Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Stability and causality constraints on frequency response coefficients applied for non-parametric $\mathscr{H}_2$ and $\mathscr{H}_\infty$ control synthesis.

A.J. den Hamer, S. Weiland, M. Steinbuch and G.Z Angelis

Abstract— **The value of norm-based control synthesis methodologies heavily depends on the quality of the model at hand. The acquisition of low-order control oriented models however is often a non-trivial task. This paper pursuits the non-parametric synthesis of optimal controllers while omitting parametrization of the plant. As a result, the actual parametrization is performed on the controller such that no data-reduction is performed without knowledge of closed-loop relevant behavior.**

The synthesis of frequency response coefficients of an optimal controller for a sampled version of the mixed-sensitivity problem is considered. To convexify the problem, the actual optimization is performed over the frequency response coefficients of the Youla parameter Q_i. Using the Youla parameter, the set of stabilizing controllers is mapped onto the set of stable transfer functions. The main contribution of this paper is the derivation of algebraic constraints over Q_i that guarantee the existence of a rational stable interpolant over the points Q_i. Simulations shows that the frequency response coefficients found via the proposed approach show similar behavior as model based methods.

I. Introduction

A vast theoretical framework of powerful and mathematically well developed norm-based control synthesis tools exist. However, the practical value of these methodologies heavily depends on the accuracy of the plant model at hand. In practice, it appears that control oriented modeling is often a non-trivial effort requiring iterative procedures [2], [4]. One of the reasons for these iterative procedures is the fact that the mismatch between plant and model is not directly coupled to performance loss of the closed-loop system in a straightforward manner.

To overcome this problem, several approaches are known [1], [5], [7], [9], [10], [14], [17] that pursuit controller synthesis based on plant input-output data directly, i.e. omitting plant identification. By doing so, order-reduction/data interpolation is applied during or after control synthesis such that knowledge of closed-loop relevant behavior is taken into account in the controller parametrization. This results in an approach which is more straightforward than model-based approaches since iterations of sequential modeling and controller synthesis are circumvented.

Nevertheless several drawbacks exist for controller synthesis based on plant input-output data. Controller synthesis using FIR predictors as described in [1], [5], [14] intrinsically suffers from a finite prediction horizon. This makes the proof of asymptotic stability a non-trivial task. Other elaborated techniques such as Quantitative Feedback Theory [9], [17] lack the ability to incorporate optimality in the sense of minimizing a closed-loop system norm.

The approach presented in this paper pursuits the synthesis of frequency response coefficients of a controller that minimizes a criterion that resembles a sampled $\mathscr{H}_\infty$ or $\mathscr{H}_2$ problem. To convexify the problem, the actual optimization is performed over the frequency response coefficients of the Youla parameter denoted by Q_i. Using the Youla parameter, the set of stabilizing controllers is mapped onto the set of stable transfer functions. The main contribution of this paper is the derivation of a set of algebraic conditions on the frequency response coefficients Q_i that guarantee the existence of a rational low-order interpolant through the frequency response coefficients that is causal and stable. Under this constraint, controller synthesis can be performed directly on the frequency response coefficients of the Youla parameter.

The outline of the paper is as follows: Sec.II introduces the control synthesis problem under consideration and introduces the Youla parameter $Q(s)$. Sec.III proves that stability and causality of the Youla parameter, i.e. closed-loop stability, can be guaranteed via an integral constraint on the frequency response function $Q(j\omega)$. Sec.IV describes the approximation of this integral constraint via a Riemann-sum. Convergence is proved and an upper-bound for the approximation error is given. In Sec.V and VI, the obtained algebraic condition on Q_i is applied for control synthesis.

II. Control synthesis framework

The control problem considered in this paper is the following:
Given evaluations of an unknown stable transfer function $P(s)$ at the frequency points $j\omega_i$ belonging to a discrete real-valued frequency grid $\omega_i \in \Omega,\ i = 1, \ldots, N$, synthesize complex numbers $C_i \in \mathbb{C},\ i = 1, \ldots, N$ such that:

- there exist a $C(s) := \mathscr{I}(C_1, \ldots, C_N)$ that internally stabilizes $P(s)$,
- $C(s) := \mathscr{I}(C_1, \ldots, C_N)$ solves the optimization problem given in (1) below,

where $\mathscr{I}$ is an interpolation map $\mathscr{I} : (\mathbb{C}^{1\times 1})^N \rightarrow \mathscr{R}_p(s)$ such that $C(s) := \mathscr{I}(C_1, \ldots, C_N)$ satisfies $C(j\omega_i) = C_i\ \forall\ \omega_i \in \Omega$. $\mathscr{R}_p(s)$ represents the class of proper rational transfer functions.

Without loss of generality, the SISO mixed-sensitivity control synthesis problem is considered in this paper to illustrate

This work is supported by PHILIPS Apptech

A.J. den Hamer and M. Steinbuch are with the Control Systems Technology Group, Eindhoven University of Technology, The Netherlands.
S. Weiland is with the Control Systems Group, Eindhoven University of Technology, The Netherlands.
G.Z. Angelis is with PHILIPS Apptech, The Netherlands.

