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# Optimal Control of a Voice–Coil–Motor with Coulombic Friction

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***Abstract*— The voice-coil-motor is a widely used mechatronic device, which represents a typical electrodynamic actuator for machine tool axes, bonding machines and hydraulic/pneumatic valve drives. One principal task consists in steering the system precisely to a prescribed target in minimal time or with minimal energy. To achieve this goal, we formulate an optimal control problem using a dynamical system derived in Zirn [19]. Since Coulombic friction is modelled by a jump function depending on the sign of the velocity, the optimal control problem belongs the class of *nonsmooth* optimization problems. We show that time-optimal controls are bang-bang for all physically reasonable control bounds. Switching times are directly optimized by nonlinear programming methods, which also allow to compute parametric sensitivity derivatives. Energy-optimal solutions are presented for several fixed final times.**

## I. Introduction

We study the servo drive shown in Figure 1, which is a typical electrodynamic actuator for small high dynamic machine tool axes as well as wire bonding machines and hydraulic/pneumatic valve drives. As this type of actuator is very common for loudspeakers it is called voice–coil–motor. A dynamical model for the voice-coil-motor was proposed by Zirn [19] who validated the model on the testbench displayed in Figure 1 and developed automatic control techniques for steering the system to a prescribed target. However, these techniques were deficient with regard to the accuracy in reaching the desired final position and the process duration. Moreover, one could observe overshooting in the positions and velocities. The goal of this paper is to improve on these deficiences by applying optimal control methods.

The optimal control model is introduced in section II. Basically, the dynamical model is linear in the state and control variables. However, a challenging nonlinearity in the dynamic system arises by modelling static Coulombic friction via a jump function depending on the sign of the velocity. This leads to a *nonsmooth* optimal control problem. In section III, we apply discretization and optimization techniques to compute time–optimal controls for a range of control bounds. It is shown that time-optimal controls are bang-bang with the number of switching times matching the number of terminal conditions. Necessary and sufficient conditions are discussed on the basis of optimal multiprocess control problems [6], [7], [2]. Section IV presents some results on sensitivity derivatives of switching times under parameter variations. In Section V, we demonstrate the excellent agreement between the computed (predicted) optimal control solutions and the experimental results using 1000 control signals. Energy-optimal control solutions are briefly discussed in Section VI.

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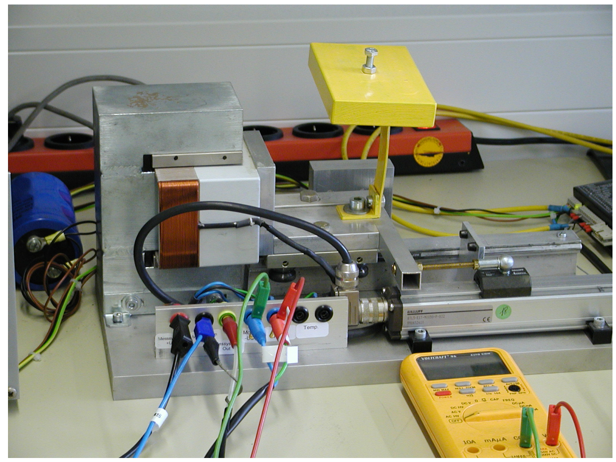

Fig. 1. Voice-coil-motor with real-time-system-control and flexible load. Test bench in the mechatronics laboratory, Gießen University of Applied Sciences.

## II. Optimal control model of the voice–coil motor

Though the servo drive system shown in Figure 1 is a rather simple drive system, it incorporates all main characteristics of servo drives with feedback controlled motors in combination with flexible transmission devices and machine structures. The stator of the voice–coil–motor is an iron core with rare earth permanent magnetic excitation; cf. Figure 2. A copper coil is guided in the air gap on a slider; the coil and slider mass is denoted by $m_1$. The linear guide produces the Coulombic friction force $F_R$, which acts in the direction opposite to the velocity. A load mass $m_2$ is mounted on the slider with a spring $k$ that has negligible damping. A coil current $I$ induces the actuating force $F$ (so called Lorenz force) given by the equation $F = K_F \cdot I$. The moving coil with the velocity $v_1$ generates a voltage $U$ (also called back-EMF) according to $U = K_s \cdot v_1$. The system parameters are given in Table I.

