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A Small-Gain Condition for Integral Input-to-State Stability of Interconnected Retarded Nonlinear Systems

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Abstract— **In this paper, interconnected retarded nonlinear systems are considered. Both the constant discrete and distributed time-delays in the subsystems and the interconnections are addressed. A sufficient small-gain type condition for integral input-to-state stability with respect to external inputs is provided in the framework of Lyapunov-Krasovskii functionals.**

I. INTRODUCTION

Time delay has been a major topic in the area of systems control since it is often a source of instability. Delays are unavoidable in practice for interconnected systems and networks which are spatially distributed. This paper addresses the stability of interconnected retarded nonlinear systems with discrete as well as distributed time-delays in both the subsystems and the interconnecting channels. A small-gain condition is proposed to verify integral input-to-state stability with respect to external inputs. The condition may result providing new delay-dependent or delay-independent stability criteria.

In the literature of nonlinear delay-free control theory, a great deal of effort has been put into the problem of finding useful conditions under which interconnected systems are stable. A major development which plays an important role in nonlinear system analysis and design is the input-to-state stable small-gain theorem (see [9], [20], [18]), which is applicable to the interconnection of input-to-state stable (ISS) subsystems. In the paper [7], small-gain type theorems for interconnected systems involving integral input-to-state stable (iISS) subsystems are derived from the state dependent scaling formulation.

Small-gain type conditions for interconnected nonlinear delay-free systems, with time-delays in the interconnecting channels, have been considered in [3], [16] recently. In [3] delay-independent stability of a feedback interconnection of nonlinear delay-free systems with finite $\mathcal{L}_2$-gain are studied. Time-varying delays are also considered. In [16] a trajectory-based version of the input-to-output stability small-gain theorem is presented. In [4], a small gain theorem for monotone systems without external signals is developed in an abstract Banach space setting, which can yield global attractivity of time-delay systems as a special case under the assumption of the existence of bounded trajectories. An IOS/ISS small-gain theorem has recently been introduced in [10] for a wide class of time-varying systems which may include hybrid, impulsive and retarded systems. In the literature of time-delay systems, the idea of decomposing a single system into an interconnection of dynamic and static systems has been also popular, and small-gain arguments have also been utilized there. In [2], the relationship between the Lyapunov-Krasovskii method and a delay-independent version of the small-gain type criterion is studied for stability of systems with time-delay and memoryless linear growth nonlinearity. For linear time-delay systems, the equivalence of several Lyapunov-based stability conditions and the application of the scaled small-gain technique to an uncertain comparison system is shown in [22]. The effects of time-delays and disturbances were treated in [21], where a Razumikhin-type theorem that guarantees ISS of retarded nonlinear systems is established using the ISS small-gain theorem. In [12] the ISS small gain arguments are applied to the problem of stabilization of nonlinear systems in the presence of quantization and bounded communication delays. In [15], Lyapunov criteria employing functionals which provide sufficient conditions for the ISS and the iISS of retarded systems are given.

In this paper, we extend considerably the results in [7] to more general systems with time-invariant non-commensurate discrete as well as distributed time delays. With respect to [10], we focus attention on Lyapunov-Krasovskii functionals and iISS property of retarded systems. This paper provides a small-gain type condition for retarded systems which are the feedback interconnection of retarded subsystems admitting suitable storage functionals and supply rates. The main features of the result are:

(i) discrete as well as distributed time-delays can appear not only in the interconnecting channels, but also in the individual subsystems;

(ii) Lyapunov-Krasovskii functionals characterizing stability of interconnected systems are constructed, on the basis of Lyapunov-Krasovskii functionals for each subsystem;

(iii) interconnections of iISS subsystems are considered;

(iv) we address stability with respect to external signals, and include global asymptotic stability as a special case.

To the best of the authors' knowledge, there have been no studies achieving all the four points. It is widely known that the notion of ISS introduced by Sontag in [17] allows us to remove too restrictive requirements of finite $\mathcal{L}_2$-gain systems and the $\mathcal{L}_2$-gain technique. The class of ISS systems is a strict subset of iISS systems which can cover a much larger class of practically important systems than the ISS ones [1]. The characterization of ISS and iISS in terms of

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Lyapunov-Krasovskii functionals developed in [15] is here exploited. Proofs are omitted due to the space limitation.

NOTATION: The symbol $|\cdot|$ stands for the Euclidean norm of a real vector. The interval $[0,\infty)$ in the space of real numbers $\mathbb{R}$ is denoted by $\mathbb{R}_+$. A measurable function $u:\mathbb{R}_+\to\mathbb{R}^m$, m positive integer, is said to be essentially bounded if $ess\sup_{t\geq 0}|u(t)|<+\infty$, where $ess\sup_{t\geq 0}|u(t)|=\inf\{a\in[0,+\infty]:\lambda(\{t\in\mathbb{R}_+:|u(t)|>a\})=0\}$, λ denoting the Lebesgue measure. For a measurable and essentially bounded function $u:\mathbb{R}_+\to\mathbb{R}^m$, $\|u\|_\infty=ess\sup_{t\geq 0}|u(t)|$. For given times $0\leq T_1<T_2$, we indicate with $u_{[T_1,T_2)}:\mathbb{R}_+\to\mathbb{R}^m$ the function given by $u_{[T_1,T_2)}(t)=u(t)$ for all $t\in[T_1,T_2)$ and $=0$ elsewhere. A function u is said to be *locally essentially bounded* if, for any $T>0$, $u_{[0,T)}$ is essentially bounded. A function $\omega:\mathbb{R}_+\to\mathbb{R}_+$ is said to be positive semidefinite and denoted by $\omega\in\mathcal{P}_0$ if it is continuous and satisfies $\omega(0)=0$. A function $\omega\in\mathcal{P}_0$ is said to be positive definite if $\omega(s)>0$ holds for all $s>0$, and written as $\omega\in\mathcal{P}$. A function is of class $\mathcal{K}$ if it belongs to $\mathcal{P}$ and is strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. A function $\beta:\mathbb{R}_+^2\to\mathbb{R}_+$ is of class $\mathcal{KL}$ if for each fixed t the function $s\to\beta(s,t)$ is of class $\mathcal{K}$ and for each fixed s the function $t\to\beta(s,t)$ is non-increasing and goes to zero as $t\to+\infty$. For given (maximum involved time-delay) $\Delta>0$, n_1, n_2 positive integers, $\mathcal{C}_i$, $i=1,2$, denote the spaces of continuous functions mapping the interval $[-\Delta,0]$ into $\mathbb{R}^{n_i}$ and for $\phi_i\in\mathcal{C}_i$, $\|\phi_i\|_\infty=\sup_{-\Delta\leq\theta\leq 0}|\phi_i(\theta)|$. $\mathcal{C}$ denotes the space of continuous functions mapping the interval $[-\Delta,0]$ into $\mathbb{R}^{n_1+n_2}$ and, again, for $\phi\in\mathcal{C}$, $\|\phi\|_\infty=\sup_{-\Delta\leq\theta\leq 0}|\phi(\theta)|$. Let $M_{a,i}:\mathcal{C}_i\to\mathbb{R}_+$ and $M_a:\mathcal{C}\to\mathbb{R}_+$, $i=1,2$, be continuous functionals such that there exist $\underline{\gamma}_{a,i}$, $\overline{\gamma}_{a,i}$, $\underline{\gamma}_a$, $\overline{\gamma}_a\in\mathcal{K}_\infty$ such that

