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Functional Differential Inclusion-based Approach to Control of Discontinuous Nonlinear Systems with Time Delay

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Abstract— **This paper presents functional differential inclusion based approach to investigate the stabilization of discontinuous nonlinear systems with time delay. First, the conception of Filippov solution for ordinary differential equations with discontinuous right-hand side is extended to discontinuous systems with time delay, which is a solution of functional differential inclusions determined by the Filippov set-valued functional. With this conception and the strong stability, it is shown that the Lyapunov stability framework can be easily extended to discontinuous systems with time delay. Then, the feedback stabilization problem for a class of discontinuous nonlinear systems with time delay is investigated with the proposed functional differential inclusion-based framework. It is shown that for the systems, a stabilization controller can be provided by a by means of a system-related function satisfying HJI inequality. Finally, to demonstrate the design process of the proposed approach, an application example with automotive system background is addressed.**

I. INTRODUCTION

The investigations on discontinuous dynamical systems have been a concerned subject for several decades (see the early contributions of Clark [1], Filippov [2], and others [3]-[7]). The dynamics of such systems is determined by discontinuous vector fields, for which the existence of a continuous differentiable solution is not guaranteed. Thus, a lot of challenges have been encountered to provide analysis and synthesis tools in the control theory community, especially on studying of nonlinear systems. To study the Lyapunov stability, the classical Lyapunov theory for systems described by continuous differential equations can not be applied directly to discontinuous dynamical systems. Consequently, there have been efforts made on establishing generalized framework. Some results can be seen in [3], [4] that are suitable for qualitative analysis of a class of special discontinuous systems, and in [5], [6], differential inclusion-based approaches have been discussed with nonsmooth Lyapunov conditions.

On the other hand, time-delay phenomena also always arise in system dynamics [8]. Since time delay has significant impact on the performance of control systems, there is continuous interest on investigating systems with time delay, such as the attentions focused on stabilizing nonlinear time-delay systems [9], [10]. Moreover, it is noticed that the Lyapunov stability results of [3] have been extended in [11], [12] to discuss discontinuous dynamical systems with time delay. However, to the best known of our knowledge, there is not any report on providing a general framework to analyze and synthesize discontinuous nonlinear systems with time delay.

As well known, the framework of differential inclusions, such as Filippov [2], is an useful tool to interpret discontinuous differential equations. With the notion of differential inclusion, new definition of solution for discontinuous differential equations is given, the corresponding properties are studied for existence, uniqueness and continuity, and the generalized Lyapunov theories are also developed [5], [6]. Meanwhile, it is known that systems with time delay are usually described by functional differential equations (FDEs), and it is noticed that functional differential inclusion (FDI), which can be discussed with respect to FDEs is actually a general type of differential inclusions. A systematic study for solution properties of FDIs has been shown in [13], the existence results have been offered in [14] for convex FDIs, and a few arguments for stability analysis have been given in [15].

The mentioned previous works provide the basic motivations to pay attention to FDIs for control and analysis of dynamical systems. The objective of this paper is to establish Lyapunov stability results from the viewpoint of FDIs, which are applied to address the feedback stabilization problem of a class of discontinuous nonlinear systems with time delay. First, the Filippov framework is extended to FDI that provides a new definition of the solution. Then, as a trivial extension of the Lyapunov-Krasovskii theorem [16] for FDEs, the corresponding theorem is also presented to evaluate the behavior of FDIs. With these preliminaries, a feedback design approach is shown that guarantees the strong asymptotic stability of the closed-loop control systems. Finally, the speed control problem of spark ignition (SI) engines is used as an application example of the proposed techniques, where both the intake-to-power delay and the disturbances due to the transitions among different operation modes are taken into account.