978-1-4244-3124-3/08/$25.00 ©2008 IEEE

control synthesis via non-parametric stability constraints:

$$\min_{C(s)\in\mathcal{C}} \left\| \begin{bmatrix} W_1 S \\ W_2 R \end{bmatrix} \right\|_\infty \tag{1}$$

where $S \triangleq \frac{1}{1+PC}$ and $R \triangleq \frac{C}{1+PC}$ represent the sensitivity- and control sensitivity function respectively and $W_1, W_2 \in \mathscr{RH}_\infty$ the corresponding weighting filters. $\mathcal{C}$ represents the set of internally stabilizing rational controllers in $\mathscr{R}_p(s)$.

Using the Youla parametrization [6], [18], the optimization problem over the set of stabilizing controllers $\mathcal{C}$ is mapped into a convex optimization problem. Assume that $P(s)$ is stable. Then the Youla parametrization defines a bijection between $\mathcal{C}$ and a set $\mathcal{Q}$ in the sense that (negative feedback is assumed):

$$\mathcal{C} = \{C = \frac{Q}{1-PQ} \mid Q \in \mathcal{Q}\} \tag{2}$$

where:

$$\mathcal{Q} = \{\frac{C}{1+CP} \mid C \in \mathcal{C}\} \tag{3}$$

It is well known [6] that $\mathcal{Q} = \mathscr{RH}_\infty$. Substitution of $Q(s)$ into (1) gives:

$$\min_{Q\in\mathscr{RH}_\infty} \left\| \begin{bmatrix} W_1(1-PQ) \\ W_2 Q \end{bmatrix} \right\|_\infty \tag{4}$$

The key observation is that the optimization problem is now performed over the set of stable rational functions, $\mathscr{RH}_\infty$.

Since $P(j\omega_i) = P_i$ is only given on the grid points $\omega_i \in \Omega$, (4) is approximated by a sampled $\mathscr{H}_\infty$ problem:

$$\begin{aligned} \min_{(Q_1,\ldots,Q_N)} \max_{1\le i\le N} & \left\| \begin{bmatrix} W_1(j\omega_i)(1-P_iQ_i) \\ W_2(j\omega_i)Q_i \end{bmatrix} \right\| \\ \text{s.t.:} \quad & \exists\, \mathscr{I}(Q_1,\ldots,Q_N) \in \mathscr{RH}_\infty \end{aligned} \tag{5}$$

where $\|x\| = \sqrt{x^\dagger x}$, and $.^\dagger$ represents the complex conjugate. The optimization is performed over the points $(Q_1,\ldots,Q_N)$.

The translation of the condition: $\exists\, \mathscr{I}(Q_1,\ldots,Q_N) \in \mathscr{RH}_\infty$ into an algebraic constraint over Q_i is the main focus of this paper and will be discussed in Section III and Section IV. Once this condition can be guaranteed, the following algorithm is applied for controller synthesis:

1) the optimal solution of (5) is computed.
2) the resulting points Q_i are mapped to C_i via pointwise evaluation of (2), i.e. $C_i = \frac{Q_i}{1-P_iQ_i}$. The existence of $\mathscr{I}(Q_1,\ldots,Q_N) \in \mathscr{RH}_\infty$ implies the existence of $\mathscr{I}(C_1,\ldots,C_N) \in \mathcal{C}$.
3) define $C(s)$ as $C(s) = \mathscr{I}(C_1,\ldots,C_N)$.

The interpolation step described in 3) is not considered in this paper and is a topic of current research.

III. NON-PARAMETRIC STABILITY CONSTRAINTS

Contrary to model-based approaches where the poles of the model are directly accessible, it is less straightforward to guarantee stability of a transfer function $Q(s)$ based on a finite number of frequency evaluations $Q_i := Q(j\omega_i)$ of $Q(s)$.

Several integral relations exist that represent the analytical properties of a stable transfer function. It is well known in literature [3], [12], [13], [15] that the frequency response function of a proper stable rational function $H(s)$ satisfies the Hilbert transform. An alternative formulation of the Hilbert transform are the Kramers-Kronig equations, also known as the dispersion formulas or Plemelj formulas [12], that link the real and imaginary part of a frequency response function. These relations represent the analytical properties of a rational function and are closely related to the well known Bode gain phase relations [8].

Although several publications can be found on the relation between analyticity, causality and integral constraints [13], [15], these results appear scattered over several application domains, e.g. spectography and wave-analysis, and are not particulary focussed on control synthesis. As a result, stability of a system is often assumed rather than guaranteed by the relation. In this section a relation similar to the Kramers-Kronig equations will be derived that guarantees stability and causality of the transfer function under consideration.

Cauchy's residue theorem [11] is one of the fundamental relations in complex function theory. It relates evaluations of a transfer function $H(s)$ on a contour γ to the residues of the poles contained in γ via:

$$\oint_\gamma H(s)ds = 2\pi j \sum_{a_i\in\mathcal{A}(\gamma)} \text{Res}(H(s), a_i) \tag{6}$$

where the residue is defined as $\text{Res}(H(s), a_i) \triangleq \lim_{s\to a_i} H(s)\cdot(s-a_i)$. $\mathcal{A}(\gamma)$ represents the set of poles contained in the contour γ. γ is chosen to be a closed path.