The dynamic process for the voice–coil–motor is studied in the time interval $t \in [0, t_f]$ with $t$ measured in seconds; the final time $t_f > 0$ is either fixed or free. The state variables are the motor mass position $x_1(t)$, the motor mass velocity $v_1(t)$,

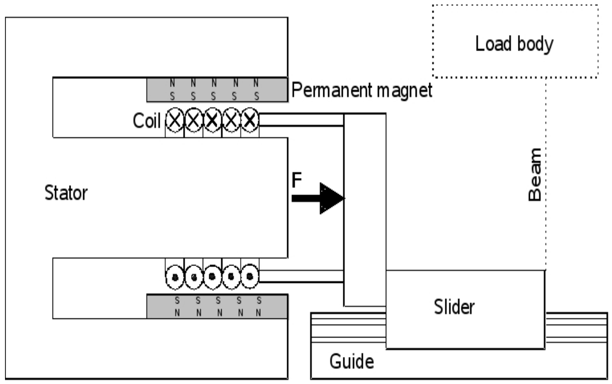


Fig. 2. Physical model of the voive-coil-motor

| | | | |
|---|---|---|---|
| DSPACE sampling time | $Ts$ | $=$ | 0.1 ms |
| Amplifier switching frequency | $fPWM$ | $=$ | 50 kHz |
| Amplifier intermediate voltage | $U_{\max}$ | $\leq$ | 10 V |
| Coil resistance | $R$ | $=$ | 2 $\Omega$ |
| Coil inductivity | $L$ | $=$ | 2 mH |
| Force constant | $K_F$ | $=$ | 12 N/A |
| Motor mass (slider, guide, coil) | $m_1$ | $=$ | 1.03 kg |
| Load mass | $m_2$ | $=$ | 0.56 kg |
| Spring constant | $k$ | $=$ | 2.4 kN/m |
| Guide friction force | $F_R$ | $=$ | 2.1 N |
| Linear encoder resolution | $\Delta x$ | $=$ | 1 $\mu$m |

TABLE I
PHYSICAL PARAMETERS

the load mass position $x_2(t)$, the load mass velocity $v_2(t)$ and the electric current $I(t)$. The input variable (control) of the motor is the voltage $U(t)$. The dynamic equations are given by the following linear differential system, where as usual the dot denotes the time derivative.

$$\dot{x}_1 = v_1\,, \tag{1}$$

$$\dot{v}_1 = \frac{1}{m_1}\left[K_F \cdot I - K \cdot (x_1 - x_2) - F_R \cdot sign(v_1)\right], \tag{2}$$

$$\dot{x}_2 = v_2\,, \tag{3}$$

$$\dot{v}_2 = \frac{K}{m_2} \cdot (x_1 - x_2)\,, \tag{4}$$

$$\dot{I} = \frac{1}{L}\left[U - R \cdot I - K_s \cdot v_1\right]. \tag{5}$$

The Coulombic friction force is modelled by the expression $-F_R \cdot sign(v_1)$ in equation (2). Here, the sign function is defined by

$$sign(v_1) = \left\{ \begin{array}{rl} 1, & \text{if} \quad v_1 > 0 \\ 0, & \text{if} \quad v_1 = 0 \\ -1, & \text{if} \quad v_1 < 0 \end{array} \right\}.$$

Note that the Coulombic friction force $-F_R \cdot sign(v_1)$ induces a state-dependent jump in (2) and thus leads to an ODE system with a non–differentiable right hand side. Therefore, the optimal control problem formulated below falls into the class of *nonsmooth* optimization problems.

The ODE (2) is slightly inexact and simplifies the real behaviour of the motor. It does not reflect accurately the static friction for the case $v_1 = 0$. To actuate the slider from a position in rest, the absolute value of the accelerating force

$$F_a = K_F \cdot I - K \cdot (x_1 - x_2)$$

has to exceed the static Coulombic friction force $F_R$. This deficiency can be removed by adding the term

$$\min\left\{-F_a, -F_R \cdot \frac{F_a}{|F_a|}\right\}, \quad \text{when} \quad v_1 = 0, \tag{6}$$

in the bracket on the right hand side of equation (2). To simplify the analysis we ignore this term in the following.