$$\underline{\gamma}_{a,i}(|\phi_i(0)|)\leq M_{a,i}(\phi_i)\leq\overline{\gamma}_{a,i}(\|\phi_i\|_\infty),\ \forall\phi_i\in C_i \quad (1)$$

$$\underline{\gamma}_a(|\phi(0)|)\leq M_a(\phi)\leq\overline{\gamma}_a(\|\phi\|_\infty),\qquad\forall\phi\in\mathcal{C} \quad (2)$$

hold. For any continuous function $x_i(s)$, $i=1,2$, defined on $-\Delta\leq s<a$, $a>0$, and any fixed t, $0\leq t<a$, the standard symbol $x_{i,t}$ will denote the element of $\mathcal{C}_i$ defined by $x_{i,t}(\theta)=x_i(t+\theta)$, $-\Delta\leq\theta\leq 0$.

II. PRELIMINARIES

Let us consider an interconnected system described by

$$\begin{aligned}\dot{x}_1(t)&=f_1(x_{1,t},x_{2,t},r_1(t)),\\ \dot{x}_2(t)&=f_2(x_{2,t},x_{1,t},r_2(t)),\\ x_{1,0}&=\xi_{1,0},\ x_{2,0}=\xi_{2,0},\end{aligned}\quad (3)$$

where, for $i=1,2$, $x_i(t)\in\mathbb{R}^{n_i}$; $r_i(t)\in\mathbb{R}^{m_i}$ is an external input (measurable, locally essentially bounded); n_i and m_i are positive integers; $\xi_{i,0}\in\mathcal{C}_i$; $f_i:\mathcal{C}_i\times\mathcal{C}_{3-i}\times\mathbb{R}^{m_i}\to\mathbb{R}^{n_i}$ is a functional which is Lipschitz on any bounded set[1]. We use the notation combining vectors such as $x(t)=[x_1(t)^T,x_2(t)^T]^T\in\mathbb{R}^n$, $r(t)=[r_1(t)^T,r_2(t)^T]^T\in\mathbb{R}^m$, $\xi_0=[\xi_{1,0}^T,\xi_{2,0}^T]^T\in\mathcal{C}$, $f()=[f_1()^T,f_2()^T]^T$, $\phi=[\phi_1^T,\phi_2^T]^T\in\mathcal{C}$. It is assumed that $f_i(0,0,0)=0$, $i=1,2$, thus ensuring that $x(t)=0$ is the solution corresponding to zero input and zero initial conditions (i.e. the trivial solution). Note that the formulation (3) accepts non-commensurate discrete as well as distributed time-delays not only in the interconnecting channels, but also in the individual subsystems. This paper does not assume the exact knowledge of functionals f_i, $i=1,2$. Instead, we associate each subsystem described by f_i with supply rates and assume the knowledge of dissipation inequalities as follows:

Assumption 1: There exist locally Lipschitz functionals $V_i:\mathcal{C}_i\to\mathbb{R}_+$, $i=1,2$, such that

$$\underline{\alpha}_i(M_{a,i}(\phi_i))\leq V_i(\phi_i)\leq\overline{\alpha}_i(M_{a,i}(\phi_i)), \quad (4)$$

$$D^+V_i(\phi_i,\phi_{3-i},r_i)\leq\rho_i(\phi_i,\phi_{3-i},r_i), \quad (5)$$

$$\forall\ \phi_j\in\mathcal{C}_j,\ j=1,2,\forall\ r_i\in\mathbb{R}^{m_i},$$

where

$$D^+V_i(\phi_i,\phi_{3-i},r_i)=\limsup_{h\to 0^+}\frac{V_i(\phi_i^h)-V_i(\phi_i)}{h} \quad (6)$$

$$\phi_i^h(s)=\begin{bmatrix}\phi_i(s+h), & s\in[-\Delta,-h),\\ \phi_i(0)+(s+h)f_i(\phi_i,\phi_{3-i},r_i), & s\in[-h,0];\end{bmatrix}$$

$\underline{\alpha}_i,\overline{\alpha}_i$ are $\mathcal{K}_\infty$ functions, and $\rho_i:\mathcal{C}_i\times\mathcal{C}_{3-i}\times\mathbb{R}^{m_i}\to\mathbb{R}$ are continuous functionals satisfying $\rho_i(0,0,0)=0$.

Each subsystem described by f_i is said to be dissipative with respect to storage functional V_i and supply rate ρ_i.

