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Decentralized Model Predictive Control via Dual Decomposition

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Abstract— **This paper proposes a decentralized model predictive control method based on a dual decomposition technique. A model predictive control problem for a system with multiple subsystems is formulated as a convex optimization problem. In particular, we deal with the case where the control outputs of the subsystems have coupling constraints represented by linear equalities. A dual decomposition technique is applied to this problem in order to derive the dual problem with decoupled equality constraints. A projected subgradient method is used to solve the dual problem, which leads to a decentralized algorithm. In the algorithm, a small-scale problem is solved at each subsystem, and information exchange is performed in each group consisting of some subsystems. Also, it is shown that the computational complexity in the decentralized algorithm is reduced if the dynamics of the subsystems are all the same. Numerical examples are given to show the effectiveness of the proposed method.**

I. INTRODUCTION

Research on control, estimation, and consensus under distributed and networked computing environments has received significant attention in recent years [3], [4], [5], [7], [8]. The common concept of such research is that an overall system achieves a goal while multiple subsystems interact with one another.

Keviczky *et al.* [4] have proposed a decentralized receding horizon control method and studied its stability conditions. Receding horizon control is a basic idea for model predictive control (MPC), and is based on optimization for control systems at each sampling time. The MPC was applied to systems with slow dynamics such as chemical plants. Nowadays, however, it is applicable to systems with fast dynamics such as mechanical systems because of performance progress of computers.

On the other hand, Samar *et al.* [8] have proposed a distributed estimation method via dual decomposition. The dual decomposition is a method for breaking a large-scale optimization problem into multiple small-scale optimization subproblems. Although information exchange among the subproblems is needed to solve the original problem, the decomposition method is useful for computational efficiency under distributed computing environments. The dual decomposition method has been also applied to trajectory optimization [6] and communication systems [9].

In this paper, we propose a decentralized MPC method based on the dual decomposition technique. We formulate an original MPC problem for a system with multiple subsystems as a convex optimization problem. In particular, we deal with the case where the control outputs of the subsystems have a coupling constraint represented by linear equalities. For instance, such an output constraint corresponds to formation of multiple vehicles. We next derive the dual problem associated with the original problem by applying a dual decomposition technique. We use a projected subgradient method to solve the dual problem, which leads to a decentralized algorithm.

This paper is organized as follows. In Section II, we formulate an MPC problem, and in Section III, we present a decentralized MPC algorithm derived from the dual decomposition technique. In Section IV, we discuss some cases of problem structures. Section V provides numerical examples to show the effectiveness of the proposed method.

II. PROBLEM FORMULATIOM

A. Model Predictive Control

We consider a system consisting of N subsystems represented by an SISO discrete-time linear time-invariant model:

$$\begin{aligned} x_{k+1}^{(i)} &= A^{(i)} x_k^{(i)} + B^{(i)} u_k^{(i)} \\ y_k^{(i)} &= C^{(i)} x_k^{(i)} \end{aligned} \tag{1}$$

where $u_k^{(i)} \in \Re$, $x_k^{(i)} \in \Re^{n_x^{(i)}}$, and $y_k^{(i)} \in \Re$ are the control input, the state variable, and the control output, respectively, of subsystem $S^{(i)}$, $i = 1, \ldots, N$, and $A^{(i)}$, $B^{(i)}$, and $C^{(i)}$ are the coefficient matrices.