The control constraint is given by

$$-U_{\max} \leq U(t) \leq U_{\max}\,, \quad 0 \leq t \leq t_f, \tag{7}$$

where $U_{\max} \leq 10\,V$ for mechanical reasons. For the state vector $x = (x_1, v_1, x_2, v_2, I)^* \in I\!R^5$, the initial and terminal boundary conditions are chosen as

$$x(0) = (0,0,0,0,0)^*, \quad x(t_f) = (0.01, 0, 0.01, 0, 0)^*, \tag{8}$$

where positions are measured in meters. The system (1)-(5) can be written as

$$\dot{x} = f(x, U) = Ax + BU + C \cdot sign(v_1), \tag{9}$$

with the $5 \times 5$–matrix $A$ and vectors $B$, $C \in I\!R^5$ defined by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{k}{m_1} & 0 & \frac{k}{m_1} & 0 & \frac{K_F}{m_1} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & 0 & 0 \\ 0 & -\frac{K_S}{L} & 0 & 0 & -\frac{R}{L} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{L} \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ -F_R \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We consider two cost functionals: either the time-optimal case,

$$\text{minimize the final time} \quad t_f\,, \tag{10}$$

or a criterion with a quadratic penalty on the control variable corresponding to the "energy–optimal" case,

$$\text{minimize} \quad \int_0^{t_f} U(t)^2 dt \quad \text{for fixed} \quad t_f > 0\,. \tag{11}$$

Of course, the fixed final time $t_f$ in (11) must be larger than the minimal time in (10). To avoid large oscillations in the mechatronic system, it is desiderable to impose state constraints of the form

$$-c_v \leq v_1(t) - v_2(t) \leq c_v\,, \quad c_v > 0, \tag{12}$$

$$-c_x \leq x_1(t) - x_2(t) \leq c_x\,, \quad c_x > 0, \tag{13}$$

with appropriate constants $c_v$, $c_x$. A detailed study of optimal solutions under such state constraints will be carried out in a future paper. For large final times $t_f$, computations of energy–optimal solutions show that the state constraints (12) and (13) are satisfied with bounds $c_v, c_x$ relevant in practice.

## III. TIME–OPTIMAL CONTROL

In the time-optimal case, computations show that optimal solutions are concatenations of finitely many bang-bang arcs with atmost one subarc with negative velocity $v_1(t) < 0$. This structure allows us to apply the necessary optimality conditions for *multiprocess optimal control problem*; cf. Clarke, Vinter [6], [7], Augustin, Maurer [2]. The minimum principle involves the adjoint variable (row vector) $\lambda = (\lambda_{x_1}, \lambda_{v_1}, \lambda_{x_2}, \lambda_{v_2}, \lambda_I)$, which sastifies the adjoint equation $\dot{\lambda} = -\lambda A$ :

$$\begin{array}{rclrcl} \dot{\lambda}_{x_1} & = & \frac{k}{m_1}\lambda_{v_1} - \frac{k}{m_2}\lambda_{v_2}, & \dot{\lambda}_{v_1} & = & -\lambda_{x_1} + \frac{K_s}{L}\lambda_I, \\ \dot{\lambda}_{x_2} & = & -\frac{k}{m_1}\lambda_{v_1} + \frac{k}{m_2}\lambda_{v_2}, & \dot{\lambda}_{v_2} & = & -\lambda_{x_2}, \\ \dot{\lambda}_I & = & -\frac{K_F}{m_1}\lambda_{v_1} + \frac{R}{L}\lambda_I. & & & \end{array} \tag{14}$$

No boundary conditions are prescribed for $\lambda \in I\!R^5$, since the intial and terminal conditions (8) are specified. The optimal control $U(t)$ minimizes the Hamiltonian function

$$H(x(t), \lambda(t), U) = 1 + \lambda(t)(Ax(t) + B \cdot U + C\,\mathrm{sign}\,(v_1(t))),$$

which gives the control law

$$U(t) = -\mathrm{sign}\,(\lambda_I(t))\,U_{\max}\,. \tag{15}$$

The linear system (9) is completely controllable, since the $5 \times 5$ Kalmann matrix

$$C = (B, AB, A^2B, A^3B, A^4B)$$

has maximal rank 5. Hence, the time-optimal control $U(t)$ is of bang–bang type.