For a locally Lipschitz functional $V_{cl}:\mathcal{C}\to\mathbb{R}_+$, $D^+V_{cl}(\phi,r)$, where $\phi\in\mathcal{C}$, $r\in\mathbb{R}^m$, is defined as follows:

$$D^+V_{cl}(\phi,r)=\limsup_{h\to 0^+}\frac{V_{cl}(\phi^h)-V_{cl}(\phi)}{h},$$

$$\phi^h(s)=\begin{bmatrix}\phi(s+h), & s\in[-\Delta,-h),\\ \phi(0)+(s+h)f(\phi,r), & s\in[-h,0].\end{bmatrix}$$

Using this derivative, in this paper, the interconnected system (3) is said to be dissipative with respect to the storage functional V_{cl} and a supply rate ρ_{cl} if

$$D^+V_{cl}(\phi,r)\leq\rho_{cl}(\phi_1,\phi_2,r_1,r_2),\quad\forall\phi\in\mathcal{C},\ \forall r\in\mathbb{R}^m \quad (7)$$

holds for a continuous functional $\rho_{cl}:\mathcal{C}_1\times\mathcal{C}_2\times\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\to\mathbb{R}$ satisfying $\rho_{cl}(0,0,0,0)=0$. The final goal of this paper is the study of iISS and ISS properties of the interconnected system (3). These properties can be characterized as special cases of the dissipativity introduced above. For this sake, we borrow the following definitions from [17], [1], [15].

Definition 1: If the solution $x(t)$ of the interconnected system (3) exists for all $t\geq 0$ and furthermore satisfies

$$|x(t)|\leq\beta(\|x_0\|_\infty,t)+\gamma_r(\|r_{[0,t)}\|_\infty), \quad (8)$$

for all $t\geq 0$, with $\beta\in\mathcal{KL}$, $\gamma_r\in\mathcal{K}$, the system (3) is said to be ISS with respect to input r and state x.

Definition 2: If the solution $x(t)$ of the interconnected system (3) exists for all $t\geq 0$ and furthermore satisfies

$$\chi(|x(t)|)\leq\beta(\|\xi_0\|_\infty,t)+\int_0^t\gamma_r(|r(\tau)|)d\tau, \quad (9)$$

for all $t\geq 0$, with $\beta\in\mathcal{KL}$, $\chi\in\mathcal{K}_\infty$, $\gamma_r\in\mathcal{K}$, the system (3) is said to be iISS with respect to input r and state x.

Definition 3: A locally Lipschitz functional $V_{cl}:\mathcal{C}\to\mathbb{R}_+$ such that the inequalities

$$\underline{\alpha}_{cl}(|\phi(0)|)\leq V(\phi)\leq\overline{\alpha}_{cl}(M_a(\phi)),\forall\phi\in\mathcal{C}, \quad (10)$$

$$D^+V_{cl}(\phi,r)\leq-\alpha_{cl}(M_a(\phi))+\sigma_r(|r|),\forall\phi\in\mathcal{C},r\in\mathbb{R}^m \quad (11)$$

[1]To guarantee that the system (3) admits a unique maximal solution $x(t)$ which is locally absolutely continuous[5], [11].

hold for some $\underline{\alpha}_{cl}$, $\overline{\alpha}_{cl}$, $\alpha_{cl} \in \mathcal{K}_\infty$, $\sigma_r \in \mathcal{P}_0$ is said to be an ISS Lyapunov-Krasovskii functional for system (3).

Definition 4: A locally Lipschitz functional $V_{cl} : \mathcal{C} \to \mathbb{R}_+$ such that the inequalities

$$\underline{\alpha}_{cl}(M_a(\phi)) \le V(\phi) \le \overline{\alpha}_{cl}(M_a(\phi)), \forall\phi \in \mathcal{C}, \quad (12)$$

$$D^+V_{cl}(\phi, r) \le -\alpha_{cl}(M_a(\phi)) + \sigma_r(|r|), \forall\phi\in\mathcal{C}, r\in\mathbb{R}^m \quad (13)$$

hold for some $\underline{\alpha}_{cl}$, $\overline{\alpha}_{cl} \in \mathcal{K}_\infty$, $\alpha_{cl} \in \mathcal{P}$, $\sigma_r \in \mathcal{P}_0$ is said to be an iISS Lyapunov-Krasovskii functional for system (3).

The following theorem is given in [15] (functionals are chosen locally Lipschitz according to results in [13], [14]).

Theorem 1: If there exists an ISS (iISS) Lyapunov-Krasovskii functional for system (3), then system (3) is ISS (iISS, respectively) with respect to input r and state x.

III. DISSIPATIVITY

This section shows that the problem of verifying dissipativity of retarded interconnected systems can be formulated into the following.

Problem 1: Given locally Lipschitz functionals $V_i : \mathcal{C}_i \to \mathbb{R}_+$ and continuous functionals $\rho_i : \mathcal{C}_i \times \mathcal{C}_{3-i} \times \mathbb{R}^{m_i} \to \mathbb{R}$, find continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\lambda_i(s) > 0 \;\forall\; s \in (0, +\infty), \quad \int_1^{+\infty} \lambda_i(s)ds = +\infty, \quad (14)$$

for $i = 1, 2$ such that

$$\begin{aligned} \lambda_1(V_1(\phi_1))\rho_1(\phi_1, \phi_2, r_1) + \lambda_2(V_2(\phi_2))\rho_2(\phi_2, \phi_1, r_2) \\ \le \rho_e(\phi_1, \phi_2, r_1, r_2) \\ \forall\phi_j \in \mathcal{C}_j,\ \forall r_j \in \mathbb{R}^{m_j},\ j = 1, 2 \end{aligned} \quad (15)$$

holds for some continuous functional $\rho_e : \mathcal{C}_1 \times \mathcal{C}_2 \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}$ satisfying

$$\begin{aligned} \rho_e(\phi_1, \phi_2, 0, 0) \le \\ -\rho(|\phi_1(0)| + |\phi_2(0)|) \;\forall\phi_1\in\mathcal{C}_1, \phi_2\in\mathcal{C}_2 \end{aligned} \quad (16)$$

with a function $\rho \in \mathcal{P}$.