Theorem 1: A transfer function $H(s)$ is stable and causal if and only if the real and imaginary part of $H(s)$ satisfy:

$$\text{Im}(H(j\omega_0)) = \frac{1}{\pi}\mathcal{P}\oint_\gamma \frac{\text{Re}(H(s))}{s-j\omega_0}ds \quad \forall\omega_0\in\mathbb{R} \tag{7}$$

where $\text{Re}(H(j\omega))$ and $\text{Im}(H(j\omega))$ represent the real and imaginary part of $H(j\omega)$ respectively. γ is chosen to be a closed path as depicted in Fig.1 with a sufficiently large radius of the semicircle.

$\mathcal{P}$ represents Cauchy's principle value, i.e. the evaluation of the contour integral while omitting the singularity at $j\omega_0$.

To prove Theorem 1, we need the following Lemma:

Lemma 2: Let p_i be the poles of $H(s)$ and $\alpha_i = \text{Res}(H(s), p_i)$ the corresponding residues, then the partial fraction expansion of the function:

$$F(s) = \frac{H(s)}{s-j\omega_0} = \sum_{i=1}^{n} \frac{\alpha_i}{(s-p_i)}\cdot\frac{1}{(s-j\omega_0)} \tag{8}$$

equals:

$$F(s) = H(s_0)\frac{1}{s-j\omega_0} - \sum_{i=1}^{n}\frac{\alpha_i}{(j\omega_0-p_i)}\cdot\frac{1}{s-p_i} \tag{9}$$

Proof: Let A_i and B_i represent the residues of the i^{th} term in (8): $\frac{\alpha_i}{s-p_i}\cdot\frac{1}{s-j\omega_0}$, then:

$$\frac{\alpha_i}{(s-j\omega_0)(s-p_i)} = \frac{A_i}{s-j\omega_0} + \frac{B_i}{s-p_i} \tag{10}$$

Equating the terms over the powers of s gives:

$$A_i = \frac{\alpha_i}{j\omega_0 - p_i}, \qquad B_i = -\frac{\alpha_i}{j\omega_0 - p_i} \tag{11}$$

Summing all terms A_i and B_i over $i = 1, \ldots, N$ gives:

$$F = \sum_{i=1}^{n} \frac{\alpha_i}{j\omega_0 - p_i} \cdot \frac{1}{s - j\omega_0} - \sum_{i=1}^{n} \frac{\alpha_i}{j\omega_0 - p_i} \cdot \frac{1}{s - p_i} \quad (12)$$

which finishes the proof of Lemma 2. ■

We now continue with the proof of Theorem 1.

Proof: (only if) (7) follows directly from stability and causality of $H(s)$ via Cauchy's residue theorem given in (6). Choose γ to be the Nyquist D-contour as described in Theorem 1. Due to stability of $H(s)$, no poles are present in γ. Hence, the right hand side of (6) equals zero.

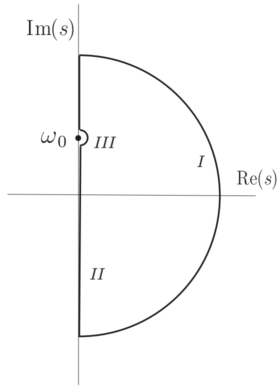

Fig. 1. Contour γ

The left hand side of (6) is spitten in three parts that are described in (13). It can be derived that term III converges to $-\pi j H(j\omega_o)$. Contrary to the common derivation of the Hilbert transform, the first term is not omitted in order to maintain a closed contour. As a result, manipulation of (13) causes a slightly different formulation of the Hilbert transform which is given in (14):

$$\begin{aligned} \frac{1}{2\pi j}\oint_\gamma \frac{H(s)}{s - j\omega_0} ds = \frac{1}{2\pi j} \lim_{r\to\infty} \Big(& \underbrace{rj \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{H(re^{j\phi})}{re^{j\phi} - j\omega_0} d\phi}_{I} \\ & + \underbrace{\lim_{\epsilon \downarrow 0} \big(j \int_{-r}^{\omega_0 - \epsilon} \frac{H(j\omega)}{j\omega - j\omega_0} d\omega + j \int_{\omega_0 + \epsilon}^{r} \frac{H(j\omega)}{j\omega - j\omega_0} d\omega \big)}_{II} \\ & + \underbrace{\lim_{\epsilon \downarrow 0} \epsilon j \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{H(j\omega_0 + \epsilon e^{j\phi})}{\epsilon e^{j\phi}} d\phi}_{III} \Big) = 0 \end{aligned} \quad (13)$$

gives:

$$H(j\omega_0) = \frac{1}{\pi j} \mathcal{P} \oint_\gamma \frac{H(s)}{s - j\omega_0} ds \quad (14)$$

Note the appearance of $\mathcal{P}$ which indicates that term III is omitted. The key observation is the appearance of j in (14) that couples the real part of the left-hand side of the equation with the imaginary part of the right-hand side. As a result:

$$\text{Im}(H(j\omega_0)) = \frac{1}{\pi} \mathcal{P} \oint_\gamma \frac{\text{Re}(H(s))}{s - j\omega_0} ds \quad (15)$$

which proves the only if part of Theorem 1.