To solve the optimal control problem, we first discretize the problem using Euler's method or Heun's second order integration method. The resulting large-scale optimization problem is formulated using the modeling language AMPL (Fourer et al. [8], [9]) and is solved by either the optimization code IPOPT (Wächter [16]) or LOQO (Vanderbei [17], [18]). Using $N = 20000$ grid points, our computations show that for all values of $U_{\max} > 0$ the control has the following structure with 4 switching times $0 < t_1 < t_2 < t_3 < t_4 < t_f$ and the free final time $t_5 := t_f$,

$$U(t) = \left\{ \begin{array}{lll} U_{\max} & \text{for} & 0 \le t < t_1 \\ -U_{\max} & \text{for} & t_1 < t < t_2 \\ U_{\max} & \text{for} & t_2 < t < t_3 \\ -U_{\max} & \text{for} & t_3 < t < t_4 \\ U_{\max} & \text{for} & t_4 < t \le t_5 \end{array} \right\} \tag{16}$$

This control structure is not surprising, since one intuitively expects that five degrees of freedom, namely the five variables $t_1, t_2, t_3, t_4, t_f$, suffice to satisfy the five terminal conditions in (8). This discretization and optimization approach provides switching times that are correct up to 3-4 decimals. After determining the correct control structure, we apply a refined numerical method for computing the switching times with high precision. Due to the structure (16), the bang-bang control problem is equivalent to an optimization problem, where the switching times $t_i, i = 1,2,3,4$, and the free final time $t_f$ figure as optimization variables; cf. Agrachev et al. [1], Osmolovskii, Maurer [15]. Instead of optimizing the switching times directly, we use the arc-parametrization method in Maurer, Büskens, Kim, Kaya [12]) to optimize the *arclengths* of the bang–bang arcs defined by

$$\xi_j = t_j - t_{j-1}\,,\ (j = 1,2,3,4,5), \quad t_0 := 0, \quad t_5 := t_f\,.$$

This method can be implemented using the Fortran code NUDOCCCS developed by Büskens [3].

The sign distribution of the motor mass velocity $v_1(t)$ in $[0, t_f]$ depends crucially on the value $U_{\max}$ of the control bound. We can summarize our numerical results as follows. There exist two limiting control bounds

$$U_{\max}^1 := 1.85476, \quad U_{\max}^2 = 2.38327$$

with the following property: for all bounds $U_{\max}$ with

$$U_{\max}^1 := 1.85476 < U_{\max} < 2.38327 =: U_{\max}^2 \tag{17}$$

we have $v_1(t) > 0$ for all $0 < t < t_f$, while for

$$\begin{array}{l} U_{\max} < 1.85476 := U_{\max}^1 \quad \text{or} \\ U_{\max}^2 := 2.38327 < U_{\max} \end{array} \tag{18}$$

the velocity $v_1(t)$ has the sign distribution

$$v_1(t) = \left\{ \begin{array}{lll} > 0 & \text{for} & 0 < t < t_1^v \\ < 0 & \text{for} & t_1^v < t < t_2^v \\ > 0 & \text{for} & t_2^v < t < t_f \end{array} \right\}. \tag{19}$$

The intermediate times $t_1^v, t_2^v$ satisfy

$$t_1 < t_1^v < t_2 < t_2^v < t_3\,.$$

For bounds $U_{\max}$ given in (18) we thus encounter a *multiprocess* control problem with two different dynamical systems defined by the friction force $F_R$ or $-F_R$ in equation (2). The velocity $v_1(t)$ is zero at the points $t_1^v$ and $t_2^v$, which gives two additional interior conditions

$$v_1(t_1^v) = 0, \quad v_1(t_2^v) = 0. \tag{20}$$

Applying the necessary conditions in [7], [2], we find that the adjoint variable $\lambda_{v_1}$ may have jumps according to

$$\lambda_{v_1}((t_k^v)^+) = \lambda_{v_1}((t_k^v)^-) + \rho_k, \qquad k = 1,2, \tag{21}$$

where $\rho_k,\ k = 1,2$, are multipliers obtained from the transversality conditions. Since the Hamiltonian $H$ is *continuous* at $t_k^v,\ k = 1,2,$, we have

$$\begin{array}{rcl} 0 & = & H((t_k^v)^+) - H((t_k^v)^-) \\ & = & [\lambda_{v_2}((t_k^v)^+) - \lambda_{v_2}((t_k^v)^-)] \cdot F_R = \rho_k \cdot F_R. \end{array}$$

This implies $\rho_k = 0$ for $k = 1,2$. Hence, the adjoint variable $\lambda_{v_1}(t)$ is continuous at $t_k^v$ for $k = 1,2$.