When $\mathcal{C}_1$ and $\mathcal{C}_2$ are replaced by $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, the functionals ρ_1, ρ_2 and ρ_e become functions and Problem 1 reduces to the problem proposed by [7] for delay-free systems. The following lemma confirms a chain-type rule for the upper bound of the upper right-hand derivative of a composite mapping, which links Problem 1 to Lyapunov-Krasovskii functionals of retarded interconnected system (3).

Lemma 1: Given a locally Lipschitz functional $V_i : \mathcal{C}_i \to \mathbb{R}_+$ and a continuous function $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, let $W_i : \mathcal{C}_i \to \mathbb{R}_+$ be a continuous functional defined as $W_i(\phi_i) = \int_0^{V_i(\phi_i)} \lambda_i(s)ds$. Then,

$$D^+W_i(\phi_i, \phi_{3-i}, r_i) \le \lambda_i(V_i(\phi_i))D^+V_i(\phi_i, \phi_{3-i}, r_i), \quad (17)$$

where

$$D^+W_i(\phi_i, \phi_{3-i}, r_i) = \limsup_{h\to 0^+} \frac{W_i(\phi_i^h) - W_i(\phi_i)}{h} \quad . \quad (18)$$

We are now in position to link Problem 1 to dissipativity associated with global asymptotic stability of the interconnected system (3).

Theorem 2: If there is a solution $\{\lambda_1, \lambda_2\}$ to Problem 1, then the trivial solution of the interconnected system (3) with $r(t) \equiv 0$ is globally asymptotically stable. Moreover, the locally Lipschitz functional $V_{cl} : \mathcal{C} \to \mathbb{R}_+$ defined as

$$V_{cl}(\phi) = \int_0^{V_1(\phi_1)} \lambda_1(s)ds + \int_0^{V_2(\phi_2)} \lambda_2(s)ds \quad (19)$$

is such that, for some $\underline{\alpha}_{cl}, \overline{\alpha}_{cl} \in \mathcal{K}_\infty$,

$$\begin{aligned} \underline{\alpha}_{cl}(M_{a,1}(\phi_1) + M_{a,2}(\phi_2)) \le V_{cl}(\phi) \le \\ \overline{\alpha}_{cl}(M_{a,1}(\phi_1) + M_{a,2}(\phi_2)) \end{aligned} \quad (20)$$

holds, and the interconnected system (3) satisfies

$$D^+V_{cl}(\phi, r) \le \rho_e(\phi_1, \phi_2, r_1, r_2),\ \forall\phi \in \mathcal{C}, \forall r \in \mathbb{R}^m. \quad (21)$$

The extension to local stability properties is possible by removing the second condition in (14). In fact, without the second condition in (14), we can still obtain Lemma 1 and Theorem 2 with $\underline{\alpha}_{cl} \in \mathcal{K}$.

Some dissipativity properties which are stronger than GAS, such as ISS and iISS, are not guaranteed by (16). The next section elaborates on this point and shows another Lyapunov-Krasovskii functional replacing (19).

IV. MAIN RESULTS: iISS

In order to derive the iISS property, this section solves Problem 1 so that the supply rate ρ_e of the interconnected system (3) is in the form of (13). Since the subsystems only satisfy Assumption 1, we cannot manipulate the supply rates of the subsystems as freely as we can do with delay-free ISS subsystems [19]. However, we may be still able to obtain a particular supply rate which is useful in achieving our goal. The following lemma provides us with such a way.

Lemma 2: Consider, for $i = 1, 2$,

$$\alpha_i, \sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{i,h_i} \in \mathcal{K}, \qquad \sigma_{r,i} \in \mathcal{P}_0, \quad (22)$$

with h_i a positive integer, and a non-decreasing continuous function $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$. Assume that

$$\lim_{s\to\infty} \alpha_i(s) < \infty \;\Rightarrow\; \lim_{s\to\infty} \lambda_i(s) < \infty \quad (23)$$

holds. If functionals $V_i, M_{a,i} : \mathcal{C}_i \to \mathbb{R}_+$ satisfy

$$\underline{\alpha}_i(M_{a,i}(\phi_i)) \le V_i(\phi_i) \le \overline{\alpha}_i(M_{a,i}(\phi_i)), \quad \forall\phi_i \in \mathcal{C}_i, \quad (24)$$

for some $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$, it holds that

$$\begin{aligned} \lambda_i(V_i(\phi_i)) \left\{ -\alpha_i(M_{a,i}(\phi_i)) + \sum_{j=1}^{h_i} \sigma_{i,j}(w_{i,j}) + \sigma_{r,i}(z_i) \right\} \\ \le -\tilde{\alpha}_i(M_{a,i}(\phi_i)) + \sum_{j=1}^{h_i} \tilde{\sigma}_{i,j}(w_{i,j}) + \tilde{\sigma}_{r,i}(z_i) \\ \forall\phi_i \in \mathcal{C}_i,\ \forall w_{i,j}, z_i \in \mathbb{R}_+, j = 1, 2, \ldots, h_i, \end{aligned} \quad (25)$$

where $\tilde{\alpha}_i, \tilde{\sigma}_i \in \mathcal{K}$ and $\tilde{\sigma}_{ri} \in \mathcal{P}_0$ are

$$\tilde{\alpha}_i(s) = \delta\left(1 - \frac{1}{\tau_i}\right)\lambda_i(\underline{\alpha}_i(s))\alpha_i(s) \quad (26)$$

$$\tilde{\sigma}_{i,j}(s) = \begin{cases} \lambda_i(\theta_{i,j}(s))\sigma_{i,j}(s) \\ \qquad \text{if } \lim_{v\to\infty} \alpha_i(v) \ge \tau_i h_i \sigma_{i,j}(s) \\ \lim_{v\to\infty} \lambda_i(v)\sigma_{i,j}(s) \quad \text{otherwise} \end{cases} \quad (27)$$

