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Delay-Dependent Rendezvous and Flocking of Large Scale Multi-Agent Systems with Communication Delays

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***Abstract*— We study the stability of multi-agent system (MAS) formations with delayed exchange of information between the agents. The agents are described by second order systems. They communicate via a symmetric connected communication topology with constant, heterogeneous, symmetric delays between any two neighboring agents. We consider two different tasks for the MAS: rendezvous, where all agents meet at an arbitrary point, and flocking, where all agents reach a given formation and move in a predefined direction. Therefore, we propose a decentralized control algorithm with position coupling gains k_{ji}. We prove that the MAS achieves rendezvous for any constant delay if the communication topology is connected and the coupling gain is sufficiently small. For larger gains, rendezvous and flocking are delay-dependent, i.e., they are reached for any delay smaller than a bound which depends on k_{ji}. Thereby, the controllers can be tuned in a totally decentralized fashion, i.e., only based on the communication delays to their neighbors and not considering the delays in the rest of the network. For the analysis, we use both frequency and time domain methods to prove delay-independent and delay-dependent rendezvous and flocking, respectively.**

***Index Terms*— Multi-agent systems, rendezvous, flocking, communication delay.**

I. INTRODUCTION

The analysis and control of large groups of autonomous systems is one of the big challenges of modern engineering science. Examples of networked systems appear in a diverse range of research areas, such as biochemical reaction networks, animal flocking behavior, internet congestion control, and coordination of robots, to name just a few. Accordingly, scientists from physics, biology, and engineering are trying to understand the collective group behavior of these systems.

In this work, we propose control laws for second order MAS with delayed communications and develop decentralized conditions that guarantee rendezvous and flocking, respectively. Rendezvous refers to agents meeting at an arbitrary point in space, and flocking describes MAS that reach a given formation and move in a certain direction. In both cases, we assume that the communication graph is undirected and connected and that the communication delays between any two neighboring agents are constant, heterogeneous, and symmetric. Both control laws contain *position gains* k_{ji} as design parameters. Adapting the results from [1], we show first that rendezvous is achieved *independent of delay* for sufficiently small gains k_{ji}. Then, we extend the set of possible controller parameters k_{ji} by introducing *delay-dependent* rendezvous conditions, which is the main contribution of this paper. This result is then extended to delay-dependent flocking conditions. The controllers can be tuned in a totally decentralized fashion, i.e., only based on the communication delays to their neighbors and not considering the delays in the rest of the network. To prove our results, we apply both frequency domain and time domain arguments. In particular, this is the first work on higher order large scale MAS with heterogeneous communication delays that uses time-domain arguments and obtains delay-dependent conditions on the group behavior.

In recent years, multi-agent systems (MAS) with first order subsystem dynamics have been studied extensively. An overview is provided for example in [2], [3]. However, many applications exhibit higher order subsystems. If we consider for example point masses with actuators applying forces to the subunits, then Newton's second law requires at least a second order differential equation for the position of the subunits. Hence, second order subsystems have attracted an increasing attention. Typical tasks for second order MAS are rendezvous, e.g., [4], [5], and flocking, e.g., [6]–[10]. Higher order subsystems and more complex group tasks have been investigated in [11]–[19] to cite a few.

It is remarkable that there are very few results for higher order MAS with delays in the communication. For consensus problems, i.e., if first order subsystems have to agree on a certain value, the influence of communication delays has been studied thoroughly, see for example [20]–[27]. However, for large scale MAS of higher order subsystems, only high gain arguments have been used so far, e.g., [1], [28]. Second order MAS with delays are also used to model car following problems where delays turned out to be crucial to describe certain phenomena, see for example [29]–[31]. Yet, in these contributions, the drivers only "communicate" with one or two cars in front of them; yet, more complex communication topologies are not studied.

The paper is organized as follows: We first review different stability arguments for functional differential equations and algebraic graph theory in Section II. The problem statement is given in Section III. Then, we present delay-independent rendezvous in Section IV, delay-dependent rendezvous in Section V and delay-dependent flocking in Section VI. The results are illustrated in an example in Section VII before the paper is concluded in Section VIII.

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II. Preliminaries

Systems with time-delays, like MAS with delayed communication, can be represented by retarded functional differential equations (RFDE). In this section, we provide a review on stability arguments for RFDEs as well as some basics on algebraic graph theory.