Let us select the control bounds $\mathbf{U}_{\max} = \mathbf{2}$ and $\mathbf{U}_{\max} = \mathbf{3}$ to illustrate the different control strategies described in (17) and (18). Fig. 3 displays the optimal state and control variables for $\mathbf{U}_{\max} = \mathbf{2}$. Recall from (17) that $v_1(t)$ remains positive for $0 < t < t_f$. The switching times and final time are computed as

$$\begin{array}{l} t_1 = 0.074140, \quad t_2 = 0.0820268, \quad t_3 = 0.101444, \\ t_4 = 0.110420, \quad \mathbf{t_f = 0.111184} \end{array}$$

(a) (b) (c) (d)

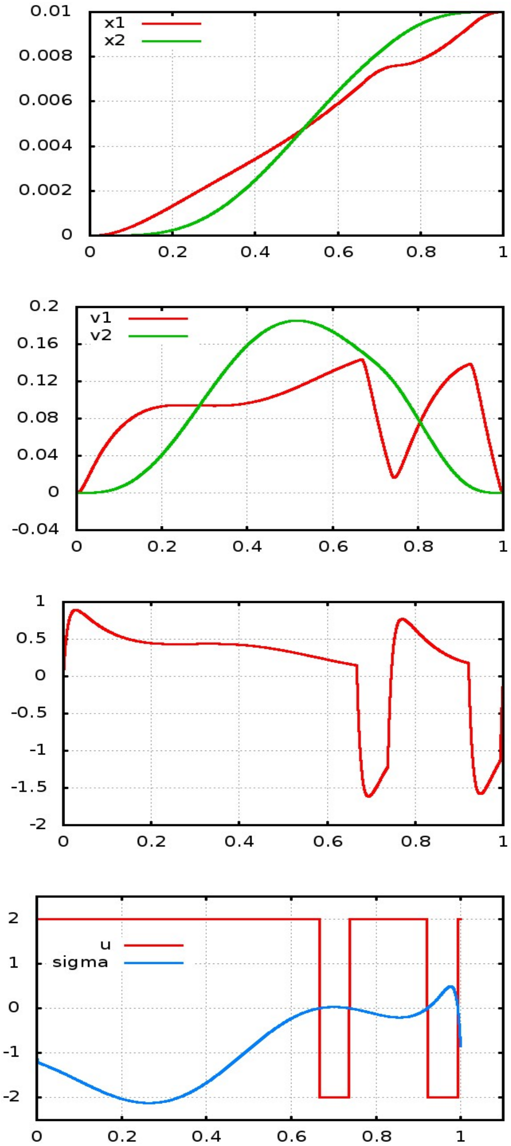


Fig. 3. $\mathbf{U_{max}} = \mathbf{2}$: time–optimal solution on normalized time interval $[0, 1]$; (a) positions $x_1(t), x_2(t)$; (b) velocities $v_1(t), v_2(t)$; (c) electric current $I(t)$; (d) control $U(t)$ and switching function $\sigma(t)$.

(a) (b) (c) (d)

