left(-k_0 x_0^d + \frac{\sum_{i=1}^2 \gamma_i \bar{x}_i^2}{x_0^{2-d}}\right)\right) \le 1-2d$$
$$D\left(\frac{\partial x_2^*}{\partial x_1}\dot{x}_1\right) = 1-2d.$$

Therefore, by $D(\bar{x}_1) = -1$, $D\left(\frac{\bar{x}_2(\gamma_2+\sigma_2)}{x_0^{1-d}}\right) = -d$, and (19), it is obtained that

$$D(x_3^*) = 1-2d,\ D(\dot{x}_2) = 1-2d. \quad (25)$$

In order to guarantee that x_3^* is bounded, we select the value of d such that $\frac{1}{2} \le d < 1$ and suppose this fact forward.

Finally, we compute the value of the degree indicator of the controller (16) and show that the controller is bounded under (9). After transforming the drift term $f_3(x_1,x_2,x_3)$ into $f_3(\bar{x_1},\bar{x}_2+x_2^*,\bar{x}_3+x_3^*)$, applying (23) and (25) under $\frac{1}{2} \le d < 1$ into (21) results in

$$D(f_3(\cdot)) \le 1-2d,\ D\left(\frac{\partial^{n_1+n_2+n_3} f_3(\cdot)}{\partial x_1^{n_1}\partial x_2^{n_2}\partial x_3^{n_3}}\right) \le 0,\ \sum_{i=1}^3 n_i \ge 1.$$

To estimate the third term of the controller (16), which includes partial derivatives of x_3^* with respect to $x_i, i = 0,1,2$, we need to simplify the process and define

$$\Pi_3(x_1,x_2) =: -k_0 x_0^d + \frac{\sum_{i=1}^2 \gamma_i \bar{x}_i^2}{x_0^{2-d}}$$

and the degree indicators of the partial derivatives of $\Pi_3(\cdot)$ yield

$$D\left(\frac{\partial \Pi_3(\cdot)}{\partial x_i}\right) = \begin{cases} 1-d & \text{if } \ i=2, \\ 2(1-d) & \text{if } \ i=0,1, \end{cases} \tag{26}$$

which is based on (19), (20), and (24).

In the course of evaluating $D\left(\frac{\partial x_3^*}{\partial x_i}\right)$, $i = 0,1,2$, $\frac{\partial \Pi_3(\cdot)}{\partial x_i}$, $i=0,1,2$ are essential. To give a detailed explanation, we present the partial derivatives of x_3^* as

$$\begin{aligned}
\frac{\partial x_3^*}{\partial x_2} =& -\frac{\partial f_2(\cdot)}{\partial x_2} - \frac{(\gamma_2+\sigma_2)}{x_0^{1-d}} + \frac{\partial x_2^*}{\partial x_0}\frac{\partial \Pi_3}{\partial x_2} + \frac{\partial x_2^*}{\partial x_1}, \\
\frac{\partial x_3^*}{\partial x_1} =& -1 - \frac{\partial f_2(\cdot)}{\partial x_1} + \frac{\partial x_2^*}{\partial x_1}\frac{(\gamma_2+\sigma_2)}{x_0^{1-d}} + \frac{\partial^2 x_2^*}{\partial x_0 \partial x_1}\Pi_3 \\
&+ \frac{\partial x_2^*}{\partial x_0}\frac{\partial \Pi_3}{\partial x_1} + \frac{\partial x_2^*}{\partial x_1}\frac{\partial f_1}{\partial x_1} \\
\frac{\partial x_3^*}{\partial x_0} =& \frac{(\gamma_2+\sigma_2)}{x_0^{1-d}}\left(\frac{\partial x_2^*}{\partial x_0} - \frac{(d-1)\bar{x}_2}{x_0}\right) + \frac{\partial^2 x_2^*}{\partial x_0^2}\Pi_3 \\
&+ \frac{\partial x_2^*}{\partial x_0}\frac{\partial \Pi_3}{\partial x_0} + \frac{\partial^2 x_2^*}{\partial x_0 \partial x_1}\dot{x}_1,
\end{aligned}$$

By (20), (24), and (26), the terms $\frac{\partial x_2^*}{\partial x_0}\frac{\partial \Pi_3}{\partial x_i}, i=0,1,2$ have

$$D\left(\frac{\partial x_2^*}{\partial x_0}\frac{\partial \Pi_3}{\partial x_i}\right) = \begin{cases} 2(1-d) & \text{if } \ i=2 \\ 3(1-d) & \text{if } \ i=0,1. \end{cases} \tag{27}$$

The result shows that each $D\left(\frac{\partial \Pi_3}{\partial x_i}\right)$ makes $D\left(\frac{\partial x_2^*}{\partial x_0}\frac{\partial \Pi_3}{\partial x_i}\right)$ have the largest value of the terms of $\frac{\partial x_3^*}{\partial x_i}$ and decides the value of $D\left(\frac{\partial x_3^*}{\partial x_i}\right)$, which explains the reason why $\frac{\partial \Pi_3(\cdot)}{\partial x_i}, i = 0,1,2$ are important.