A. Stability of Functional Differential Equations

This subsection gives a brief summary of stability results for functional differential equations. The interested reader is referred to [32], [33] for details.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with the standard norm $|\cdot|$. Let $\mathscr{C}([a,b],\mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[a,b] \subset \mathbb{R}$ into $\mathbb{R}^n$ with the topology of uniform convergence. For easier notation, we drop the argument of $\mathscr{C}$ if $a = -\mathscr{T}$ and $b = 0$ for a given $\mathscr{T} > 0$, i.e., $\mathscr{C} = \mathscr{C}([-\mathscr{T},0],\mathbb{R}^n)$. The norm on $\mathscr{C}$ is defined as $\|\varphi\| = \sup_{-\mathscr{T}\leq s\leq 0} |\varphi(s)|$. Let $\rho \geq 0$ and $x \in \mathscr{C}([-\mathscr{T},\rho],\mathbb{R}^n)$, then for any $t \in [0,\rho]$, define a segment $x_t \in \mathscr{C}$ of x such that $x_t(s) = x(t+s), s \in [-\mathscr{T},0]$.

Let Ω be a subset of $\mathscr{C}$, $f : \Omega \to \mathbb{R}^n$ a given function, and '$\dot{}$' represent the right-hand Dini derivative. Then, we call

$$\dot{x}(t) = f(x_t) \tag{1}$$

an autonomous Retarded Functional Differential Equation (RFDE) on Ω. Given $\varphi \in \Omega$ and $\rho > 0$, a function $x(\varphi) \in \mathscr{C}([-\mathscr{T},\rho],\mathbb{R}^n)$ is said to be a solution to (1) with initial condition φ, if $x_t(\varphi) \in \Omega$, $x(\varphi)(t)$ satisfies (1) for $t \in [0,\rho)$, and $x_0(\varphi) = \varphi$. Such a solution exists and is unique if f is continuous and $f(\varphi)$ is Lipschitzian in each compact set in Ω. Note that $x(\varphi)(t) \in \mathbb{R}^n$, whereas $x_t(\varphi) \in \mathscr{C}$. We denote the value of the segment $x_t(\varphi)$ at time s where $s \in [-\mathscr{T},0]$ as $x_t(\varphi)(s) = x(\varphi)(t+s)$. For easier notation, we often drop the initial condition φ of x and x_t.

An element $\phi \in \mathscr{C}$ is called a steady-state or equilibrium of (1) if $x_t(\phi) = \phi$ for all $t \geq 0$. Without loss of generality we assume that $\phi = 0$ is an equilibrium of (1). The stability of (1) around such a steady-state is defined in a way similar to the stability of nonlinear Ordinary Differential Equations (ODE) using an ε-δ argument, see [32].

The stability of RFDEs can be analyzed in the time-domain using Lyapunov-type arguments. Since the state in RFDEs is a segment of trajectory x_t, the corresponding Lyapunov function is a functional, the so-called Lyapunov-Krasovskii functional $V(x_t)$. The derivative of V, $\dot{V}(x_t)$, is the right-hand derivative along the solutions of (1).

Theorem 1 ([32]): Suppose $V : \mathscr{C} \to \mathbb{R}$ is continuous and there exist nonnegative functions u, v such that $u(s) \to \infty$ as $s \to \infty$ and

$$u(\|\phi(0)\|) \leq V(\phi), \qquad \dot{V}(\phi) \leq -v(\|\phi(0)\|).$$

Then the trivial solution $x(t) = 0$ of (1) is stable and every solution is bounded. If, in addition, $v(s) > 0$ for $s > 0$, then every solution approaches zero as $t \to \infty$.

In this work, we only consider autonomous RFDEs where f is completely continuous. In this case, we can conclude the attractivity of a positively invariant set using a result similar to LaSalle's theorem for Ordinary Differential Equations.

Definition 1 ([32]): We say $V : \mathscr{C} \to \mathbb{R}$ is a Lyapunov functional *on a set G in $\mathscr{C}$ relative to (1) if V is continuous on $\bar{G}$ (the closure of G) and $\dot{V} \leq 0$ on G. Define*

$$S = \{\phi \in \bar{G} : \dot{V}(\phi) = 0\}$$
$$M = \textit{Largest set in } S \textit{ that is invariant with respect to } (1).$$

Then, we have the following theorem:

Theorem 2 ([32]): If V is a Lyapunov functional on G and $x_t(\phi)$ is a bounded solution of (1) that remains in G, then $x_t(\phi)$ tends to M as $t \to \infty$.