Notations: R^n denotes the n-dimensional Euclidean space with the norm of x in R^n denoted by $\parallel x \parallel$. $R^{n\times m}$ denotes $n\times m$-dimensional matrix. Let r be a given positive number. $C_r = \{\phi \mid \phi : [0,r] \to R^n\}$ and the norm of an element ϕ in C_r is designated by $\parallel \phi \parallel_c := \sup_{0\le\tau\le r} \parallel \phi(\tau) \parallel$. x is a function taking $[-r,\infty)$ into R^n where $x(t)$ denotes the value of x at t. x_t denotes an element of C_r defined by $x_t(\tau) = x(t-\tau)$, $\tau \in [0,r]$ where $x_t(\tau)$ denotes the value of x_t at τ. Without the confusion, we use symbol x denoting the variable $x(t)$. Here, we recall that a continuous function $W_i(\cdot) : [0,a) \to [0,\infty)$ is said to belong to class $\mathscr{K}$ if it is strictly increasing and

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$W_i(0) = 0$. $L_f V = \frac{\partial V}{\partial x} f$ denotes the Lie derivative of $V(x)$ with respect to $f(x)$.

II. MATHEMATICAL FRAMEWORK

It is known that for the systems described by differential equations with discontinuous right-hand side, the Filippov framework is well developed to analyze the behavior of the dynamical systems. For discontinuous systems with time delay, the dynamics is represented by FDEs with discontinuous right-hand side. A feasible way to analyze the behavior of discontinuous FDEs is to extend the Filippov framework. Based on the extension, the generalized Lyapunov stability criteria can be developed.

A. Definitions

Consider the following FDE

$$\dot{x}(t) = f(x_t) \tag{1}$$

where $\dot{x}(t)$ is the right-hand derivative of $x(t)$ and $f(x_t) : C_r \to R^n$ is locally bounded. Suppose that $f(x_t)$ is discontinuous at a given surface S_f defined by

$$S_f = \{x_t \mid S(x_t(0)) = 0\} \tag{2}$$

where $S(x_t(0))$ is a smooth function.

Definition 1. *With an initial time t_0, a function $x(\cdot) : [-r+t_0, t_1] \to R^n$ $(t_1 > t_0)$ is called a Filippov solution of (1), if $x(\cdot)$ is absolutely continuous on $[-r+t_0, t_1]$ and*

$$\dot{x}(t) \in \mathbf{K}[f](x_t) \tag{3}$$

holds for almost every $t \in [t_0, t_1]$, where $\mathbf{K}[f](x_t)$: $C_r \to \Omega \subset B(R^n)$ denotes the set-valued map defined by

$$\mathbf{K}[f](x_t) = \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \overline{co}\{f(B(x_t, \delta)/S)\} \tag{4}$$

where $\bigcap_{\mu(S)=0}$ denotes the intersection over all sets S of Lebesgue measure zero and

$$B(x_t, \delta) := \{x_{t'} \in C_r \mid \| x_{t'} - x_t \|_c < \delta\}$$

From Definition 1, it is clear that the Filippov solution of (1) is the solution of FDI (3). In the following, we denote the solution of the FDI (3) with an initial function $x_0(\tau) := \phi(\tau)$ as $x_t(\phi)$, where $\tau \in [0, r]$, $\phi \in C_r$.

Remark 1. It is clear that the set-valued map defined by (4) is a trivial extension of the Filippov set-valued map, which is originally presented by Filippov for discontinuous differential equations [2], however, it should be noticed that $\mathbf{K}[f](x_t)$ is a set-valued functional.

Remark 2. If we consider piecewise continuous FDEs, and notice that an equivalent definition for the Filippov set-valued map has been given in [17], then an equivalent definition of (4) can also be given by

$$\mathbf{K}[f](x_t) = co\{\lim_{i\to\infty} f(x_t^i) \mid x_t^i \to x_t, x_t^i \notin S_f\} \tag{5}$$

where S_f denotes the set of points where f is nondifferentiable, and is measure zero.

Example 1: Consider the following system

$$\dot{x}(t) = -ax(t) + b\text{sign}(x(t))x(t-\tau) \tag{6}$$

with $\tau \in [0, r]$. Using the notation

$$f(x_t) = -ax_t(0) + b\text{sign}(x_t(0))x_t(\tau)$$

it is clear that functional $f(x_t)$ is discontinuous at the surface

$$S_f = \{x_t \mid S(x_t(0)) = x_t(0) = 0\} \tag{7}$$

According to Definition 1 and Remark 2, the associated set-valued map is calculated as follows

$$\mathbf{K}[f](x_t) = \begin{cases} \{-ax_t(0) + bx_t(\tau)\}, & x_t(0) > 0 \\ \{-ax_t(0) + \lambda \cdot bx_t(\tau) \mid \lambda \in [-1,1]\}, & x_t(0) = 0 \\ \{-ax_t(0) - bx_t(\tau)\}, & x_t(0) < 0 \end{cases} \tag{8}$$

In this paper, we investigate the Lyapunov stability of system (1) with FDI (3). Here we discuss the strong stability with respect to all the solutions, since in general the solution of a FDI is not unique. Without loss of generality, we suppose that $0 \in \mathbf{K}[f](0)$.