The if part of Theorem 1 will be proven now. Titchmarsh's theorem [12](page 27), [16] proves that (7) implies causality of $H(s)$ and states that (7) implies the dual expression:

$$\text{Re}(H(j\omega_0)) = \frac{1}{\pi} \mathcal{P} \oint_\gamma \frac{\text{Im}(H(j\omega))}{j\omega - j\omega_0} ds. \quad (16)$$

As a result, (7) implies that $H(s)$ satisfies (14). The derivation made in the only if part of this proof shows that if a function $H(s)$ satisfies (14), the residues contained in the corresponding contour sum up to zero. This however does not exclude the appearance of poles in the right-half plane that have canceling residues, i.e.:

$$\sum_{p_i \in \mathcal{A}(\gamma)} \text{Res}(H(s), p_i) = 0 \overset{\not\Rightarrow}{\Leftarrow} \text{Res}(H(s), a_i) = 0, \ \forall p_i \in \mathcal{A}(\gamma) \quad (17)$$

However, (9) shows that $\text{Res}(\frac{H(s)}{s - j\omega_0}, p_i)$ depends on ω_0. This means that cancelation of residues can only occur for distinct ω_0. Consequently:

$$\begin{aligned} \sum_{p_i \in \mathcal{A}(\gamma)} \text{Res}(\frac{H(s)}{s - j\omega_0}, p_i) = 0, \ \forall \omega_0 \\ \Leftrightarrow \ \text{Res}(\frac{H(s)}{s - j\omega_0}, a_i) = 0 \ \forall p_i \in \mathcal{A}(\gamma) \end{aligned} \quad (18)$$

Ergo, a transfer function that satisfies (7) for all ω_0 is stable. This ends the proof of Theorem 1. ■

Remarks: (7) connects all frequency points and therefore represent the analytical properties of a transfer function. These relations can be seen as a replacement for the analytical properties that are naturally embedded in the structure of a model, e.g. a fraction of polynomials in s.

IV. Discrete approximation

It is assumed that measured frequency response data of the plant, $P(j\omega_i) = P_i$, is only available on a finite discrete grid of frequency points $j\omega_i$ with $\omega_i \in \Omega$ and Ω a finite set. This section proves that the continuous integral given in Theorem 1 can be approximated using a left Riemann-sum. The resulting algebraic relation guarantees that $\exists \mathscr{I}(Q_1, \ldots, Q_N) \in \mathscr{RH}_\infty$. This relations will be combined with (5) and used in Sec. V for control synthesis.

The approach is the following: the partial fraction expansion described in (9) is substituted in the integrant of (14). Due to linearity of the sum and integral operator, the analysis can be performed separately over every term in the partial fraction expansion and summed afterwards. Substitution gives:

$$\begin{aligned} H(j\omega_0) = -\frac{1}{j\pi} \sum_{i=1} \frac{\alpha_i}{j\omega_0 - p_i} \mathcal{P} \oint_\gamma \frac{1}{s - p_i} ds \\ + H(j\omega_0) \frac{1}{\pi j} \mathcal{P} \oint \frac{1}{s - j\omega_0} ds \end{aligned} \quad (19)$$

To prove convergence, the error between the left-Riemann sum and the integral of the partial fraction will be considered. The error for the partial fraction corresponding to a pole β is defined as:

$$\text{E}(\frac{1}{s - \beta}, \gamma, \mathcal{N}, \Delta) \triangleq \mathcal{P} \oint_\gamma \frac{1}{s - \beta} ds - \text{R}(\frac{1}{j\omega - \beta}, \mathcal{N}, \Delta) \quad (20)$$

where R represents the Riemann-sum which is defined as:

$$\begin{aligned} \text{R}(\frac{1}{j\omega - \beta}, \mathcal{N}, \Delta) = \sum (\frac{1}{j\omega_n - \beta}) j\Delta, \\ \omega_n = \{\text{Im}(\beta) + n\Delta \mid n \in \mathcal{N}\} \end{aligned} \quad (21)$$

Fig.2 depicts a typical error coarse. It can be observed that both positive and negative errors appear. As a result, the errors partly cancel out if summed over the entire contour γ. As a consequence, the error analysis is performed over the entire integration interval for one partial fraction.

In Prop. 3 and 4, the errors induced by approximation will be derived for the separate terms of the partial fraction expansion.

Proposition 3: The error defined in (20) satisfies following equality for $\frac{1}{s-j\omega_0}$*:*

$$\mathrm{E}(\frac{1}{j\omega - j\omega_0}, \gamma, \mathcal{Z}, \Delta) = j\pi \tag{22}$$

where the $\mathcal{N} = [-N, \dots, -1, -1, \dots, N]$(zero excluded) and $N \in \mathbb{N}$.

In fact $\mathcal{N}$ can be interpreted as the discrete equivalent of Cauchy's principle value since the singularity is excluded.