B. Algebraic Graph Theory

The topology of the communication network between the agents is represented by a graph. A graph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ consists of a set of vertices (nodes) $\mathscr{V} = \{v_i\}, i \in \mathscr{I} = \{1,\ldots,N\}$, which represent the agents, and a set of edges (links) $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, which represent the communication channels between the agents. If $v_i, v_j \in \mathscr{V}$ and $e_{ij} = (v_i, v_j) \in \mathscr{E}$, then there is an edge (a directed arrow) from node v_i to node v_j, i.e., agent j can receive data from agent i. In this paper, we assume that the graph $\mathscr{G}$ is *undirected*, i.e., $e_{ij} \in \mathscr{E}$ if and only if $e_{ji} \in \mathscr{E}$. We also assume that the network topology does not contain self-loops, i.e., $e_{ii} \notin \mathscr{E}$.

The *graph adjacency matrix* $A = [a_{ij}], A \in \mathbb{R}^{N\times N}$, is such that $a_{ij} = 1$ if $e_{ij} \in \mathscr{E}$ and $a_{ij} = 0$ if $e_{ij} \notin \mathscr{E}$. If $e_{ij} \in \mathscr{E}$, then agents i and j are neighbors. The number of neighbors of agent i, also called the *valence* or *degree* of vertex v_i, is denoted by n_i. The diagonal *valency matrix* is $\mathscr{N} = \mathrm{diag}(n_i)$. A *path* from v_i to v_j is a sequence of edges from $\mathscr{E}$ that takes the following form (v_i, v_{i_1}), $(v_{i_1}, v_{i_2}), \ldots, (v_{i_p}, v_j)$. If there exists a path between two vertices, then these vertices are connected. A graph $\mathscr{G}$ is *connected*, if any two vertices of $\mathscr{G}$ are connected. More details on algebraic graph theory can be found for example in [34].

III. Problem Statement

In this paper, we consider two different control problems. First, we want to design a controller such that the MAS achieves *rendezvous* of the agents, i.e., all agents eventually meet at an arbitrary point. The second controller has to achieve *flocking* of the agents, i.e., all agents asymptotically converge to a formation and move in a certain direction, preserving this formation. Thereby, the desired formation is given by the distance matrix $D = D^T = [d_{ji}] \in \mathbb{R}^{N\times N}$, i.e., the desired positions of the agents is $r_i(t) - r_j(t) = d_{ji}$ where $r_i(t)$ and $r_j(t)$ are the position of agent i and j at time t. Clearly, the matrix D has to be assigned such that the desired distances are consistent. Here, we only consider flocking with a given reference speed $v^* \in R$.

If the input of the agents is an external force, single integrators cannot represent the agents' speed and position dynamics properly. Therefore, we consider a MAS consisting of N subunits described by second order systems with dynamics

$$\begin{aligned}\dot{r}_i(t) &= v_i(t)\\ \dot{v}_i(t) &= -cv_i(t)+u_i(t),\end{aligned} \tag{2}$$

$i \in \mathscr{I}$, where $r_i \in \mathbb{R}$ is the position, $v_i \in \mathbb{R}$ is the speed of agent i, $-cv_i(t)$ is a friction drag term, and $u_i(t)$ is an external force considered as input; all agents are assumed to be identical. For simplicity, we discuss only dynamics in a 1D space. Yet, our results can also be applied to 2D and 3D problems if the dynamics of the agents are decoupled in all coordinates. The communication network between the agents is given by a communication graph with adjacency matrix $A = [a_{ji}]$. The communication delay from agent j to agent i is $\tau_{ji} \in \mathbb{R}_+$. We assume symmetric communication, i.e., $a_{ji} = a_{ij}$ and $\tau_{ji} = \tau_{ij}$. The control tasks are particularly difficult because of the communication delays. Due to these delays, only out-dated position data of the neighboring agents can be used for control.