Definition 2. [6] *The solution $x = 0$ of (3) is said to be strongly stable if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that all the solutions $\|x_t(\phi)\|_c \leq \varepsilon$, $t \geq t_0$ for any initial condition ϕ satisfying $\|\phi\|_c \leq \delta$. Furthermore, the solution $x = 0$ is said to be strongly asymptotically stable if it is strongly stable and all $x_t(\phi) \to 0$ as $t \to \infty$.*

As well known, the right upper Dini's derivative is used in time-delay systems for evaluating the non-increasing property of a functional along the solution of the system as shown in the following definition.

Definition 3. [16] *If $V(x_t) : C_r \to R$ is continuous and $x_t(\phi)$ is a solution of (3) starting from ϕ, the directional derivative of V along $x_t(\phi)$ is defined by*

$$\dot{V}(x_t(\phi)) := \limsup_{h\to 0^+} \frac{V(x_{t+h}(\phi)) - V(x_t(\phi))}{h} \tag{9}$$

B. Stability condition

For a given solution of FDI (3), the boundedness and convergence can be evaluated by investigating the non-increasing property of a Lyapunov-Krasovskii functional on t along the solution. In fact, note that the solution of FDI is absolutely continuous, then the function on t determined by the Lyapunov-Krasovskii functional with respect to system states will be an absolutely continuous function, if we suppose the functional is absolutely continuous on x_t [18]. Hence, the Lyapunov-Krasovskii functional is differentiable on t almost everywhere. Furthermore, if the time derivative of the functional is nonpositive almost everywhere along all the solutions, then the strong stability of the system can be guaranteed [19]. As mentioned in [20], this time derivative is equivalent to the Dini's derivative almost everywhere. As a conclusion, we can investigate the strong stability of

FDIs (3) by evaluating the derivative of Lyapunov-Krasovskii functional along all the solutions, which is summarized as follows.

Theorem 1. *Suppose $f(x_t)$: $C_r \to R^n$ is bounded. If there exists an absolutely continuous functional $V(x_t)$: $C_r \to R$ and $W_1(\cdot), W_2(\cdot) \in \mathscr{K}$ such that*

$$W_1(|x_t(0)|) \leq V(x_t(\phi)) \leq W_2(\| x_t(\phi) \|_c) \tag{10}$$

$$\dot{V}(x_t(\phi)) \leq 0 \tag{11}$$

hold for all $x_t(\phi)$, then $x = 0$ is a strongly stable solution of (3). In addition, if there exists a function $W_3(\cdot) \in \mathscr{K}$ such that

$$\dot{V}(x_t(\phi)) \leq -W_3(|x_t(0)|) \tag{12}$$

then, the solution $x = 0$ of (3) is strongly asymptotically stable.

Remark 3. It should be noted that the time derivative requested in Theorem 1 is difficult to calculate when the solution is not given. For evaluating a time derivative of a nonsmooth function along a solution of differential inclusion, the generalized time derivative, which is a set-valued map and includes the time derivative almost everywhere, is introduced in [5], [6] with Clark's generalized gradient [1]. In the next section, for the FDI-based control design, we will also introduce a set-valued map denoted by $\dot{\hat{V}}$, which contains the time derivative of the Lyapunov-Krasovskii functional candidate almost everywhere.