Proof: Evaluation of $\mathrm{E}(\frac{1}{j\omega - j\omega_0}, \gamma, \mathcal{N}, \Delta)$ gives:

$$\mathrm{E}(\frac{1}{j\omega - j\omega_0}, \gamma, \mathcal{N}, \Delta) = \underbrace{\oint_\gamma \frac{1}{j\omega_0 - j\omega} ds}_{=0}$$
$$- \lim_{\epsilon \to 0^+} \epsilon j \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{1}{(\epsilon e^{j\phi} + j\omega_0) - j\omega_0} d\phi - \frac{1}{\pi j} \sum_{n \in \mathcal{N}} \frac{j\Delta}{nj\Delta} \tag{23}$$

The series at the right-hand side of (23) satisfy:

$$\sum_{n \in \mathcal{N}} \frac{j\Delta}{nj\Delta} = \sum_{n \in \mathcal{N}} \frac{1}{n} = 0 \tag{24}$$

As a result:

$$\mathrm{E}(\frac{1}{j\omega - j\omega_0}, \gamma, \mathcal{N}, \Delta) = \pi j \tag{25}$$

which ends the proof of Proposition 3 ■

Proposition 4: The error defined in (20) satisfies the following equality for the partial fraction $\frac{1}{s-p_i}$*:*

$$\lim_{\frac{\mathrm{Re}(p_i)}{\Delta} \to \infty} \mathrm{E}(\frac{1}{s - p_i}, \gamma, \mathbb{N}, \Delta) = -\pi j \tag{26}$$

Proof: Two cases are considered separately. Either $\mathrm{Re}(p_i) > 0$, i.e. p_i is an instable pole or $\mathrm{Re}(p_i) < 0$, i.e. p_i is a stable pole. First stability of p_i is assumed. Both the contribution of the continuous and discrete part of (20) will be computed for the function $\frac{1}{j\omega_i - p_i}$.

Since no poles occur in or on the contour γ, the continuous part can be computed via Cauchy's residue theorem:

$$\oint_\gamma \frac{1}{s - p_i} ds = \mathcal{P} \oint_\gamma \frac{1}{s - p_i} ds = 0 \tag{27}$$

Redefinition of the axis by $j\omega_n = n\Delta + \mathrm{Im}(p_i), n \in \mathbb{Z}$ as proposed in (21) gives:

$$\sum_{n \in \mathbb{Z}} \frac{1}{j\omega_n - p_i} j\Delta = \sum_{n \in \mathbb{Z}} \frac{j\Delta}{nj\Delta - \mathrm{Re}(p_i)} \tag{28}$$

$$= \sum_{n \in \mathbb{Z}} \frac{1}{n + j\frac{\mathrm{Re}(p_i)}{\Delta}} \tag{29}$$

Given stability of the pole p_i, i.e. $\mathrm{Re}(p_i) < 0$, and $\Delta > 0$, numerical evaluation shows that this series converges to $j\pi$ for $|\frac{\mathrm{Re}(p_i)}{\Delta}| \gg 1$ (see Fig.3), i.e.:

$$\lim_{\frac{\mathrm{Re}(p_i)}{\Delta} \to -\infty} \sum_{n \in \mathbb{Z}} \frac{1}{n + j\frac{\mathrm{Re}(p_i)}{\Delta}} = j\pi \tag{30}$$

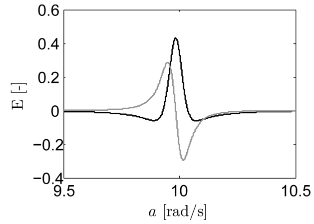

Fig. 2. $\mathrm{E}(\frac{1}{s-(10+5j)}, [a, a + 0.05], \frac{a-5}{\Delta}, 0.05)$, (-) real part and (-) imaginary part.

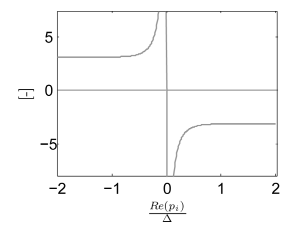

Fig. 3. (-) Real and (-) imag part of (29)

We will now consider the case that p_i is instable. In this case the pole p_i is encircled by γ such that the Cauchy residue theorem gives: (note the clockwise evaluation of the contour resulting in a minus sign):

$$\mathcal{P} \int_\gamma \frac{1}{s - p_i} ds = -2\pi j \tag{31}$$

For an unstable pole, (29) converges to $-\pi j$. Substitution of (30) and (31) into (20) shows:

$$\mathrm{E}(\frac{1}{s - p_i}, \gamma, \mathbb{Z}, \Delta) = -\pi j \tag{32}$$

As a result, the cumulative error converges towards $-\pi j$ regardless of the stability of the pole which ends the proof of Proposition 4 ■

We are now ready to derive the main result of this section. Using Proposition 3 and Proposition 4, the cumulative error for the sum of terms described in (19) can be computed.

Theorem 5: The integral imposed by Theorem 1 satisfies the following equivalence which relates the continuous integral to a discrete sum:

$$\frac{1}{\pi j} \mathcal{P} \oint_\gamma \frac{H(j\omega)}{j\omega - j\omega_0} dj\omega = \frac{1}{\pi j} \sum_{n \in \mathcal{N}} \frac{H(j\omega_n)}{(j\omega_n - j\omega_0)} j\Delta$$
$$+ \Big(H(j\omega_0) - \sum_i \frac{\alpha_i}{j\omega_0 - p_i} \sum_{n \in \mathcal{N}} \frac{1}{n + \frac{j\,\mathrm{Re}(p_i)}{\Delta}} \Big) \tag{33}$$