For subsystem $S^{(i)}$, $i = 1, \ldots, N$, we define the following performance index at time k:

$$J_k^{(i)} = \sum_{j=1}^{m} \left(y_{k+j}^{(i)} \right)^2 + w^{(i)} \sum_{j=0}^{m-1} \left(u_{k+j}^{(i)} \right)^2$$

where $w^{(i)} > 0$ is a weight, and m is the prediction horizon. Our control problem is now represented by the following optimization problem:

$$\begin{aligned} \underset{u_{k+j}^{(i)}, i=1,\ldots,N, j=0,\ldots,m-1}{\text{minimize}} \quad & \sum_{i=1}^{N} J_k^{(i)} \\ \text{subject to} \quad & y_{k+j}^{(1)} + p_{k+j}^{(1)} = \cdots = y_{k+j}^{(N)} + p_{k+j}^{(N)}, \\ & \quad j = 1, \ldots, m \\ & u_{k+j}^{(i)} \in \mathcal{U}_{k+j}^{(i)}, \ i = 1, \ldots, N, \\ & \quad j = 0, \ldots, m-1 \end{aligned} \tag{2}$$

where $\mathcal{U}_{k+j}^{(i)}$ is a convex set, and $p_{k+j}^{(i)}$ specifies a relative difference to the outputs of the other subsystems. We assume that $p_{k+j}^{(i)}$ is smaller than $y_{k+j}^{(i)}$ to avoid contradiction between the objective and the constraint. For instance, the output constraint can represent a formation constraint of vehicles.

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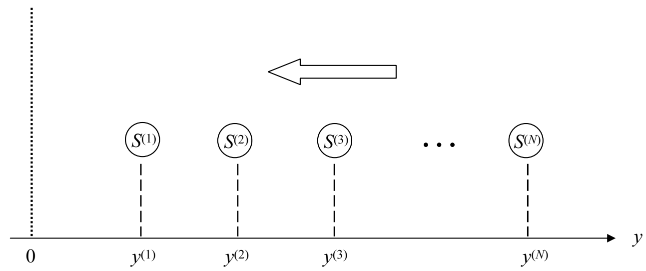


Fig. 1. An example of formation control of N subsystems.

The problem (2) is a convex optimization problem since the objective function is quadratic and convex, and the constraints are convex.

In an MPC scheme, the optimization problem (2) is solved at each sampling time. Then the first control input $u_k^{(i)*}$ from the optimal control input sequence $u_{k+j}^{(i)*}, j = 0, \ldots, m-1$ is implemented. This procedure is repeated at time $k+1$. The goal represented by the problem (2) is that the inputs and outputs of the overall system go to zero while the specified output formation constraint and the control input constraint are satisfied. Fig. 1 shows an example of formation control of N subsystems $S^{(1)}, \ldots, S^{(N)}$.

B. Reformulation by Vector and Matrix Representation

For convenience, we express system (1) in terms of vectors and matrices. From system (1), we obtain

$$\begin{bmatrix} y_{k+1}^{(i)} \\ \vdots \\ y_{k+m}^{(i)} \end{bmatrix} = \begin{bmatrix} C^{(i)}A^{(i)} \\ \vdots \\ C^{(i)}(A^{(i)})^m \end{bmatrix} x_k^{(i)} + \begin{bmatrix} C^{(i)}B^{(i)} & 0 & \cdots & 0 \\ C^{(i)}A^{(i)}B^{(i)} & C^{(i)}B^{(i)} & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ C^{(i)}(A^{(i)})^{m-1}B^{(i)} & \cdots & \cdots & C^{(i)}B^{(i)} \end{bmatrix} \cdot \begin{bmatrix} u_k^{(i)} \\ \vdots \\ u_{k+m-1}^{(i)} \end{bmatrix}. \quad (3)$$

By denoting

$$\hat{y}_k^{(i)} = \begin{bmatrix} y_{k+1}^{(i)} \\ \vdots \\ y_{k+m}^{(i)} \end{bmatrix}, F^{(i)} = \begin{bmatrix} C^{(i)}A^{(i)} \\ \vdots \\ C^{(i)}(A^{(i)})^m \end{bmatrix}$$

$$H^{(i)} = \begin{bmatrix} C^{(i)}B^{(i)} & 0 & \cdots & 0 \\ C^{(i)}A^{(i)}B^{(i)} & C^{(i)}B^{(i)} & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ C^{(i)}(A^{(i)})^{m-1}B^{(i)} & \cdots & \cdots & C^{(i)}B^{(i)} \end{bmatrix}$$

$$\hat{u}_k^{(i)} = \begin{bmatrix} u_k^{(i)} \\ \vdots \\ u_{k+m-1}^{(i)} \end{bmatrix}, \hat{p}_k^{(i)} = \begin{bmatrix} p_{k+1}^{(i)} \\ \vdots \\ p_{k+m}^{(i)} \end{bmatrix},$$