$$\tilde{\sigma}_{ri}(s) = \begin{cases} \lambda_i(\theta_{ri}(s))\sigma_{ri}(s) \\ \qquad \text{if } \lim_{v\to\infty} \alpha_i(v) \ge \tau_{ri}\sigma_{ri}(s) \\ \lim_{v\to\infty} \lambda_i(v)\sigma_{ri}(s) \quad \text{otherwise} \end{cases} \quad (28)$$

defined with

$$\theta_{i,j}(s) = \overline{\alpha}_i \circ \alpha_i^{-1} \circ \tau_i h_i \sigma_{i,j}(s)$$
$$\theta_{ri}(s) = \overline{\alpha}_i \circ \alpha_i^{-1} \circ \tau_{ri} \sigma_{ri}(s)$$

for any $\delta \in (0,1)$ and $\tau_i, \tau_{ri} \in (1,\infty)$ satisfying

$$1 - \frac{1}{\tau_i} - \frac{1}{\tau_{ri}} \geq \delta \left(1 - \frac{1}{\tau_i}\right) \quad .$$

The next lemma provides particular functions $\tilde{\alpha}_i$ and $\tilde{\sigma}_{i,j}$ having a nice property by selecting λ_1 and λ_2.

Lemma 3: Given (22) and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ satisfying $\underline{\alpha}_i(s) \leq \overline{\alpha}_i(s)$, $\forall s \in \mathbb{R}_+$, for $i = 1, 2$, we assume that one of

(A1) $\lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \alpha_1(s) = \infty$
(A2) $\lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty$
(A3) $\lim_{s\to\infty} \sigma_1(s) < \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty$

holds, where

$$\sigma_i(s) = h_i \max_{j=1,2,\ldots,h_i} \{\sigma_{i,j}(s)\} \tag{29}$$

Consider $\tilde{\alpha}_i, \tilde{\sigma}_i \in \mathcal{K}$ and $\tilde{\sigma}_{ri} \in \mathcal{P}_0$ given by (26), (27) and (28). Let $M_{a,i}, M_{\sigma,3-i,j} : \mathcal{C}_i \to \mathbb{R}^+$, $i = 1, 2$, be functionals fulfilling

$$M_{a,1}(\phi_1) \geq M_{\sigma,2,j}(\phi_1),\ \forall \phi_1 \in \mathcal{C}_1,\ j = 1, 2, \ldots, h_2 \tag{30}$$
$$M_{a,2}(\phi_2) \geq M_{\sigma,1,j}(\phi_2),\ \forall \phi_2 \in \mathcal{C}_2,\ j = 1, 2, \ldots h_1 \tag{31}$$

If there exist $c_i > 1$, $i = 1, 2$ such that

$$c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \leq \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s),\ \forall s \in \mathbb{R}^+ \tag{32}$$

is satisfied, then there exist $\alpha_{cl,i} \in \mathcal{K}$, $i = 1, 2$, such that

$$\sum_{i=1}^{2} \Big\{ -\tilde{\alpha}_i(M_{a,i}(\phi_i)) + (1+\epsilon_i) \sum_{j=1}^{h_1} \tilde{\sigma}_i(M_{\sigma,i,j}(\phi_{3-i})) + \tilde{\sigma}_{r,i}(|r_i|) \Big\} \leq \sum_{i=1}^{2} \left\{ -\tilde{\alpha}_{cl,i}(M_{a,i}(\phi_i)) + \tilde{\sigma}_{r,i}(|r_i|) \right\},$$
$$\forall \phi_1 \in \mathcal{C}_1,\ r_1 \in \mathbb{R}^{m_1},\ \forall \phi_2 \in \mathcal{C}_2,\ r_2 \in \mathbb{R}^{m_2} \tag{33}$$

holds for each $\epsilon_i \in [0, c_i - 1)$ with

$$\lambda_1(s) = \left[\alpha_2 \circ \hat{\mu}_1^{-1} \circ \frac{1}{\tau_1} \alpha_1 \circ \overline{\alpha}_1^{-1}(s)\right] \left[\frac{1}{\tau_1} \alpha_1 \circ \overline{\alpha}_1^{-1}(s)\right]^{\psi} \tag{34}$$
$$\lambda_2(s) = \frac{k_2}{(k_2 - 1)} \sqrt{\frac{k_1}{\tau_1}} \left[\hat{\mu}_1 \circ \underline{\alpha}_2^{-1}(s)\right]^{\psi+1} \tag{35}$$

for any k_1, k_2, τ_1, δ, $\psi \in \mathbb{R}$ and any $\hat{\mu}_1 \in \mathcal{K}$ satisfying

$$\mu_i(s) = (1+\epsilon_i)\sigma_i(s), \quad k_i > 1, \quad i = 1, 2 \tag{36}$$
$$0 < \sqrt{\tau_1/k_1} < \delta < 1 \tag{37}$$
$$0 \leq \psi,\ 1 < \tau_1,\ \left(\frac{\tau_1}{k_1}\right)^{\psi} \leq (\tau_1 - 1)(k_2 - 1) \tag{38}$$
$$\mu_1(s) \leq \hat{\mu}_1(s),\ \forall s \in \mathbb{R}^+ \tag{39}$$
$$\lim_{s\to\infty} \alpha_1(s) \leq \lim_{s\to\infty} \hat{\mu}_1(s),\ \forall s \in \mathbb{R}^+ \tag{40}$$
$$k_1\hat{\mu}_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ k_2\mu_2(s) \leq \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s),\ \forall s \in \mathbb{R}^+ \tag{41}$$

Furthermore, $\tilde{\alpha}_{cl,1}, \tilde{\alpha}_{cl,2} \in \mathcal{K}_\infty$ holds if $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

The existence of k_1, k_2, τ_1, δ, $\psi \in \mathbb{R}$ and $\hat{\mu}_1 \in \mathcal{K}$ fulfilling (36)-(41) is guaranteed when (32) and one of *(A1)*, *(A2)* and *(A3)* is satisfied. The key point of Lemma 3 is that $\epsilon_i > 0$ is allowed in (33), which makes a remarkable contrast to the previous techniques for delay-free systems.