C. Special case of Lyapunov-Krasovskii functional

In many cases, a Lyapunov-Krasovskii functional candidate is constructed as follows [16]

$$V(x_t) = V_1(x_t(0)) + V_2(x_t) \tag{13}$$

where $V_1 : R^n \to R$ is a differentiable function and $V_2 : C_r \to R$ is a functional with the integral form

$$V_2(x_t) = \int_{t-\tau}^{t} Q(x(s))ds$$

where $\tau \in [0, r]$ and $Q(s) > 0, \forall s \neq 0, Q(0) = 0$. Due to the set-valued property of FDI, the following derivative (14) of $V(x_t)$ should be discussed when judge the stability of the system with Theorem 1.

$$\dot{V}(x_t) = \nabla V_1\big(x_t(0)\big)\xi(t) + Q\big(x(t)\big) - Q\big(x(t-\tau)\big)\ , \quad \forall \xi(t) \in \mathbf{K}[f](x_t) \tag{14}$$

Corollary 1. *Consider system (3) and an absolutely continuous Lyapunov-Krasovskii functional candidate $V(x_t)$ with the form*

$$V(x_t) = V_1(x_t(0)) + \int_{t-\tau}^{t} Q(x(s))ds$$

If there exists a class $\mathscr{K}$ function $W(\cdot)$ such that

$$\nabla V_1(x_t(0))\ \xi + Q(x(t)) - Q(x(t-\tau)) \leq -W(|x_t(0)|)$$

holds $\forall \xi \in \mathbf{K}[f](x_t)$, then the solution $x = 0$ of (3) is strongly asymptotically stable.

Example 2: Recall example 1 and suppose $b \leq \bar{b}$. Construct a Lyapunov-Krasovskii functional for system (6)

$$V(x_t) = \frac{1}{2}x_t^2(0) + \int_{t-\tau}^{t} x^2(s)ds \tag{15}$$

Along the Filippov solution of system (6), the time derivative of $V(x_t)$ is

$$\dot{V}(x_t) = x \cdot \xi + x^2 - x^2(t-\tau)$$

where $\xi(t)$ belongs to the set-valued map (8). Since

$$x \cdot \xi = -ax^2 + b|x|x(t-\tau)$$

and substituting it into (II-C), we have

$$\dot{V}(x_t) \leq -(a-1)x^2 + |b||x|\|x(t-\tau)\|_c - x_t^2$$

by Corollary 1, the trivial solution of system (6) is strongly asymptotically stable if $a > (\bar{b}^2 + 1)/4$.

III. STABILIZATION DESIGN

In this section we investigate the stabilization design problem for a class of discontinuous nonlinear time-delay system described by the form

$$\dot{x}(t) = f(x, x_t) + h(x_t)\delta(x_t) + g(x_t)u \tag{16}$$

where $x \in R^n$ represents the system state, $u \in R^m$ is the control input, $f(\cdot) : C_r \to R^n$ is a nonlinear functional with $f(0,0) = 0$, $\delta(\cdot) : C_r \to R^p$ denotes a piecewise continuous functional that is discontinuous on the surface $S_\delta(x_t) = \{x_t \mid S(x_t(0)) = 0\}$, and $h(\cdot) : C_r \to R^{n\times p}$, $g(\cdot) : C_r \to R^{n\times m}$ with $h(0) = 0$, $g(0) = 0$ are known continuous functionals which are assumed to satisfy the matching condition

$$h(x_t) = g(x_t)\eta(x_t) \tag{17}$$

where functional $\eta(\cdot) : C_r \to R^{m\times p}$ is continuous differential.

Theorem 2. *For system (16), suppose that the following conditions are satisfied*

H1: Functional $f(x, x_t)$ is continuous and can be represented by

$$f(x, x_t) = f_0(x) + f_1(x)e(x_t) \tag{18}$$

where $f_0(\cdot) : R^n \to R^n$ is a known continuous function, $e(\cdot) : C_r \to R^q$ is a known continuous functional and $f_1(\cdot) : R^n \to R^{n\times q}$ is a known matrix whose entries are continuous functions.