(3) is written as follows:

$$\hat{y}_k^{(i)} = F^{(i)} x_k^{(i)} + H^{(i)} \hat{u}_k^{(i)}. \quad (4)$$

Therefore, we express the performance index $J_k^{(i)}$ as

$$J_k^{(i)} = \|\hat{y}_k^{(i)}\|^2 + w^{(i)} \|\hat{u}_k^{(i)}\|^2$$

where $\|\cdot\|$ denotes the Euclidean norm. As a concequence, we obtain a more compact form of the optimization problem (2):

$$\begin{array}{ll} \underset{\hat{u}_k^{(i)}, i=1,\ldots,N}{\text{minimize}} & \sum_{i=1}^{N} J_k^{(i)} \\ \text{subject to} & \hat{y}_k^{(1)} + \hat{p}_k^{(1)} = \cdots = \hat{y}_k^{(N)} + \hat{p}_k^{(N)}, \\ & \hat{u}_k^{(i)} \in \hat{\mathcal{U}}_k^{(i)}, \ i = 1, \ldots, N \end{array} \quad (5)$$

where $\hat{\mathcal{U}}_k^{(i)} = \mathcal{U}_k^{(i)} \times \cdots \times \mathcal{U}_{k+m-1}^{(i)}$. Although the objective function in (5) is decomposed into the individual performance index for each subsystem, the optimization problem (5) is not solved individually for each subsystem because of the coupling equality constraint on the control outputs. Therefore, the optimization problem (5) must be solved at one computer by gathering information on all subsystems, which implies that (5) must be solved in a manner of centralized control.

III. DECENTRALIZED CONTROL VIA DUAL DECOMPOSITION

In this section, we present a decentralized algorithm for solving the optimization problem (5) by using a dual decomposition technique.

A. Dual Problem

We consider the case where the subsystems communicate with their neighbors as shown in Fig. 2. More concretely, subsystem $S^{(i)}$ performs information exchange with subsystem $S^{(i-1)}$ and $S^{(i+1)}$. We refer to such a pair as a *group* for information exchange. This system has a chain structure.

Corresponding to the $N-1$ groups, we express the equality constraint in (5) as the following equivalent equality constraints:

$$\begin{aligned} \hat{y}_k^{(1)} + \hat{p}_k^{(1)} &= \hat{y}_k^{(2)} + \hat{p}_k^{(2)} \\ \hat{y}_k^{(2)} + \hat{p}_k^{(2)} &= \hat{y}_k^{(3)} + \hat{p}_k^{(3)} \\ &\vdots \\ \hat{y}_k^{(N-1)} + \hat{p}_k^{(N-1)} &= \hat{y}_k^{(N)} + \hat{p}_k^{(N)}. \end{aligned} \quad (6)$$

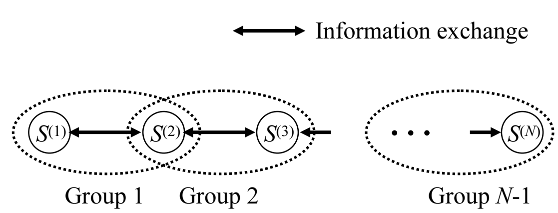


Fig. 2. Groups for information exchange.