Remark 1: It is stressed that (32) with c_1, $c_2 > 1$ requires

$$\lim_{s\to\infty} \alpha_2(s) = \infty \quad \vee \quad \lim_{s\to\infty} \alpha_2(s) > \lim_{s\to\infty} \sigma_2(s) \tag{42}$$

Also note that (23) is fulfilled by (34) and (35) with the help of (40) when one of *(A1)*, *(A2)* and *(A3)* is satisfied.

The following theorem is the main result of this paper.

Theorem 3: Suppose that supply rate functionals ρ_i, $i = 1, 2$, are as follows:

$$\rho_i(\phi_i, \phi_{3-i}, r_i) = -\alpha_i(M_{a,i}(\phi_i)) + S_{i,0}\sigma_{i,0}(M_{a,3-i}(\phi_{3-i})) + \sum_{j=1}^{h} S_{i,j}\sigma_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(-\Delta_j)|)\right) + \sigma_{r,i}(|r_i|), \tag{43}$$

where h is a positive integer and, for $i = 1, 2$, α_i are functions of class $\mathcal{K}$, $\sigma_{r,i} \in \mathcal{P}_0$, $\Delta_j \in (0, \Delta]$, $j = 1, 2, \ldots, h$, $\sigma_{i,j}$, $j = 0, 1, \ldots, h$, are functions of class $\mathcal{K}$, $S_{i,k}$ belongs to $\{0, 1\}$, $k = 0, 1, \ldots, h$. Assume that one of the following three conditions holds:

(H1) $\lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \alpha_1(s) = \infty,$
(H2) $\lim_{s\to\infty} \alpha_2(s) = \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty,$
(H3) $\lim_{s\to\infty} \sigma_1(s) < \infty \quad \wedge \quad \lim_{s\to\infty} \sigma_2(s) < \infty,$

where σ_i are defined as

$$\sigma_i(s) = \left(\sum_{k=0}^{h} S_{i,k}\right) \max_{j=0,1,\ldots,h} S_{i,j}\sigma_{i,j}(s) \tag{44}$$

Suppose that there exist $c_i > 1$, $i = 1, 2$, such that

$$c_1\sigma_1 \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \alpha_2^{-1} \circ c_2\sigma_2(s) \leq \alpha_1 \circ \overline{\alpha}_1^{-1} \circ \underline{\alpha}_1(s),\ \forall s \in \mathbb{R}_+ \tag{45}$$

Then, the interconnected system (3) is iISS with respect to input r and state x. In addition, it is ISS with respect to input r and state x in the case of *(H1)*. Furthermore, an iISS (ISS in the *(H1)* case) Lyapunov-Krasovskii functional for (3) is

$$V_{cl}(\phi) = \int_0^{V_1(\phi_1)} \lambda_1(s)ds + \int_0^{V_2(\phi_2)} \lambda_2(s)ds + \sum_{j=1}^{h} \int_{-\Delta_j}^{0} F_{1,j}(\tau)S_{1,j}\tilde{\sigma}_{1,j}\left(\underline{\gamma}_{a,2}(|\phi_2(\tau)|)\right)d\tau + \sum_{j=1}^{h} \int_{-\Delta_j}^{0} F_{2,j}(\tau)S_{2,j}\tilde{\sigma}_{2,j}\left(\underline{\gamma}_{a,1}(|\phi_1(\tau)|)\right)d\tau \tag{46}$$

where λ_1, λ_2 and $\tilde{\sigma}_{i,j}$ are given in (34), (35) and (27), and $F_{i,j} : [-\Delta_j, 0] \to [1, 1+\epsilon_i]$ is defined for $0 < \epsilon_i < c_i - 1$ as

$$F_{i,j}(\tau) = \frac{-\tau}{\Delta_j} + (1+\epsilon_i)\frac{\tau + \Delta_j}{\Delta_j} \tag{47}$$

Theorem 3 is a natural generalization of the delay-free systems results in [7] to time-delay systems. In fact, as the maximum involved delay $\Delta \to 0$, taking into account the inequalities (1), the supply rate (43) reduces, for $x_i \in \mathbb{R}^{n_i}, r_i \in \mathbb{R}^{m_i}, i = 1, 2$, into

$$\rho_i(x_i, x_{3-i}, r_i) = -\hat{\alpha}_i(|x_i|) + \hat{\sigma}_i(|x_{3-i}|) + \sigma_{r,i}(|r_i|),$$
$$\hat{\alpha}_i(s) = -\alpha_i(\underline{\gamma}_{a,i}(s)),$$
$$\hat{\sigma}_i(s) = S_{i,0}\sigma_{i,0}\left(\overline{\gamma}_{a,3-i}(s)\right) + \sum_{j=1}^{h} S_{i,j}\sigma_{i,j}\left(\overline{\gamma}_{a,3-i}(s)\right)$$

The inequality (45) is a small-gain condition.

Time-invariant non-commensurate discrete as well as distributed time-delays in both the subsystems and the interconnecting channels can be covered effectively by the dissipative inequalities (5) and the supply rates (43). The inequality (45) by itself is independent of the delays. Thus, if the subsystems accept the supply rates (43), $i = 1, 2$, for arbitrary time delays, the stability condition given by Theorem 3 is delay-independent. It can be also intuitively understood that a delay-independent small-gain condition can guarantee iISS of the interconnection of two delay-free subsystems regardless of (discrete as well as distributed) delays in interconnecting channels.