IV. Delay-Independent Rendezvous of Large Scale MAS

First, we design a controller that achieves rendezvous independent of delay, i.e., the rendezvous is asymptotically attracting for any τ_{ji}. The proposed control of agent i is

$$u_i(t) = -\sum_{j=1}^{N} \frac{k_i}{n_i} a_{ji}\left(r_i(t) - r_j(t-\tau_{ji})\right) \tag{3}$$

where k_i is the *position gain* of agent i, $A = [a_{ji}]$ is the adjacency matrix of the communication network, n_i is the degree of agent i, and τ_{ji} is the communication delay between agent j and agent i. For convenience, we introduce the normalized adjacency matrix $\tilde{A} = [\tilde{a}_{ji}] = \mathscr{N}^{-1}A$, where $\mathscr{N}$ is the diagonal valency matrix, see Section II-B. Note that $\tilde{A}$ is a stochastic matrix, i.e., the row and column sums equal one. Moreover, we know that that the spectral radius of $\tilde{A}$ is $\rho(\tilde{A}) = 1$.

Theorem 3: Given a MAS consisting of N agents with dynamics (2) *and control* (3)*, where the communication network is connected and symmetric, i.e., $a_{ji} = a_{ij}$ and $\tau_{ji} = \tau_{ij}$, then rendezvous is asymptotically reached, i.e., $r_i(t) - r_j(t) \to 0$ and $v_i \to 0$ for $t \to \infty$ and all $i, j \in \mathscr{I}$, if $k_i < \frac{c^2}{2}$ for all $i \in \mathscr{I}$.*

Proof: The proof follows immediately from [1] and is presented here for completeness. The closed loop dynamics of agent i is

$$\begin{aligned}\dot{r}_i(t) &= v_i(t)\\ \dot{v}_i(t) &= -cv_i(t) - k_i r_i(t) + \sum_{j=1}^{N} k_i \tilde{a}_{ji} r_j(t-\tau_{ji}).\end{aligned}$$

The open loop transfer function $G_i(s)$ of agent i is

$$G_i(s) = \frac{k_i}{s^2 + cs + k_i}. \tag{4}$$

The feedback loop contains the communication topology $\tilde{A}$ and delays τ_{ji}. Clearly, the feedback loop has gain 1 because $\rho(\tilde{A}) = 1$ and following the arguments in [1], G_i has to satisfy

$$|G_i(j\omega)| < 1 \qquad \text{for all } \omega \neq 0 \text{ and} \tag{5}$$

$$\lim_{\omega \to 0} |G_i(j\omega)| = 1. \tag{6}$$

Clearly, $G_i(0) = 1$ is satisfied for all i and we have

$$\left|\frac{k_i}{k_i - \omega^2 + jc\omega}\right| < 1 \quad \Leftrightarrow \quad \frac{k_i}{c^2} < \frac{1}{2},\ \forall i \in \mathscr{I},$$

i.e., rendezvous is reached for any τ_{ji} if $k_i < \frac{c^2}{2}$ for all i. ■

Note that the controller design is totally decentralized because each agent can choose its k_i independently as long as it satisfies $k_i < \frac{c^2}{2}$. Clearly, this rendezvous condition is independent of the delays τ_{ji}.

V. Delay-Dependent Rendezvous of Large Scale MAS

In the previous section, we derived an upper bound for the position gain k_i such that the MAS reaches rendezvous for any delays τ_{ji}. However, we often know that the delays τ_{ji} are bounded by some value $\overline{\tau}_{ji}$, which might be quite small. In this case, we expect that rendezvous is also reached for some gains $k_i \geq \frac{c^2}{2}$. In this section, we provide a far less restrictive *delay-dependent* rendezvous condition for MAS (2). Therefore, we propose the following controller for agent i

$$u_i(t) = -\sum_{j=1}^{N} \frac{k_{ji}}{n_i} a_{ji}\left(r_i(t) - r_j(t-\tau_{ji})\right) \tag{7}$$

where k_{ji} is the *position gain* of agent i when comparing his position with the position of agent j, $A = [a_{ji}]$ is the adjacency matrix of the communication network, n_i is the degree of agent i, and τ_{ji} is the communication delay between agent j and agent i. The rendezvous property of (2) with controller (7) is stated in the following theorem:

Theorem 4: Given a MAS consisting of N agents with dynamics (2) *and control* (7)*, where the communication network is connected and symmetric, i.e., $a_{ji} = a_{ij}$ and $\tau_{ji} = \tau_{ij}$, then rendezvous is asymptotically reached, i.e., $r_i(t) - r_j(t) \to 0$ and $v_i \to 0$ for $t \to \infty$ and all $i, j \in \mathscr{I}$, if $k_{ji} = k_{ij} > 0$ and $c > k_{ji}\tau_{ji}$ for all $i, j \in \mathscr{I}$.*