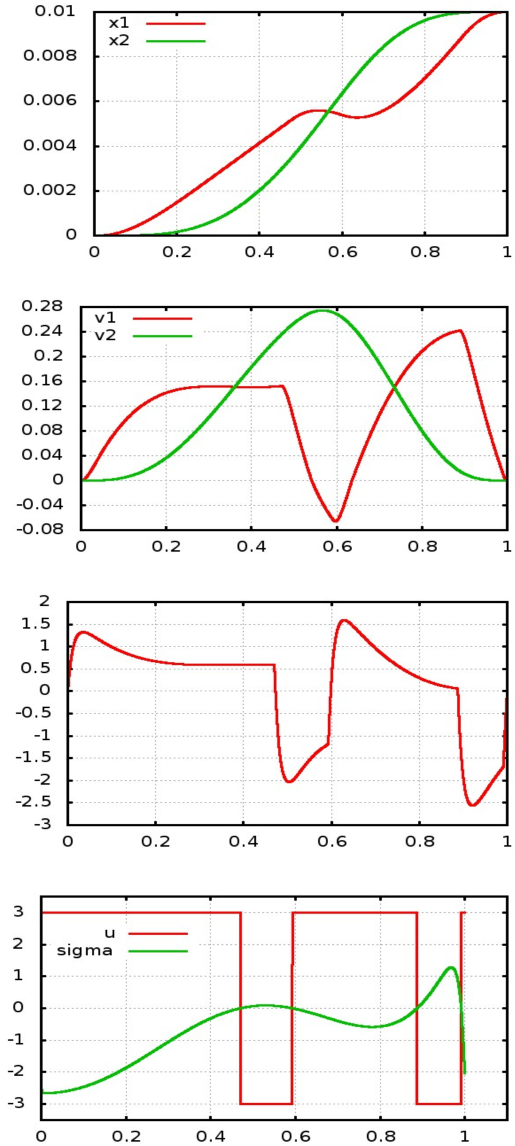


Fig. 4. $\mathbf{U_{max}} = \mathbf{3}$: time–optimal solution on normalized time interval $[0, 1]$; (a) positions $x_1(t), x_2(t)$; (b) velocities $v_1(t), v_2(t)$; (c) electric current $I(t)$; (d) control $U(t)$ and switching function $\sigma(t)$.

The initial value of the adjoint variable $\lambda(t) \in I\!R^5$ satisfying the adjoint equation (14) is given by $\lambda(0) = (-4.82918, -0.100808, -4.09481, -0.057766, -0.001074)$. With these values the reader may verify that the switching function $\sigma(t) := H_U(t) = \lambda_I(t)/L$ obeys the control law (15) with high accuracy. The local optimality of this trajectory follows from the fact that the Jacobian $5 \times 5$ matrix of the terminal conditions computed with respect to the switching times and final time is a regular matrix. Hence, first order sufficient conditions hold for this time-optimal control problem; cf. Maurer, Osmolovskii [13], [15].

For $\mathbf{U}_{\max} = 3$, the optimal state and control variables are depicted in Fig. 4. In view of (18) and (19) we have $v_1(t) < 0$ for $t_1^v < t < t_2^v$. Here, the times $t_1^v, t_2^v$ are treated as additional optimization variables which allows us to apply again the arc-parametrization method in [12]. We obtain the switching times, the two intermediate times and the final time

$$\begin{array}{lll} t_1 = 0.0416854, & t_1^v = 0.04800526, & t_2 = 0.0525894, \\ t_2^v = 0.5635593, & t_3 = 0.0786491, & t_4 = 0.0878590, \\ \mathbf{t_f} = \mathbf{0.0886180}. & & \end{array}$$

The computed initial value of the adjoint variable is $\lambda(0) = (-4.40300, -0.065128, 1.34424, -0.005169, -0.00692)$. The switching function $\sigma(t) := \lambda_I(t)/L$ satisfies the control law (15) precisely.

## IV. Sensitivity analysis of arclengths

We give a brief outlook on sensitivity analysis of optimal solutions when system parameters are subject to perturbations. For purpose of demonstration, we choose the bound $\mathbf{U_{max}} = \mathbf{3}$. The corresponding optimal control has 7 subarcs with arclengths $\xi_1 = t_1$, $\xi_2 = t_1^v - t_1$, $\xi_3 = t_2 - t_1^v$, $\xi_4 =$

| $i$ | $\xi_i$ | $d\xi_i/dm_2$ | $d\xi_i/dR$ |
|---|---|---|---|
| 1 | 0.041685 | 0.008917 | 0.010355 |
| 2 | 0.0063199 | $-0.003495$ | 0.001788 |
| 3 | 0.0045841 | 0.002253 | $-0.003302$ |
| 4 | 0.0037666 | 0.007321 | $-0.002233$ |
| 5 | 0.022931 | 0.001764 | 0.000936 |
| 6 | 0.0092098 | 0.001446 | 0.003410 |
| 7 | 0.00075901 | $0.2879e-6$ | $-0.000449$ |

TABLE II

SENSITIVITY DERIVATIVES OF ARCLENGTHS: PARAMETERS $m_2$ AND $R$.