H2: There exists a C^1 nonnegative function $V_1(x_t(0)) : R^n \to R$ with $V_1(0) = 0$ and a positive-definite function $Q(x)$ such that

$$L_{f_0}V_1 + \frac{1}{2}\|L_{f_1}^T V_1\|^2 + \frac{1}{2}\|e(x)\|^2 \leq -Q(x) \tag{19}$$

Then, system (16) is strongly asymptotically stabilized by the feedback control law

$$u = \alpha(x_t) = -\text{sign}(L_g V_1)\,|\eta(x_t)|\,\gamma_t^*(\phi) \tag{20}$$

with

$$\gamma_t^*(\phi) = \max_{\gamma_t(\phi)\in\mathbf{K}[\delta](x_t)} \{|\gamma_t(\phi)|\} \tag{21}$$

Proof: Let $F(x_t) = f(x,x_t) + h(x_t)\delta(x_t) + g(x_t)u$,then, according to definition 1, the Filippov solution of the closed loop system (16) consisting with (20) is the solution of the following FDI

$$\dot{x} \in \mathbf{K}[F](x_t) \subset f(x,x_t) + h(x_t)\mathbf{K}[\delta](x_t) + g(x_t)\mathbf{K}[\alpha](x_t) \tag{22}$$

Choose a candidate of Lyapunov-Krasovskii functional in the form of (13) as follows

$$V(x_t) = V_1(x_t(0)) + \frac{1}{2}\int_{t-\tau}^{t} \|e(x(s))\|^2 ds \tag{23}$$

and define a set for the time derivative of V_1 with respect to (22) by

$$\begin{aligned}\hat{\dot{V}}_1(x_t(\phi)) = \Big\{ &\lambda_t(\phi) \mid \lambda_t(\phi) = L_f V_1 + L_h V_1 \cdot \gamma_t(\phi) + \\ &L_g V_1 \cdot \nu_t(\phi), \forall \gamma_t(\phi) \in \mathbf{K}[\delta](x_t), \forall \nu_t(\phi) \in \mathbf{K}[\alpha](x_t) \Big\}\end{aligned} \tag{24}$$

Then, along any solution $x_t(\phi)$ of FDI (22), there exists a $\lambda_t(\phi) \in \hat{\dot{V}}(x_t(\phi))$ such that

$$\dot{V}(x_t(\phi)) = \lambda_t(\phi) + \frac{1}{2}\|e(x)\|^2 - \frac{1}{2}\|e(x_t)\|^2 \tag{25}$$

and from the assumptions *H1-H2*, it is easy to deduce that

$$\begin{aligned}&\dot{V}(x_t(\phi)) \\ &\quad = L_{f_0} V_1 + L_{f_1} V_1 \cdot e(x_t) + L_g V_1 \cdot \eta(x_t)\gamma_t(\phi) + L_g V_1 \cdot \nu_t(\phi) \\ &\quad \le -Q(x) - \tfrac{1}{2}\|L_{f_1}^T V_1 - e(x_t)\|^2 - \|e(x_t)\|^2\end{aligned} \tag{26}$$

Hence, by Theorem 1, the solution $x = 0$ of the system (16) connecting with (20) is strongly asymptotically stable. ■

Corollary 2. *Suppose that the functional $f(x,x_t)$ satisfies the decomposition of (19) but $f_0 : R^n \to R^n$ is piecewise continuous, and the following condition*

$$\frac{\partial V_1}{\partial x}\gamma_t'(\phi) + \tfrac{1}{2}\|L_{f_1}^T V_1\|^2 + \tfrac{1}{2}\|e(x)\|^2 \le -Q(x), \quad \forall \gamma_t'(\phi) \in \mathbf{K}[f_0](x) \tag{27}$$

holds with a C^1 positive-definite function $V_1(x_t(0))$ and a positive-definite function $Q(x)$. Then, system (16) is strongly asymptotically stable under the feedback control law (20).

Remark 4. Theorem 2 shows that the nonlinear discontinuous time-delay system (16) can be stabilized by a discontinuous delay-dependent feedback controller, however, it should be noted that due to the assumptions *H1* and *H2*, the system under consideration is a special case of (16). Then, we will show that the proposed design method can be extended and applied to more general systems. First, we consider the case when system (16) can be rendered by a state feedback compensation to satisfy the condition (19), then the whole control law can be obtained by the combination of the compensation and the controller given by Theorem 2. Furthermore, note that a smooth function $f(x,x_t)$ with respect to x and x_t can be always decomposed by [21]

$$f(x,x_t) = f_0(x) + f_1(x,x_t) \tag{28}$$

and for a given $V_1(x) \in C^1$, the Lie derivative $L_{f_1}V_1(x,x_t)$ is a continuous function which satisfies [22]

$$|L_{f_1}V_1(x,x_t)| \le \nu_a(x)\nu_b(x_t) \tag{29}$$

where $\nu_a(x) > 1$ and $\nu_b(x_t) > 1$ are smooth. According to these properties, if suppose that $f(x,x_t)$ in system (16) is smooth, then an argument similar to Theorem 2 obtains the stabilizing controller with a new Lyapunov-Krasovskii functional candidate.