Proof: First, we reformulate the control (7) using

$$x_i(t-\tau) = x_i(t) - \int_{-\tau}^{0} \dot{x}_i(t+\eta)\, d\eta. \tag{8}$$

Moreover, we introduce $r_{ji}(t) = r_i(t) - r_j(t)$. With this, the closed loop system is

$$\begin{aligned}\dot{r}_i(t) &= v_i(t)\\ \dot{v}_i(t) &= -cv_i(t) - \sum_{j=1}^{N} \frac{k_{ji}}{n_i} a_{ji}\left(r_{ji}(t) + \int_{-\tau_{ji}}^{0} v_j(t+\eta)\, d\eta\right),\end{aligned}$$

where $\dot{r}_{ji}(t) = v_i(t) - v_j(t)$. In order to prove rendezvous, we have to show that $r_{ji}(t) \to 0$ and $v_i(t) \to 0$ for $t \to \infty$. Therefore, consider the Lyapunov-Krasovskii candidate $V =$

$V_1+V_2+V_3$ with

$$V_1 = \frac{1}{2}\sum_{i=1}^{N} n_i c v_i^2(t), \tag{9}$$

$$V_2 = \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N} k_{ji}a_{ji}c(r_{ji}(t))^2, \tag{10}$$

$$V_3 = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{-\tau_{ji}}^{0} k_{ji}^2 a_{ji}\tau_{ji}\int_{\eta}^{0} v_i^2(t+\xi)d\xi d\eta. \tag{11}$$

Differentiating along the solutions of the MAS, we get with $k_{ji}=k_{ij}$

$$\begin{aligned}\dot V_1 &= -\sum_{i=1}^{N} n_i c^2 v_i^2(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} k_{ji}a_{ji}cv_i(t)\\ &\quad\times\left(r_{ji}(t)+\int_{-\tau_{ji}}^{0} v_j(t+\eta)d\eta\right),\\ \dot V_2 &= \sum_{i=1}^{N}\sum_{j=1}^{N} k_{ji}a_{ji}cv_i(t)r_{ji}(t),\\ \dot V_3 &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{-\tau_{ji}}^{0} k_{ji}^2 a_{ji}\tau_{ji}\left(v_i^2(t)-v_i^2(t+\eta)\right)d\eta.\end{aligned}$$

and therefore

$$\begin{aligned}\dot V &= -\sum_{i=1}^{N} n_i c^2 v_i^2(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} k_{ji}a_{ji}cv_i(t)\int_{-\tau_{ji}}^{0} v_j(t+\eta)d\eta\\ &\quad+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{-\tau_{ji}}^{0} a_{ji}k_{ji}^2\tau_{ji}\left(v_i^2(t)-v_i^2(t+\eta)\right)d\eta\\ &= -\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}c^2v_i^2(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}k_{ji}^2\tau_{ji}^2 v_i^2(t)\\ &\quad-\sum_{i=1}^{N}\sum_{j=1}^{N} k_{ji}a_{ji}cv_i(t)\int_{-\tau_{ji}}^{0} v_j(t+\eta)d\eta\\ &\quad-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{-\tau_{ji}}^{0} a_{ji}k_{ji}^2\tau_{ji}v_i^2(t+\eta)d\eta\\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}\Big(\left(c^2-\tau_{ji}^2k_{ji}^2\right)v_i^2(t)\\ &\quad+\int_{-\tau_{ji}}^{0}\frac{c^2}{\tau_{ji}}v_i^2(t)+2k_{ji}cv_i(t)v_j(t+\eta)\\ &\quad+k_{ji}^2\tau_{ji}v_i^2(t+\eta)d\eta\Big)\\ &= -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}(c^2-\tau_{ji}^2k_{ji}^2)v_i^2(t)\\ &\quad-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{-\tau_{ji}}^{0} a_{ji}\left(\frac{c}{\sqrt{\tau_{ji}}}v_i(t)+k_{ji}\sqrt{\tau_{ji}}v_j(t+\eta)\right)^2 d\eta.\end{aligned}$$