Remark 2: Hypotheses on the supply rates (43) of Theorem 3 are sufficient conditions for each subsystem to be iISS with respect to input $(x_{3-i,t}, r_i)$ and state x_i. In fact, the choice (43) implies that

$$\rho_i = -\alpha_i(M_{a,i}(\phi_i)) + \sigma_i(\overline{\gamma}_{a,3-i}(\|\phi_{3-i}\|_\infty)) + \sigma_{r,i}(|r_i|) \quad (48)$$

is also a supply rate for the x_i subsystem, which yields

$$\chi_i(|x_i(t)|) \le \beta_i(\|\xi_{i,0}\|_\infty, t) + \int_0^t \gamma_i\left(\|x_{3-i,\tau}\|_\infty\right) d\tau + \int_0^t \gamma_{r,i}(|r_i(\tau)|) d\tau$$

for all $t \ge 0$ with $\beta_i \in \mathcal{KL}$, $\chi_i \in \mathcal{K}_\infty$, and γ_i, $\gamma_{r,i} \in \mathcal{K}$. In the case of *(H1)*, the x_i subsystem is ISS with respect to input $(x_{3-i,t}, r_i)$ and state x_i, i.e. ,

$$|x_i(t)| \le \beta_i(\|\xi_{i,0}\|_\infty, t) + \gamma_i\left(\sup_{\tau\in[0,t)} \|x_{3-i,\tau}\|_\infty\right) + \gamma_{r,i}(\|(r_i)_{[0,t)}\|_\infty)$$

for all $t \ge 0$ with $\beta_i \in \mathcal{KL}$ and γ_i, $\gamma_{r,i} \in \mathcal{K}$.

Remark 3: If the x_2 subsystem accepts a supply rate in the form of (43) with $S_{2,0} = S_{2,1} = ... = S_{2,h} = 0$, the interconnected system (3) becomes a cascade. The condition (45) can be always met by choosing small $\sigma_{2,j}(s)$'s.

Remark 4: If there exist positive numbers $G_{i,j}$, $j = 0, 1, ..., h$, such that $G_{3-i,j}\alpha_i = \sigma_{3-i,j}$ hold for each $i = 1, 2$ in (43), the ISS Lyapunov-Krasovskii functional of the whole system becomes simple. For example, if

$$\rho_i(\phi_i, \phi_{3-i}, r_i) = -(M_{a,i}(\phi_i))^p + S_{i,0}G_{i,0}(M_{a,3-i}(\phi_{3-i}))^p + \sum_{j=1}^{h} S_{i,j}G_{i,j}\left(\underline{\gamma}_{a,3-i}(|\phi_{3-i}(-\Delta_j)|)\right)^p + \sigma_{r,i}(|r_i|)$$

holds for $i = 1, 2$ with some $p > 0$, the ISS Lyapunov-Krasovskii functional (46) becomes

$$V_{cl}(\phi) = V_1(\phi_1) + \lambda_2 V_2(\phi_2) + \sum_{j=1}^{h}\int_{-\Delta_j}^{0} F_{1,j}S_{1,j}G_{1,j}\left(\underline{\gamma}_{a,2}(|\phi_2(\tau)|)\right)^p d\tau + \lambda_2\sum_{j=1}^{h}\int_{-\Delta_j}^{0} F_{2,j}(\tau)S_{2,j}G_{2,j}\left(\underline{\gamma}_{a,1}(|\phi_1(\tau)|)\right)^p d\tau,$$

where λ_2 is any positive real number satisfying $G_1 < \lambda_2 < 1/G_2$ and $G_i = \sum_{k=0}^{h} S_{i,k}G_{i,k}$. Note that the existence of $c_i > 1$, $i = 1, 2$ fulfilling (45) can be replaced by $G_1G_2 < 1$ in this case.

Remark 5: In the absence of the external signals r_1 and r_2, the Lyapunov-Krasovskii functional can be chosen as (46) with $\epsilon_1 \ge 0$ and $\epsilon_2 \ge 0$.

V. Illustrative Examples

Example 1) Consider

$$\dot{x}_1(t) = \frac{1}{1 + x_1^2(t)}\left(-x_1(t) + x_2(t - \Delta_1) + r_1(t)\right)$$
$$\dot{x}_2(t) = -\gamma x_2(t) + x_2(t - \Delta_2) + \frac{x_1(t - \Delta_3)}{1 + x_1^2(t - \Delta_3)} \quad (49)$$

where $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$, $r_1(t) \in \mathbb{R}$, and γ, Δ_i, $i = 1, 2, 3$ are positive reals. Let, for $\phi_1 \in \mathcal{C}_1, \phi_2 \in \mathcal{C}_2$,

$$V_1(\phi_1) = \phi_1^2(0),$$
$$V_2(\phi_2) = \phi_2^2(0) + \int_{-\Delta_2}^{0}\left(\frac{-\tau}{\Delta_2} + (1+\delta)\frac{\tau + \Delta_2}{\Delta_2}\right)\phi_2^2(\tau) d\tau \quad (50)$$

where a positive real δ has yet to be chosen. Define

$$M_{a,1}(\phi_1) = \phi_1^2(0), \ M_{a,2}(\phi_2) = \phi_2^2(0) + \int_{-\Delta_2}^{0} \phi_2^2(\tau) d\tau \quad (51)$$

Then, $\underline{\alpha}_1(s) = \overline{\alpha}_1(s) = s$, $\underline{\alpha}_2(s) = s$, $\overline{\alpha}_2(s) = (1 + \delta)s$ and $\underline{\gamma}_{a,1}(s) = \overline{\gamma}_{a,1}(s) = s^2$, $\underline{\gamma}_{a,2}(s) = s^2$, $\overline{\gamma}_{a,2}(s) = (1 + \Delta_2)s^2$. Using the Young's inequality, we obtain

$$D^+V_1 \le (-1+\epsilon)\frac{\phi_1^2(0)}{1+\phi_1^2(0)} + \frac{\phi_2^2(-\Delta_1)}{1+\phi_1^2(0)} + \frac{r_1^2}{\epsilon(1+\phi_1^2(0))},$$

for $1 > \epsilon > 0$. Thus, the x_1 subsystem is iISS with respect to input (x_2, r_1) and state x_1. A supply rate is (43), with $h = 3$ (number of all discrete delays in the overall system), $S_{1,0} = S_{1,2} = S_{1,3} = 0$, $S_{1,1} = 1$, $\alpha_1(s) = (1 - \epsilon)\frac{s}{1+s}$, $\sigma_{1,1}(s) = s$, $\sigma_{r,1}(s) = \frac{1}{\epsilon}s^2$. The Young's inequality also yields

$$D^+V_2 \le (-2\gamma + 3 + \delta)\phi_2^2(0) + \frac{\phi_1^2(-\Delta_3)}{1 + \phi_1^2(-\Delta_3)} - \frac{\delta}{\Delta_2}\int_{-\Delta_2}^{0} \phi_2^2(\tau) d\tau$$