Theorem 3. *Suppose that condition H1 holds, and if there exists a C^1 positive-definite function $V_1(x_t(0))$ such that*

$$L_{f_0}V_1 + \frac{1}{2}\left(\|L_{f_1}^T V_1\|^2 - \|L_g^T V_1\|^2\right) + \frac{1}{2}\|e(x)\|^2 < 0 \tag{30}$$

then, system (16) is strongly asymptotically stable under the feedback controller given by

$$u = \beta(x) - \mathrm{sign}(L_g V_1)\,|\eta(x_t)|\,\gamma_t^*(\phi) \tag{31}$$

with

$$\beta(x) = -\frac{1}{2}L_g^T V_1 \tag{32}$$

Proof: First, consider a subsystem of (16)

$$\dot{x} = f(x,x_t) + g(x_t)\beta(x) \tag{33}$$

Then, by taking the conditions (18) and (30) into account, we get the time derivative of V_1 along any solution of (33) satisfying

$$\begin{aligned}\dot{V}_1 &= L_{f_0}V_1 + L_{f_1}V_1 \cdot e(x_t) + L_g V_1 \cdot \beta(x) \\ &< -\tfrac{1}{2}\|L_{f_1}^T V_1\|^2 - \tfrac{1}{2}\|e(x)\|^2 + L_{f_1}V_1 \cdot e(x_t)\end{aligned} \tag{34}$$

with $\beta(x)$ being given by (32). In this case, it is clear that the remainder of the proof is straightforward from the proof of Theorem 2. ■

Theorem 4. *For system (16), suppose $f(x,x_t)$ is smooth and the following condition is satisfied*

H3: There exists a C^1 positive-definitive function $V_1(x_t(0))$ and a positive-definite function $Q(x)$ such that

$$L_{f_0}V_1 + \frac{1}{2}\nu_a^2(x) + \frac{1}{2}\nu_b^2(x) \le -Q(x) \tag{35}$$

Then, the closed-loop system (16) with (20) is strongly asymptotically stable.

Proof: First, note that by using the relations (28) and (29), the time derivative of V_1 along the trajectory of system $\dot{x} = f(x,x_t)$ satisfies

$$\begin{aligned}\dot{V}_1 &= L_{f_0}V_1 + L_{f_1}V_1 \\ &\le L_{f_0}V_1 + \nu_a(x)\nu_b(x_t) \\ &\le L_{f_0}V_1 + \tfrac{1}{2}\nu_a^2(x) + \tfrac{1}{2}\nu_b^2(x_t)\end{aligned} \tag{36}$$

and by the assumption *H3*, it yields

$$\dot{V}_1 \le -Q(x) - \frac{1}{2}\nu_b^2(x) + \frac{1}{2}\nu_b^2(x_t) \tag{37}$$

Then, choose a candidate Lyapunov-Krasovskii functional as follows

$$V(x_t) = V_1(x_t(0)) + \frac{1}{2}\int_{t-\tau}^{t} v_b^2(x(s))ds \tag{38}$$

and compute its time derivative along any Filippov solution of the closed-loop system of (16) with (20). It is straightforward to get

$$\dot{V}(x_t) \leq -Q(x) \tag{39}$$

Therefore, the strong asymptotic stability of the closed-loop system follows by Theorem 1. ■

IV. DESIGN EXAMPLE OF ENGINE SYSTEM

For the speed control of SI engines, it is well known that the intake-to-power delay is one of the main difficulties in the phases of control design and evaluating, moreover, rejecting load disturbance effectively is also an significant issue (refer to [23], [24]). In order to improve control precision, we pay attention to several typical disturbance torques which are usually known but possess switching properties according to engine speed or its acceleration direction, such as the engagement of automatic transmission and the electrical load due to the operation mode switching in hybrid electric vehicles.