Clearly, we have $V(t)>0$ if $r_{ji}(t)\neq 0$ or $v_i(t)\neq 0$ for any $i,j\in\mathscr{I}$. Moreover, $\dot V\leq 0$ if $c>k_{ji}\tau_{ji}>0$ for all i,j such that $e_{ji}\in\mathscr{E}$. Since the graph is connected, we conclude that all solutions x_t of the MAS that start in $G=\{x_t\in\mathscr{C}\mid V(x_t)\leq c\}$ for any $c>0$ remain in G for $t\geq 0$. Going back to Definition 1, we see that V is continuous on $\overline{G}$ and $\dot V\leq 0$ on G. The set S contains all solutions where all agents stop, i.e., $v_t(s)=0$ for any $s\in[-\tau,0]$ and all $t>0$, where $\tau=\max_{i,j\in\mathscr{I}}\tau_{ji}$. The maximal invariant set M in S requires in addition to $v_t(s)=0$ for any $s\in[-\tau,0]$ that $r_{ji}(t)=0$, i.e., $r_i(t)=r_j(t)$, for any t. Hence, the rendezvous is asymptotically attracting. ■

If we assume that agent i is able to measure the communication delay τ_{ji} from agent j, then one can tune its position gain such that $0<k_{ji}<\frac{c}{\tau_{ji}}$. The communication delay can be measured, for example, if all users time-stamp their positions messages or using the round trip times τ_{RTT}. Since the communication channel is symmetric, we have $\tau_{ji}=\tau_{ij}=\frac{1}{2}\tau_{RTT}$. If the delay τ_{ji} is not known but its upper bound $\overline{\tau}_{ji}\geq\tau_{ji}$ is known, then the controller gain can be chosen to be $0<k_{ji}<\frac{c}{\overline{\tau}_{ji}}$. The condition $k_{ji}=k_{ij}$ can be satisfied by choosing $k_{ji}=\alpha\frac{c}{\tau_{ji}}, \alpha\in(0,1)$, if the delay is known or by comparing the parameter k_{ji} with the neighbor.

Clearly, the bound of Theorem 4 exceeds the bound of Theorem 3 for sufficiently large τ_{ji}, as will be illustrated in an example in Section VII. An important property of Theorem 4 is that the controller can be tuned and implemented in a totally distributed fashion, i.e., without knowing the size or configuration of the complete communication network.

VI. Delay-Dependent Flocking of Large Scale MAS with Fixed Reference Speed

Finally, we derive a delay-dependent flocking condition for large scale MAS with communication delays. Flocking means that all agents converge to a formation and move in a certain direction, preserving this formation. Here, we consider flocking with a given reference speed $v^*\in R$, i.e., the direction and the speed where the agents are supposed to go is predefined and forms part of the controller.

The agents are again given by Equation (2). The desired formation is given by the distance matrix $D=D^T=[d_{ji}]\in\mathbb{R}^{N\times N}$, i.e., the desired positions of the agents are $r_i(t)-r_j(t)=d_{ji}$. Assuming that the delays τ_{ji} are known to agent i, the MAS achieves $r_i(t)-r_j(t)\to d_{ji}$ for $t\to\infty$. The corresponding controller is

$$u_i(t) = cv^* - \sum_{j=1}^{N}\frac{k_{ji}}{n_i}a_{ji}\left(r_i(t)-r_j(t-\tau_{ji})-d_{ji}^*\right) \tag{12}$$

where c is the damping parameter of (2) and $d_{ji}^*=d_{ji}+v^*\tau_{ji}$ results from the desired distances d_{ji}, the reference velocity v^*, and the delays τ_{ji}. The delay-dependent flocking condition is stated in the following theorem:

Theorem 5: Given a MAS consisting of N agents with dynamics (2) *and control* (12)*, a connected and symmetric communication network, i.e., $a_{ji}=a_{ij}$ and $\tau_{ji}=\tau_{ij}$, a reference speed $v^*\in\mathbb{R}$, and a distance matrix $D=D^T=[d_{ji}]$, then flocking is asymptotically reached, i.e., $r_i(t)-r_j(t)\to d_{ji}$ and $v_i\to v^*$ for $t\to\infty$ and all $i,j\in\mathscr{I}$, if $k_{ji}=k_{ij}>0$ and $c>k_{ji}\tau_{ji}$ for all $i,j\in\mathscr{I}$.*