where $\mathcal{N} = [-N, \dots, -1, 1, \dots N]$*(zero excluded) and* Ω *is an equally spaced frequency grid centered around* ω_0*, i.e.* $\omega_n \in \Omega = \{\omega_0 + n\Delta | n \in \mathcal{N}\}$*. It can be shown that the term:*

$$H(j\omega_0) - \sum_i \frac{\alpha_i}{j\omega_0 - p_i} \sum_{n \in \mathcal{N}} \frac{1}{n + \frac{jRe(p_i)}{\Delta}} \tag{34}$$

converges to zero as $|\frac{\mathrm{Re}(p_i)}{\Delta}| \to \infty$*.*

It has to be mentioned that the step-size not necessary has to converge to zero as long as $\mathrm{Re}(p_i) \gg \Delta$. Under the assumption that $\frac{Re(p_i)}{\Delta} \gg 1$, a practical usable relation equals:

$$\frac{1}{\pi j} \mathcal{P} \oint_\gamma \frac{H(j\omega)}{j\omega - j\omega_0} dj\omega = \frac{1}{j\pi} \sum_{n \in \mathcal{N}} \frac{H(j\omega_n)}{(j\omega_n - j\omega_0)} j\Delta \tag{35}$$

Proof: Substitution of (22) and (29) into (19) gives:

$$\frac{1}{\pi j}\oint_{\gamma}\frac{H(j\omega)}{(j\omega-j\omega_0)}d\omega=\frac{1}{\pi j}\sum_{i\in\mathcal{N}}\frac{H(j\omega_i)}{j\omega_i-j\omega_0}+\Big(H(j\omega_0) -\frac{1}{j\pi}\sum_i\frac{\alpha_i}{j\omega_0-p_i}\big(\sum_{n\in\mathbb{Z}}\frac{1}{n+\frac{j\,\mathrm{Re}(p_i)}{\Delta}}-\frac{1}{\frac{j\,\mathrm{Re}(p_i)}{\Delta}}\big)\Big) \tag{36}$$

In accordance with Proposition 3, the Riemann-sum is computed over $\mathcal{N}$ such that $n=0$ has to be excluded for the terms $\frac{1}{s-p_i}$. This ends the proof of Theorem 5. ■

V. Optimization

We now continue the control synthesis problem formulated in (5). Exploiting the results of Section III and IV, the condition $\mathscr{I}(Q_1,\ldots,Q_N)\in\mathscr{RH}_\infty$ can be translated into a algebraic relation on the points Q_i. This relation will be used to perform the synthesis of the optimal points Q_i^* in Subsection V-B.

For ease of notion, (35) is written as the following set of constraints:

$$\begin{bmatrix}\mathrm{Im}(Q_1)\\ \mathrm{Im}(Q_2)\\ \vdots\\ \mathrm{Im}(Q_{2N})\end{bmatrix}=\frac{1}{\pi}\Lambda\begin{bmatrix}\mathrm{Re}(Q_1)\\ \mathrm{Re}(Q_2)\\ \vdots\\ \mathrm{Re}(Q_{2N})\end{bmatrix} \tag{37}$$

where Λ equals (assuming that the grid points $j\omega_i$ are equally spaced):

$$\Lambda=\begin{bmatrix}0 & -1 & -\frac{1}{2} & -\frac{1}{3} & \cdots & -\frac{1}{2N}\\ 1 & 0 & -1 & \cdots & & \\ \vdots & \vdots & \ddots & & & \\ \frac{1}{2N} & \cdots & & & & 0\end{bmatrix} \tag{38}$$

In accordance with Theorem 5, zeros appear at the singularity, i.e. ω_0.

A. Truncation

Due to computational constraints, the vector $[Q_1,Q_2,\ldots,Q_{2N}]^T$ is of finite length. As a result, summation over $\mathcal{N}$ as described in (35) is limited. The influence of this truncation will be considered shortly.

From literature it is known that the optimal controller for a strictly proper plant satisfies properness [18] (see also (3)). Hence, $Q(j\omega_i)$ can be constrained to properness, i.e. $Q_i-Q_{i+1}\leq 0$ without loss of optimality for high values of $|\omega_i|$.

Given properness of $Q(j\omega)$, the contribution of $\frac{Q(j\omega)}{s-j\omega_0}$ uniformly converges to zero for $|\omega|\to\infty$. Consequently, truncation of the integration bounds is allowed for $|\omega_i|\to\infty$. Quantitative results are not derived yet and are a topic of current research.

B. $\mathscr{H}_\infty$ controller synthesis

To optimize the sampled $\mathscr{H}_\infty$ problem formulated in (5), (5) is written as a complex valued Second Order Cone Program (SOCP) of the form:

$$\begin{aligned}&|A_iQ_i-b_i|^2<C_iQ_i+d_i,\quad Q_i\in\mathbb{C},\ i=[1,\ldots,2N]\\ &D_iQ_i+f_i=0\end{aligned} \tag{39}$$

which can be solved via standard optimization routines.

To obtain the coefficients of the SOCP, (5) is written as:

$$(W_1-W_1P_iQ_i)^\dagger(W_1-W_1P_iQ_i)+(W_2Q_i)^\dagger W_2Q_i<\rho^2 \tag{40}$$

where $.^\dagger$ represents the complex conjugate. To obtain the optimal points $Q_1^*,\ldots,Q_N^*$, $|\rho|$ is used as objective function.