Here we introduce variables $z_k^{(i)}$ for these equalities, and express them as follows:

$$\begin{bmatrix} \hat{y}_k^{(i)} + \hat{p}_k^{(i)} \\ \hat{y}_k^{(i+1)} + \hat{p}_k^{(i+1)} \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix} z_k^{(i)}, \ i = 1, \dots, N-1. \quad (7)$$

We assume that $C^{(i)}B^{(i)} \neq 0$. Since $H^{(i)}$ is nonsingular, we obtain from (4) and (7),

$$\begin{bmatrix} \hat{u}_k^{(i)} \\ \hat{u}_k^{(i+1)} \end{bmatrix} = \begin{bmatrix} (H^{(i)})^{-1} \\ (H^{(i+1)})^{-1} \end{bmatrix} z_k^{(i)} - \begin{bmatrix} r_k^{(i)} \\ r_k^{(i+1)} \end{bmatrix},$$

where $r_k^{(i)} = (H^{(i)})^{-1}(F^{(i)}x_k^{(i)} + \hat{p}_k^{(i)})$.

Next, we form the partial Lagrangian, by introducing Lagrange multipliers only for the coupling equality constraints

$$\begin{aligned} &L(\hat{u}_k, z_k, \nu_k) \\ &= \sum_{i=1}^{N} J_k^{(i)} - \sum_{i=1}^{N-1} [(\nu_k^{(i,1)})^T \ (\nu_k^{(i,2)})^T] \left(\begin{bmatrix} \hat{u}_k^{(i)} \\ \hat{u}_k^{(i+1)} \end{bmatrix} \right. \\ &\quad \left. - \begin{bmatrix} (H^{(i)})^{-1} \\ (H^{(i+1)})^{-1} \end{bmatrix} z_k^{(i)} + \begin{bmatrix} r_k^{(i)} \\ r_k^{(i+1)} \end{bmatrix} \right) \end{aligned} \quad (8)$$

where

$$\hat{u}_k = \begin{bmatrix} \hat{u}_k^{(1)} \\ \vdots \\ \hat{u}_k^{(N)} \end{bmatrix}, \quad z_k = \begin{bmatrix} z_k^{(1)} \\ \vdots \\ z_k^{(N-1)} \end{bmatrix},$$

$$\nu_k^{(i)} = \begin{bmatrix} \nu_k^{(i,1)} \\ \nu_k^{(i,2)} \end{bmatrix}, \quad \nu_k = \begin{bmatrix} \nu_k^{(1)} \\ \vdots \\ \nu_k^{(N-1)} \end{bmatrix}.$$

Notice that the Lagrange multiplier $\nu_k^{(i)}$ is associated with the ith group, and the decomposition of the problem will be possible by this treatment of the equality constraints.

We let $q(\nu_k)$ denote the dual function:

$$q(\nu_k) = \inf_{\hat{u}_k \in \hat{\mathcal{U}}_k, z_k} L(\hat{u}_k, z_k, \nu_k)$$

where $\hat{\mathcal{U}}_k = \hat{\mathcal{U}}_k^{(1)} \times \cdots \times \hat{\mathcal{U}}_k^{(N)}$. Suppose that $\hat{u}_k'$ is a feasible solution of (5). Then there exists $z_k^{(i)\prime}$ satisfying (7), and

$$\begin{bmatrix} \hat{u}_k^{(i)\prime} \\ \hat{u}_k^{(i+1)\prime} \end{bmatrix} - \begin{bmatrix} (H^{(i)})^{-1} \\ (H^{(i+1)})^{-1} \end{bmatrix} z_k^{(i)\prime} + \begin{bmatrix} r_k^{(i)} \\ r_k^{(i+1)} \end{bmatrix} = 0$$

holds. Therefore, denoting the objective function of (5) by $f(\hat{u}_k) = \sum_{i=1}^{N} J_k^{(i)}$, we obtain the following relationship:

$$q(\nu_k) = \inf_{\hat{u}_k \in \hat{\mathcal{U}}_k, z_k} L(\hat{u}_k, z_k, \nu_k) \leq L(\hat{u}_k', z_k', \nu_k) = f(\hat{u}_k'). \quad (9)$$

This inequality implies the weak duality [2].