Hence, using (23), (24), (25), (27), and the properties (19), (20), and (21), we arrive at

$$\begin{aligned}
&D\left(-\bar{x}_2 - f_3(\cdot) - \frac{\bar{x}_2(\gamma_2+\sigma_2)}{x_0^{1-d}} + \frac{\partial x_2^*}{\partial x_0}\frac{\bar{x}_2\bar{x}_3}{x_0^{2-d}}\right) \leq 0 \\
&D\left(\frac{\partial x_3^*}{\partial x_2}\dot{x}_2\right) = 2(1-d) + (1-2d) = 3-4d \\
&D\left(\frac{\partial x_3^*}{\partial x_i}\dot{x}_i\right) = 3(1-d) - d = 3-4d, \ i=0,1
\end{aligned}$$

and

$$D(u) = 3-4d.$$

With the help of the approach above, we conclude that, if we choose d satisfying $\frac{3}{4} \leq d < 1$, the controller (16) is bounded.

Remark 1: For $n \geq 2$ system, it is very important to evaluate the partial derivatives of virtual controls since considering every term of them is a tedious and complicated work. The degree indicator (18) proposed in this paper is a very useful and efficient tool since it only notices the term of which the value of the degree indicator is positive, i.e., it may be unbounded without the condition (9) on d.

C. Proof of Theorem 1

Smoothness of the closed-loop system (1) and (10) in $\mathbb{R}^{(+,3)}$ guarantees existence and uniqueness of the solution $(x_0(t), x(t))$ as long as $x_0(t) > 0$. Hence, while $x_0(t) > 0$, the inequality (17) holds which ensures stability of the origin and the finite-time convergence of the solution into the origin. It can be shown similarly to Section II that the solution $(x_0(t), x(t))$ becomes $(0,0)$ at the same time, and before that time, $x_0(t) > 0$.

Now, by the definition of the set $\mathcal{P}$, it can be seen (as in Section II) that any solution enters $\mathcal{P}_R$ with any $R > 0$ in a finite-time. Since we have shown that the functions $f_0(x_0,x)$ and $u(x_0,x)$ of (10) are bounded on $\mathcal{P}_R$ (in the previous subsection), it is concluded that both $f_0(x_0(t), x(t))$ and $u(x_0(t), x(t))$ are bounded from the initial time to the time when the solution gets to the origin.

IV. Conclusion

This paper has proposed a smooth dynamic controller for a class of triangular nonlinear systems. Boundedness of the controller has been proved with the help of a new tool 'degree indicator,' which turned out very useful to evaluate degrees of singular terms. Although the presentation in this paper is limited to the 3rd-order system, it is extensible to general nth-order systems. Our future works include some extension to uncertain systems.

References

[1] M. Athans and P. I. Falb, *Optima Control: An Introduction to the Theory and Its Applications*, New York: McGraw-Hill, 1966.
[2] J. Back, S.G. Cheong, H. Shim, and J.H. Seo, "Nonsmooth feedback stabilizer for strict-feedback nonlinear systems that may not be linearizable at the origin," *Syst. and Contr. Letters*, vol. 56, pp. 742-752, 2007.
[3] S.P. Bhat and D.S. Berstein, "Finite-time stability of homogeneous systems," *Proc. of American Control Conf.*, pp. 2513-2514, 1997.
[4] S.P. Bhat and D.S. Berstein, "Continuous finite-time stabilization of the translational rotational double integrators," *IEEE Trans. Automat. Contr.*, vol. 43, no. 5, pp. 678-682, 1998.
[5] S.P. Bhat and D.S. Berstein, "Finite-time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751-766, 2000.
[6] R.A. Freeman and P.V. Kokotovic, *Robust Nonlinear Control Design*, Birkhauser, 1996.
[7] V.T. Haimo, "Finite-time controllers," *SIAM J. Control Optim.*, vol. 24, no. 4, pp. 760-770, 1986.
[8] Y. Hong, J. Huang, and Y. Wu, "On an output feedback finite-time stabilization problem," *IEEE Trans. Automat. Contr.*, vol. 46, no. 2, pp. 305-309, 2001.
[9] Y. Hong, "Finite-time stabilization and stability of a class of controllable systems," *Syst. and Contr. Letters*, vol. 46, pp. 231-236, 2002.
[10] Y. Hong, G. Yang, L. Bushnell, and H.O. Wang "Global finite-time stabilization: from state feedback to output feedback," *Proc. 39th IEEE Conf. on Decision and Control*, pp. 2908-2913, 2000.
[11] X. Huang, W. Lin, and B. Yang, "Global finite-time stabilization of a class of uncertain nonlinear systems," *Automatica*, vol. 41, pp. 881-888, 2005.
[12] G. Kamenkov, "On stability of motion over a finite interval of time," *J. of Applied Math. and Mechanics*, vol. 17, pp. 529-540, 1953.
[13] H.K. Khalil, *Nonlinear systems*, NJ:Prentice-Hall, 3rd edition, 2002.
[14] J. Li and C. Qian, "Global finite-time stabilization of a class of nonsmooth systems by output feedback," *Proc. of American Contr. Conf.*, pp. 4716-4721, 2005.
[15] J. Li and C. Qian, "Global finite-time stabilization by dynamic output feedback for a class of continuous nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 51, no. 5, pp. 879-884, 2006.
[16] C. Qian and J. Li, "Global finite-time stabilization by output feedback for planar systems without observable linearization," *IEEE Trans. Automat. Contr.*, vol. 50, no. 6, pp. 885-890, 2005.
[17] C. Qian and J. Li, "Global output stabilization of upper-triangular nonlinear systems using a homogeneous domination approach," *Int. J. Robust Noninear Control*, vol. 16, pp. 441-463, 2006.