A_i, b_i, C_i and d_i can be obtained via:

$$A_i=\Big(P_i^\dagger P_i\big(W_1(j\omega_i)^\dagger W_1(j\omega_i)\big)+W_2(j\omega_i)^\dagger W_2(j\omega_i)\Big)^{\frac{1}{2}}$$

$$b_i=\frac{\big(W_1(j\omega_i)P_i\big)^\dagger W_1}{A_i^\dagger},\quad C_i=0,\quad d_i=\rho^2$$

To guarantee the existence of $\mathscr{I}(Q_1,\ldots,Q_N)\in\mathscr{RH}_\infty$, (37) is added as equality constraint to the SOCP. It has to be emphasized that stability is directly imposed on Q_i, such that no stability or even rationality constraints are required for A_i and b_i. This makes the proposed approach fundamentally different that matching procedures in classical $\mathscr{H}_\infty$ approaches that require stable spectral factorizations in order to solve the problem.

C. $\mathscr{H}_2$ synthesis

In analogy to the $\mathscr{H}_\infty$ synthesis approach, a sampled version of H_2 can be obtained:

$$\min_{Q_1,\ldots,Q_N}\sum_i\left\|\begin{bmatrix}W_1(j\omega_i)(1-P(j\omega_i)Q_i)\\ W_2(j\omega_i)Q_i\end{bmatrix}\right\| \tag{41}$$

$$\text{s.t.:}\qquad \exists\,\mathscr{I}(Q_1,\ldots,Q_N)\in\mathscr{RH}_\infty \tag{42}$$

Substitution of (37) into (41) gives:

$$\begin{aligned}\min_{Q_1,\ldots,Q_N}\ &\mathrm{Re}(\mathrm{vec}(Q_i))^T\Big(\\ &\mathrm{diag}(A_i^\dagger A_i)+\Lambda^T\,\mathrm{diag}(A_i^\dagger A_i)\Lambda\Big)\,\mathrm{Re}(\mathrm{vec}(Q_i))\\ &-2\big(\mathrm{Re}(\mathrm{vec}(A_i^\dagger b_i))+\mathrm{Im}(\mathrm{vec}(A_i^\dagger b_i))\Lambda\big)\,\mathrm{Re}(\mathrm{vec}(Q_i))\\ &+\alpha(W_1,W_2)\end{aligned} \tag{43}$$

where α is a constant depending on W_1 and W_2, diag and vec are defined as:

$$\mathrm{diag}(x_i)=\begin{bmatrix}x_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & x_n\end{bmatrix}\qquad \mathrm{vec}(x_i)=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} \tag{44}$$

(43) represents a Quadratic Program (QP) which can be solved analytically or via standard algorithms. Given the optimal vector $\mathrm{vec}(\mathrm{Re}(Q_i^*))$, $\mathrm{vec}(Q_i^*)$ can be obtained via (37).

VI. Simulation results

To verify the results described in Section III and IV, the control problem described in (5) is solved via the algorithm proposed in Sec.II and V-B. The results are compared with the results obtained via the model-based `mixsyn` algorithm of matlab. As mentioned in the introduction, the actual interpolation step, $C(s)=\mathscr{I}(C_1,\ldots,C_N)$ is not considered in this paper.

The following plant (assumed to be unknown) and weighting filters are used for evaluation:

$$P(s)=\frac{1}{s^2+2\xi_1\omega_1 s+\omega_1^2}+\frac{1}{s^2+2\xi_2\omega_2 s+\omega_2^2} \tag{45}$$

where $\omega_1 = 30$, $\omega_2 = 1$, $\xi_1 = 0.05$ and $\xi_2 = 1$.

$$W_1(s) = \frac{1.58 \cdot 10^2}{s^2 + s + 1} \tag{46}$$

$$W_2(s) = \frac{0.395s^2 + 5.925s + 88.88}{225s^2 + 11250s + 562500} \tag{47}$$

Fig.4 shows the corresponding frequency response functions.

The points P_i are generated via substitution of the grid points $s = j\omega_i$ with $\omega_i = [-100, -99.9, \ldots 99.9, 100]$ into $P(s)$. It has to be mentioned that substitution is only applied for comparison with the model-based result. The points P_i are commonly directly obtained from frequency response experiments.

The SOCP described in (39) is solved for the given points P_i, $W_1(j\omega_i)$ and $W_2(j\omega_i)$ using the Sedumi solver and Yalmip interface. After optimization the optimal value of ρ appeared to be 0.7.

The resulting controller points C_i, obtained via $C_i = \frac{Q_i}{1-P_iQ_i}$, are depicted in Fig.5. For comparison, the model-based controller is also depicted. It can be observed that the obtained results exhibit similar behavior as the model based controller. Deviations between the model-based controller and the points C_i obtained via the proposed approach mainly occur in the high-frequent region. This could be explained from the error induced by truncation. Furthermore, the non-uniqueness of suboptimal controllers allow for slight deviations.