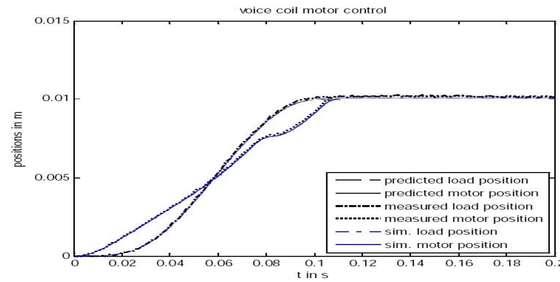


Fig. 5. $\mathbf{U_{max} = 2}$ : positions $x_1(t), x_2(t)$; predicted (solid), simulated (dashed), real-time (dashed-dot).

$t_2^v - t_2$, $\xi_5 = t_3 - t_2^v$, $\xi_6 = t_4 - t_3$, $\xi_7 = t_f - t_4$, The arc-parametrization method [12] in combination with the code NUDOCCCS [3] allows to compute the *sensitivity derivatives* $d\xi_i/dp$, $i = 1, ..., 7$, with respect to any parameter $p$ in the system. The existence of parametric sensitivity derivatives follows from the fact that second-order sufficient conditions hold for the switching time optimization problem. The precomputation of parametric sensitivity derivatives then enables us to design real-time control approximations to perturbed optimal solutions; cf. the theory and numerical approach in [4], [5].

Let us consider the following two parameters: the load mass $m_2$ with nominal value $m_2^0 = 0.56$ and the resistance $R$ with nominal value $R^0 = 2$. In Table II, we have listed the nominal values $\xi_i$ of the arclengths and their sensitivity derivatives with respect to the parameters $m_2$ and $R$.

## V. COMPARISON OF NUMERICAL SOLUTIONS AND EXPERIMENTAL SIMULATIONS

The computed optimal control solutions were implemented on the test bench in the mechatronics laboratory of the Gießen University of Applied Sciences; cf. Figure 1. Control signals are applied with the real-time-system sampling time of $T_s = 0.1$ ms ; cf. Table I. Since the computed minimal times have order of magnitude $0.1$ sec., approximately $1000$ values of the computed optimal control can be used in the experimental test bench. Both for $\mathbf{U_{max} = 2}$ and $\mathbf{U_{max} = 3}$, where $v_1(t)$ changes sign, we obtain an excellent agreement between the predicted optimal solution, the simulated solution with $1000$ control signals and the experimental solution; cf. Figures 5, 6.

The small deviations between predicted and measured positions result from friction uncertainties of the guides as well as from noise in the analogue position capturing unit. Positioning times realised at this plant by feedback position control and stepwise reference input are in the range of 0.2 s [19] if the step response should be overshoot free. This indicates that the described control method is very efficient.

## VI. ENERGY-OPTIMAL CONTROL

We consider the "energy-optimal" cost functional (11) of minimizing $\int_0^{t_f} U(t)^2 dt$ with a fixed final time $t_f > t_{min}$, where $t_{min}$ is the minimal time computed in the previous

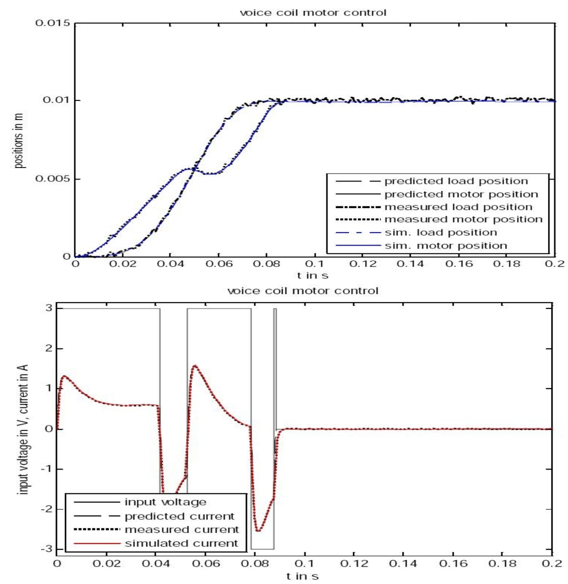


Fig. 6. $\mathbf{U_{max} = 3}$ : (a) positions $x_1(t), x_2(t)$, (b) electric current $I(t)$; predicted (solid), simulated (dashed), experimental (dashed-dot).

sections. In this case the Hamiltonian $H(x(t), \lambda(t), U) = U^2 + \lambda(t)(Ax(t) + B \cdot U + C \,\mathrm{sign}\,(v_1(t)))$ admits a unique minimizer $U(t) = Proj_{[-U_{\max}, U_{\max}]}(-\lambda_5(t)/2L)$, where $Proj$ denotes the projection onto the control set. In particular, it follows that any optimal control $U(t)$ is *continuous*. It is well known that the quadratic cost functional smoothes the structure of the optimal control. For $U_{\max} = 3$, Figure 7 depict optimal solutions for 3 final times that differ from the minimal time $t_{\min} = 0.088618$ by less than $25\%$. Note that already for the final time $t_f = 0.09$ the velocity $v_1(t)$ does not change sign.