In practical engineering, the mean value engine model developed in [24] is used widely, which includes the crankshaft rotational dynamics and the intake air dynamics. With a few minor extensions, it is described as follows

$$\dot{\omega}(t) = c_1 p_m(t-t_d) - D\omega(t) - T_l(t) \tag{40}$$

$$\dot{p}_m(t) = c_2 u_t(t) - c_3 p_m(t)\omega(t) \tag{41}$$

where t_d denotes the intake-to-power delay and T_l represents the load torque which is assumed to be modeled by

$$T_l = T_{l0} + T_{l1}\mathrm{sign}(\omega - \omega^*) \tag{42}$$

where $T_{l0} > 0$ denotes a fixed torque value and $T_{l1} > 0$ denotes the torque variation under the speed command signal ω^*. The other definitions of the parameters in the model and the expressions of the parameterized coefficients c_i $(i = 1-3)$ are shown in the appendix.

The design objective for engine speed acceleration control is as follows. Consider the case $\omega_d > \omega^*$. For any given desired speed ω_d, design a feedback controller $u_t = \alpha(\omega, \omega_d, p_m)$ for throttle opening such that $\omega \to \omega_d$ as $t \to \infty$ with any switching setting values ω^*. For simplicity, we suppose that all the system parameters and the torque values T_{l0}, T_{l1} are known and both the states ω and p_m can be measured.

Usually, the tracking control problem is solved by rendering the error between the desired trajectory and the actual output to be 0. By this motivation, we introduce the following feedback compensations first,

$$\begin{aligned} p_{md} &= \tfrac{1}{c_1}(D\omega + T_{l0} + T_{l1} - k_\omega e_\omega), \text{ and} \\ u_t &= u + \tfrac{1}{c_2}\left[c_3 p_m \omega - \tfrac{D-k_\omega}{c_1}(-k_\omega e_\omega + T_{l1}) - k_p e_p\right] \end{aligned} \tag{43}$$

where k_ω, k_p are given control parameters and u is the new control input signal, then the error dynamic system can be deduced as follows

$$\begin{cases} \dot{e}_\omega = -k_\omega e_\omega + c_1 e_p(t-t_d) + T_{l1} - T_{l1}\ \mathrm{sign}(e_\omega + \sigma) \\ \dot{e}_p = c_2 u - k_p e_p - (D-k_\omega)e_p(t-t_d) + \\ \qquad \frac{D-k_\omega}{c_1} T_{l1}\ \mathrm{sign}(e_\omega + \sigma) \end{cases} \tag{44}$$

where $\sigma = \omega_d - \omega^*$. Let $x = [e_\omega\ e_p]^T$, then, system (44) can be rewritten by

$$\dot{x} = f_0(x) + f_1(x)e_p(t-t_d) + h(x_t)\delta(x_t) + g(x_t)u \tag{45}$$

where $\delta(x_t) = \mathrm{sign}(e_\omega + \sigma)$ and

$$f_0(\cdot) = \begin{bmatrix} -k_\omega e_\omega + T_{l1} - T_{l1}\ \mathrm{sign}(e_\omega + \sigma) \\ -k_p e_p \end{bmatrix},$$

$$f_1(\cdot) = \begin{bmatrix} c_1 \\ -(D-k_\omega) \end{bmatrix},\ h(\cdot) = \begin{bmatrix} 0 \\ \frac{(D-k_\omega)T_{l1}}{c_1} \end{bmatrix},$$

$$g(\cdot) = \begin{bmatrix} 0 \\ c_2 \end{bmatrix}$$

Choose a C^1 positive-definite function $V_1(x_t(0))$ as follows

$$V_1 = \frac{1}{2}(e_\omega^2 + e_p^2) \tag{46}$$

then, it is easy to see that the condition (27) is satisfied with

$$k_\omega > \frac{c_1^2}{2} \quad \text{and} \quad k_p > 1 - \frac{1}{2}(D-k_\omega)^2$$

Therefore, by Corollary 2, the desired feedback controller for system (44) is

$$u = -\mathrm{sign}(c_2 e_p)\frac{|D-k_\omega|T_{l1}}{c_1 c_2} \tag{47}$$

The effectiveness of the proposed engine speed controller has been tested by simulation, where the initial speed value is set at $\omega(0) = 1000[rpm]$, and ω_d, ω^* are set at $3000[rpm]$ and $2000[rpm]$, respectively. With the control parameters $k_\omega = 0.2$, $k_p = 15$, we get the response results shown in Fig.1 that demonstrate the asymptotic stability of the closed-loop control system.