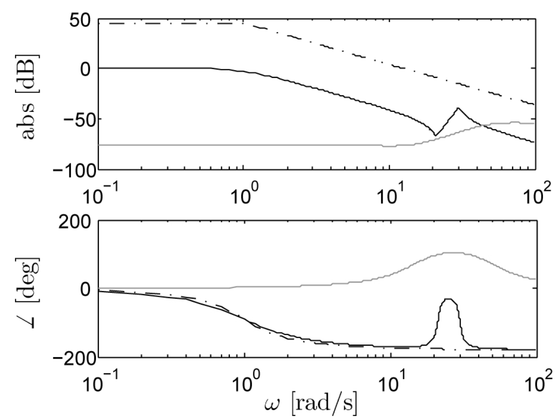

Fig. 4. Bode plot of $P(j\omega)$[-], $W_1(j\omega)$ [-.] and $W_2(j\omega)$ [-],

VII. CONCLUSIONS

This paper proposes a method to perform optimal controller synthesis via the frequency response coefficients of the plant, i.e. without plant identification. This approach has the advantage that the realization step can be performed over the controller with full knowledge of closed-loop relevant behavior.

A sampled version of the $\mathscr{H}_\infty$ and $\mathscr{H}_2$ controller synthesis problem is considered. The Youla parameter is introduced to convexify the problem and map the set of stabilizing controllers onto the set of stable transfer functions. The main contribution of this paper is the derivation of a set of algebraic relations on the frequency response coefficients of the Youla parameter that guarantee stability and causality.

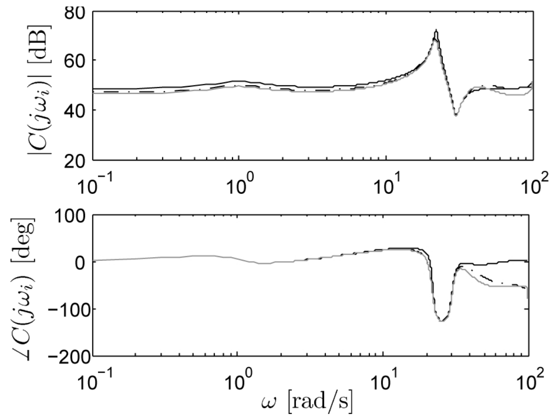

Fig. 5. Bode plot of: [-] model-based controller ($\rho = 0.6$), [-.] C_i ($\rho = 0.7$) and [-] C_i ($\rho = 0.75$).

As a result, synthesis of the controller can be performed in terms of the frequency response coefficients of the Youla parameter. Simulation show that the controller frequency response coefficients obtained via optimization over the points Q_i exhibit similar behavior as a controller obtained via model-based synthesis.

In analogy to the approach presented in this paper, similar relations can be derived for discrete systems by replacing the D-contour with the unit circle. Furthermore, the stability and causality relations described in the paper are not limited to SISO systems which makes the extension to the synthesis of MIMO controllers conceptually straightforward.

REFERENCES

[1] R.L. Kosut B. R. Woodley, J. P. How. Model free subspace based $\mathscr{H}_\infty$ control. In *Proceedings of the American Control Conference*, 2001.

[2] R.A. de Callafon. *Feedback Oriented Identification for enhanced and robust Control*. PhD thesis, Technische Universiteit Delft, 1998.

[3] L. Debnath and D. Bhatta. *Integral Transforms and Their Applications*. Second Edition.

[4] P.M.J. Van den Hof and R.J.P. Schrama. Identificationand control - closed-loop issues. *Automatica*, 31(12):1751–1770, 1995.

[5] W. Favoreel, B. de Moor, and M. Gevers. Closed-loop model-free subspace-based lqg-design. 1999.

[6] B.A. Francis. *A course in $\mathscr{H}_\infty$ Control theory*. Springer-Verlag, 1987.

[7] G.F. Franklin, J.D Powell, and A. Emami-Naeini. *Feedback Conrol of Dynamic Systems*. Addison-Wesley, 3 edition, 1994.

[8] J.S. Freudenberg and D.P. Looze. *Frequency Domain Properties of Scalar amd Multivariable Feedback Systems*. Springer-Verlag, 1985.

[9] I.M. Horowitz. *Quantitative Feedback Design Theory - QFT*. QFT Publications, 1993.

[10] L.H. Keel and S.P. Bhattacharyya. Data driven synthesis of three term digital controllers. In *American Control Conference*, 2006.

[11] S.G. Krantz. *Handbook of Complex Variables*. Birkhauser, 1999.

[12] H.M. Nussenzveig. *Causality and Disperion Relations*. Academic Press, 1972.

[13] C. Rusu and P. Kuosmanen. Phase approximation by logarithmic sampling of gain. *IEEE Trans. on Circ. and Syst.*, 50(2):93–101, 2003.

[14] R.E. Skelton and Guojun Shi. The data-based lqg control problem. *Proc. of 33rd IEEE Conference on Decision and Control*, 2:1447–1452, 1994.

[15] F.M. Tesche. On the use of the hilbert transform for processing measured cw data. *IEEE Trans. on Elec. Comp.*, 34(3), 1992.

[16] E.C Titchmarsh. *Introduction to the theory of Fourier Integrals*. Chealsea Publishing Company.

[17] O. Yaniv. *Quantitative Feedback Design of linear and Nonlinear Control Systems*. Kluwer Academic Publishers, 1999.

[18] K. Zhou, J.C Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, 1996.