Moreover, the energy-optimal controls reduce oscillations in the state variables, since the difference in positions and velocities becomes substantially smaller with increasing final time; cf. Table III. As an example, consider the energy-optimal functional, where the final time $t_f$ is increased by only $1.5$ % , $t_f = 1.015 \cdot t_{\min}$. It is remarkable that the maximum difference $||v_1 - v_2||_\infty$ in the velocities is reduced by $30$ % compared to the time-optimal case.

| $t_f$ | $\|\|x_1 - x_2\|\|_\infty$ | $\|\|v_1 - v_2\|\|_\infty$ |
|---|---|---|
| 0.088618 | 0.002979 | 0.337792 |
| 0.09 | 0.002174 | 0.238127 |
| 0.1 | 0.001940 | 0.150524 |
| 0.11 | 0.001594 | 0.111915 |

TABLE III

DIFFERENCES IN POSITIONS AND VELOCITIES FOR TIME–OPTIMAL ($t_f = 0.088618$) AND ENERGY-OPTIMAL SOLUTIONS ($t_f = 0.09, 0.1, 0.11$).

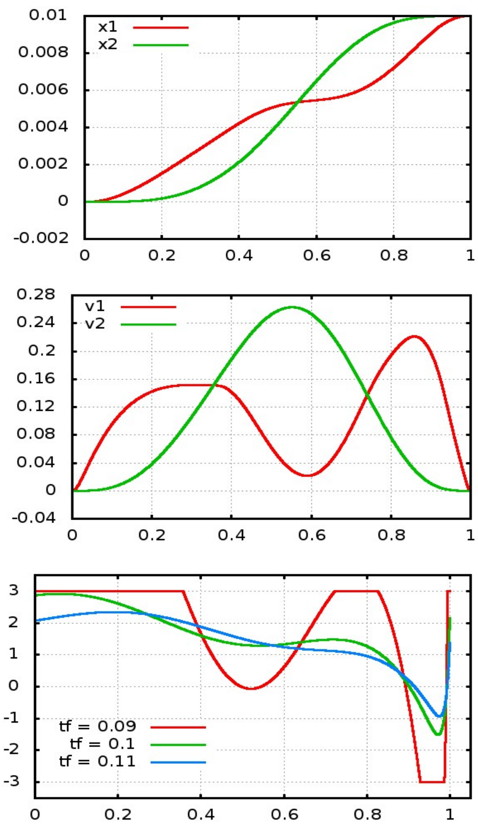


Fig. 7. $\mathbf{U_{max} = 3}$ : energy-optimal solutions; (a) positions $x_1(t), x_2(t)$ for final time $t_f = 0.09$, (b) velocities $v_1(t), v_2(t)$ for final time $t_f = 0.09$, (c) optimal control for final times $t_f = 0.09$, $t_f = 0.1$, $t_f = 0.11$.

## VII. CONCLUSION

An optimal control problem for an electrodynamical servo drive system, the voice–coil–motor, was formulated. The Coulombic friction force gives rise to state-dependent jumps in the dynamical system. This feature leads us to consider a nonsmooth control problem, when the velocity of the slider changes sign. We showed that time–optimal controls are bang-bang and determined those control bounds for which the slider velocity changed sign. The arc-parametrization method in [12] in conjunction with the routine NUDOCCCS [3], [5] were applied to directly optimizing the switching times. We could observe an excellent agreement between the computed optimal trajectories and experimental results on a test bench developed by the third author.

Oscillations in the positions and velocities can be significantly reduced by determining energy-optimal solutions, however, at the expense of a larger process time. Future work will concern a detailed study of optimal solutions under the state constraint (12), $|v_1 - v_2| \le c_v$, resp., (13), $|x_1 - x_2| \le c_x$. The intention behind imposing these state constraints is to further reduce oscillations of the slider and the flexible load mass.

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