V. CONCLUSIONS

Filippov framework for describing the behavior of the solution of differential equations is not new in the community of mathematics, and recently, the framework has been introduced to analyze and synthesize discontinuous systems [2], [5], [6]. The basic idea is to use the solution of differential inclusion determined by Filippov set-valued map. In this paper, we have shown that by introducing a Filippov set-valued functional for FDEs, the conception of Filippov solution with the stability analysis framework can be also extended to discontinuous systems with time delay. It should be noted that the proposed design approach is a kind of domination-based control, which is motivated by robust design principle, more exactly, the presented controller uniformly ensures the stability for all possible Filippov solutions, and it is fortunately benefited by the smooth Lyapunov function. The presented example of engine control shows the feasibility of the proposed control approach.

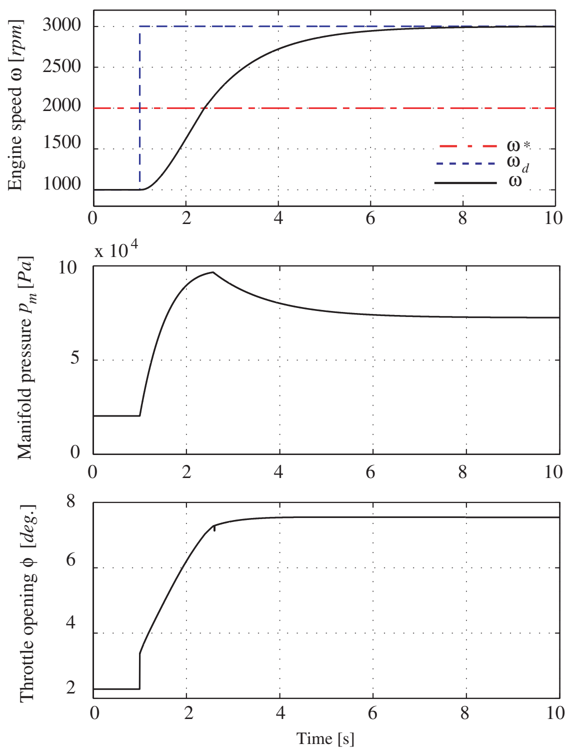

Fig. 1. Simulation results of the engine speed control problem

APPENDIX I

Parameter definitions of engine model:

$$c_1 = \frac{a\rho_a T_a V_c \eta}{4\pi T_m p_a}, \quad c_2 = \frac{RT_m s_0 f_p(p_m)}{V_m}, \quad c_3 = \frac{V_c \eta}{4\pi V_m},$$

$$D = \frac{D_0}{J}, \quad T_{l0} = \frac{\tau_{l0}}{J}, \quad T_{l1} = \frac{\tau_{l1}}{J}, \quad u = 1 - \cos\phi$$

ω	Engine speed $([rad \cdot s^{-1}])$
J	Crankshaft inertia $(0.1[kg \cdot m^2])$
D_0	Damping coefficient $(0.034[Nm \cdot s \cdot rad^{-1}])$
s_0	Throttle area $(3.5 \times 10^{-3}[m^2])$
ϕ	Throttle opening $([rad])$
V_c	Engine volume $(3.0 \times 10^{-6}[m^3])$
p_m	Manifold pressure $([Pa])$
T_m	Manifold temperature $(298.15[K])$
V_m	Manifold volume $(6.0 \times 10^{-3}[m^3])$
R	Gas constant (287)
p_a	Atmospheric pressure $(1.01 \times 10^5[Pa])$
T_a	Atmospheric Temperature $(298.15[K])$
ρ_a	Atmospheric density $(1.1837[g \cdot m^{-3}])$
τ_l	Load torque $([Nm])$
η	Volumetric efficiency $(1[-])$
a	Maximum torque capacity $(6.25 \times 10^5[-])$

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