Proof: We use again (8), as well as $r_{ji}(t)=r_i(t)-r_j(t)$ and $\tilde v_i(t)=v_i(t)-v^*$ to obtain the closed loop system dynamics with control (12)

$$\dot r_{ji}(t) = \tilde v_i(t)-\tilde v_j(t)$$

$$\dot{\tilde{v}}_i(t) = -c\tilde{v}_i(t) - \sum_{j=1}^{N} \frac{k_{ji}}{n_i} a_{ji} \left(r_{ji}(t) - d_{ji} + \int_{-\tau_{ji}}^{0} \tilde{v}_j(t+\eta) d\eta \right).$$

The Lyapunov-Krasovskii candidate is $V = V_1 + V_2 + V_3$ with

$$V_1 = \frac{1}{2} \sum_{i=1}^{N} c n_i \tilde{v}_i^2(t), \tag{13}$$

$$V_2 = \frac{1}{4} \sum_{i=1}^{N} \sum_{j=1}^{N} k_{ji} a_{ji} c (r_{ji}(t) - d_{ji})^2, \tag{14}$$

$$V_3 = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{-\tau_{ji}}^{0} a_{ji} k_{ji}^2 \tau_{ji} \int_{\eta}^{0} \tilde{v}_i^2(t+\xi) d\xi d\eta. \tag{15}$$

Similarly as in the proof of Theorem 4, we differentiate along the solutions of the MAS and get

$$\begin{aligned}
\dot{V} = &-\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji} c^2 \tilde{v}_i^2(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji} k_{ji}^2 \tau_{ji}^2 \tilde{v}_i^2(t) \\
&- \sum_{i=1}^{N}\sum_{j=1}^{N} k_{ji} a_{ji} c \tilde{v}_i(t) \int_{-\tau_{ji}}^{0} \tilde{v}_j(t+\eta) d\eta \\
&- \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \int_{-\tau_{ji}}^{0} a_{ji} k_{ji}^2 \tau_{ji} \tilde{v}_i^2(t+\eta) d\eta \\
= &-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji} \Big((c^2 - \tau_{ji}^2 k_{ji}^2) \tilde{v}_i^2(t) \\
&+ \int_{-\tau_{ji}}^{0} \left(\frac{c}{\sqrt{\tau_{ji}}} \tilde{v}_i(t) + k_{ji} \sqrt{\tau_{ji}} \tilde{v}_j(t+\eta) \right)^2 d\eta \Big).
\end{aligned}$$

Again, we have $V > 0$ if $r_{ji}(t) \neq 0$ or $\tilde{v}_i(t) \neq 0$ for any $i, j \in \mathscr{I}$, and $\dot{V} \leq 0$ if $c > k_{ji}\tau_{ji} > 0$ for all i, j such that $e_{ji} \in \mathscr{E}$. Using the same arguments as in the proof of Theorem 4, we see that all solutions converge to the set S, where $\tilde{v}_t(s) = v_t(s) - v^* = 0$ for all $s \in [-\tau, 0]$ and all $t > 0$. The maximal invariant set M in S requires in addition that $r_{ji}(t) = r_i(t) - r_j(t) = d_{ji}$ for any t. Hence, flocking is asymptotically attracting. ■

Surprisingly, the controller (12) achieves flocking of the current states, i.e., $r_i(t) - r_j(t) \to d_{ji}$, by comparing states at different points of time, namely, $r_i(t)$ and $r_j(t - \tau_{ji})$. This is achieved by introducing the additional term $v^*\tau_{ji}$, which "predicts" the position of the neighbor. This requires obviously that the communication delays τ_{ji} are known.

As in the previous section, we emphasize that the algorithm is completely decentralized, meaning that each agent only needs knowledge of the delays to its neighbors in order to perform the control task. As before, the identity $k_{ji} = k_{ij}$ can be achieved by a fixed rule depending on c and $\tau_{ji} = \tau_{ij}$ or by communicating with each neighbor.

VII. Simulation Example

We illustrate our results on a simulation example. Consider a set of four robots with dynamics (2) with $c = 1$. The robots exchange information via a communication network with homogeneous, constant delay $\tau > 0$. The normalized adjacency matrix is

$$\tilde{A} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{16}$$

i.e., the robots are communicating using a star topology with agent 1 in the center. Note that the eigenvalues of $\tilde{A}$ are $\lambda_{1,2}(\tilde{A}) = \pm 1$ and $\lambda_{3,4}(\tilde{A}) = 0$. The task for the four robots is to rendezvous. Therefore, we apply control (3) with $k_i = k$.