Thus, if $\gamma > \frac{3+\delta}{2}$, the x_2 subsystem is ISS with respect to input x_1 and state x_2. A supply rate is (43), with $S_{2,0} = S_{2,1} = S_{2,2} = 0$, $S_{2,3} = 1$, $\alpha_2(s) = \min\left\{2\gamma - 3 - \delta, \frac{\delta}{\Delta_2}\right\} s$, $\sigma_{2,3}(s) = \frac{s}{1+s}$. The condition (45) is satisfied if $(1 + \delta)/\min\left\{2\gamma - 3 - \delta, \frac{\delta}{\Delta_2}\right\} < 1 - \epsilon$. Since ϵ can be chosen arbitrarily small, we conclude that system (49) is iISS with respect to input r_1 and state x, provided that there exists a positive real δ such that the system parameters γ, Δ_2 satisfy

$$1 + \delta < \min\left\{2\gamma - 3 - \delta, \frac{\delta}{\Delta_2}\right\} \tag{52}$$

Whenever (52) is verified, for any value of positive reals Δ_1, Δ_3, the interconnected system (49) is iISS with respect to input r_1 and state x. For instance, $\gamma = 3, \Delta_2 = 0.2$ fulfill (52) with $\delta = 1/2$. An iISS Lyapunov functional of the system (49) is V_{cl} given in (46) with (27) and

$$\lambda_1 = 1, \quad \frac{1+\delta}{\min\left\{2\gamma - 3 - \delta, \frac{\delta}{\Delta_2}\right\}} < \lambda_2 < 1 \quad .$$

Example 2). Next, we consider

$$\begin{aligned}\dot{x}_1(t) &= -2x_1(t) + x_2(t)\\ \dot{x}_2(t) &= x_1(t) - 4x_2(t) - x_2(t)\frac{\rho^2(|x(t)|)}{1+|x_2(t)|\rho(|x(t)|)} + \\ &\qquad \gamma x_1(t-\Delta)x_2(t) + r_2(t),\end{aligned} \tag{53}$$

where $x_i(t) \in \mathbb{R}$, $\rho(s) = 4s + 4s^2 + s^3, s \geq 0$, $\gamma \in \mathbb{R}$ is a parameter, $\Delta \geq 0$ is a fixed time-delay. The system (53) is a particular time-invariant case of the more general time-varying control system studied in [6]. The authors of [6] investigated uniform ultimate boundedness of their time-varying system, while this paper introduces the disturbance r_2 into the time-invariant model to address stability with respect to actuator errors as well as global asymptotic stability. Let $V_1(\phi_1) = \phi_1^4(0)$, $V_2(\phi_2) = \phi_2^2(0)$, $M_{a,1}(\phi_1) = \phi_1^2(0)$, $M_{a,2}(\phi_2) = V_2(\phi_2)$, Thus $\underline{\alpha}_1(s) = \overline{\alpha}_1(s) = s^2$, $\underline{\alpha}_2(s) = \overline{\alpha}_2(s) = s$ and $\underline{\gamma}_{a,1}(s) = \overline{\gamma}_{a,1}(s) = \underline{\gamma}_{a,2}(s) = \overline{\gamma}_{a,2}(s) = s^2$. The following inequalities holds:

$$\begin{aligned}D^+V_1 &\leq -5\phi_1^4(0) + \phi_2^4(0)\\ D^+V_2 &\leq -\phi_2^2(0) + \frac{1}{7}\phi_1^2(0) + \frac{\gamma^2}{2}\phi_1^2(-\Delta_1) + \epsilon\phi_2^2(0) + \frac{r_2^2}{\epsilon}\end{aligned}$$

where $\epsilon > 0$ is arbitrary. Thus, we take $h = 1$, $S_{1,0} = 1$, $S_{1,1} = 0$, $S_{2,0} = 1$, $S_{2,1} = 1$ and

$$\begin{aligned}&\alpha_1(s) = 5s^2, \quad \alpha_2(s) = (1-\epsilon)s, \quad \sigma_{r_1}(s) = 0,\\ &\sigma_1(s) = s^2, \quad \sigma_2(s) = 2\left(\frac{1}{7} + \frac{\gamma^2}{2}\right)s, \ \sigma_{r,2}(s) = \frac{1}{\epsilon}s^2\end{aligned}$$

The condition (45) becomes in this case

$$4\left(\frac{1}{7} + \frac{\gamma^2}{2}\right)^2 < 5 \tag{54}$$

since ϵ is arbitrary. We can conclude, by Theorem 3, that, for any given value of the delay Δ, the resulting system (53) is ISS with respect to the disturbance r_2, provided that $|\gamma| < 1.3965$ holds. An ISS Lyapunov functional V_{cl} of the system (53) is (46) with (27) and

$$\lambda_1(s) = s^{\psi+0.5}, \quad \lambda_2(s) = ks^{2(\psi+1)}$$

for suitably chosen $k > 0$ and $\psi \geq 0$.

VI. Conclusion

This paper has proposed a small-gain condition for iISS of interconnected retarded nonlinear systems consisting of iISS subsystems. Constant discrete as well as distributed time-delays in both the subsystems and the interconnecting channels are covered by the formulation. Whether to result in a delay-dependent criterion or a delay-independent stability criterion depends on system structure and the choice of supply rates and storage functionals of subsystems. Lyapunov-Krasovskii functionals characterizing stability of interconnected systems are constructed from Lyapunov-Krasovskii functionals of individual subsystem.

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