First, we are interested in a delay-independent rendezvous. From Section IV, we know that rendezvous is achieved for any τ if $k < 0.5$. For this particular system with homogeneous delays, we are able to calculate the exact delay-dependent stability bound using the frequency-sweeping test, see [33]:

$$\bar{\tau} = \begin{cases} \frac{1}{\sqrt{2k-1}} \left(\pi - \arctan\left(\frac{\sqrt{2k-1}}{1-k} \right) \right) & \text{for } k < 1, \\ \frac{\pi}{2} & \text{for } k = 1, \\ \frac{1}{\sqrt{2k-1}} \left(\arctan\left(\frac{\sqrt{2k-1}}{k-1} \right) \right) & \text{for } k > 1. \end{cases} \tag{17}$$

Details are omitted due to lack of space. With Theorem 4, we obtain $k < \frac{1}{\tau}$. The resulting stability curves are depicted in Figure 1. The shaded area indicates delay-independent stability. The solid curve is the delay bound (17). The dashed line shows the delay bound from Theorem 4. Note that for this example, Theorem 4 gives a quite accurate delay bound for $\bar{\tau} \geq 2$.

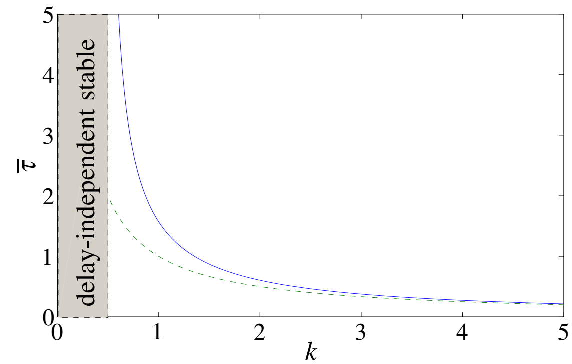
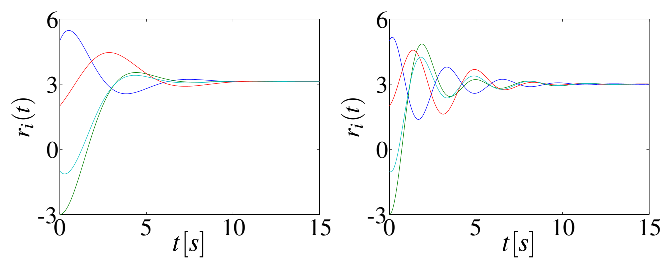


Fig. 1. Delay bound of Equation (17) (solid line) and of Theorem 4 (dashed line) for MAS (2) with $c = 1$ and control (3) with respect to position gain k.



(a) Rendezvous with $k = 0.5$. (b) Rendezvous with $k = 2$.

Fig. 2. Simulation result for rendezvous with initial condition φ.

In Figure 2, some exemplary simulations results are shown for the considered system with constant initial condition $\varphi(s) = \begin{bmatrix} 5 & 2 & -3 & 0 & 2 & 1 & -1 & -1 \end{bmatrix}^T$, $s \in [-\tau, 0]$, and communication delay $\tau = 0.1$. Figure 2(a) shows the simulations for $k = 0.5$, i.e., if the position gain just reaches the bound of delay-independent stability. Figure 2(b) shows the simulations for $k = 2$. Note that the system is delay-dependent stable for $\tau < 10$ according to Theorem 4 and $\tau < 10.33$ according to (17). The simulations for $k = 2$ show stronger oscillations but a much shorter rise time; the settling time is roughly the same for both cases.

Summarizing this example, we see that the delay-dependent rendezvous conditions derived in this paper is little conservative and at the same time easy to compute.

VIII. Conclusion

We presented delay-dependent rendezvous and flocking conditions for second order MAS with communication delays. Thereby, we assumed that the communication network is connected and that the communication delays are constant, heterogeneous, and symmetric. The virtue of the conditions is that they are totally decentralized, i.e., each subsystem may tune its position gain according to a simple rule depending on the delay to its neighbors. In particular, it is not necessary to know the delays or a delay bound of the complete communication network.

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