To find the condition for the dual function to be finite, we first minimize the Lagrangian $L(\hat{u}_k, z_k, \nu_k)$ over z_k, which results in the condition

$$[(\nu_k^{(i,1)})^T \ (\nu_k^{(i,2)})^T] \begin{bmatrix} (H^{(i)})^{-1} \\ (H^{(i+1)})^{-1} \end{bmatrix} = 0, \quad i = 1, \dots, N-1 \quad (10)$$

for $q(\nu_k) > -\infty$.

Under the condition (10), the dual function can be decomposed into the sum of the following functions for subsystem $S^{(i)}$:

$$\begin{aligned} &q^{(1)}(\nu_k^{(1,1)}) \\ &\quad = \min_{\hat{u}_k^{(1)} \in \hat{\mathcal{U}}_k^{(1)}} \{J_k^{(1)} - (\nu_k^{(1,1)})^T(\hat{u}_k^{(1)} + r_k^{(1)})\} && (11) \\ &q^{(i)}(\nu_k^{(i-1,2)}, \nu_k^{(i,1)}) \\ &\quad = \min_{\hat{u}_k^{(i)} \in \hat{\mathcal{U}}_k^{(i)}} \{J_k^{(i)} - (\nu_k^{(i-1,2)} + \nu_k^{(i,1)})^T(\hat{u}_k^{(i)} + r_k^{(i)})\}, \\ &\qquad\qquad i = 2, \dots, N-1 && (12) \\ &q^{(N)}(\nu_k^{(N-1,2)}) \\ &\quad = \min_{\hat{u}_k^{(N)} \in \hat{\mathcal{U}}_k^{(N)}} \{J_k^{(N)} - (\nu_k^{(N-1,2)})^T(\hat{u}_k^{(N)} + r_k^{(N)})\}. && (13) \end{aligned}$$

These functions are concave and nondifferentiable because they are the pointwise infimum of linear functions [2].

To sum up, the dual problem of (5) is

$$\begin{aligned} &\underset{\nu_k}{\text{maximize}} \quad q(\nu_k) = \sum_{i=1}^{N} q^{(i)} && (14) \\ &\text{subject to} \quad [(\nu_k^{(i,1)})^T \ (\nu_k^{(i,2)})^T] \begin{bmatrix} (H^{(i)})^{-1} \\ (H^{(i+1)})^{-1} \end{bmatrix} = 0, \\ &\qquad\qquad i = 1, \dots, N-1. \end{aligned}$$

Notice that the constraints are imposed for each group. Although inequality (9) implies the weak duality, the strong duality holds, *i.e.*, the duality gap reduces to zero if the Slater's constraint qualification is satisfied [2]. We assume the strong duality holds. This implies that the primal problem (2) can be equivalently solved by solving the dual problem (14).

When there is no constraint on the control input $\hat{u}_k^{(i)}$, we can analytically calculate the optimal solutions $\hat{u}_k^{(i)*}$ providing the function values of (11)–(13). Noting that the functions to be minimized in (11)–(13) are quadratic and convex, we obtain these solutions as follows:

$$\begin{aligned} \hat{u}_k^{(1)*} &= \frac{1}{2} \left((H^{(1)})^T H^{(1)} + w^{(1)} I \right)^{-1} \\ &\qquad \cdot \left(\nu_k^{(1,1)} - 2(H^{(1)})^T F^{(1)} x_k^{(1)} \right) \\ \hat{u}_k^{(i)*} &= \frac{1}{2} \left((H^{(i)})^T H^{(i)} + w^{(i)} I \right)^{-1} \\ &\qquad \cdot \left(\nu_k^{(i-1,2)} + \nu_k^{(i,1)} - 2(H^{(i)})^T F^{(i)} x_k^{(i)} \right), \\ &\qquad\qquad i = 2, \dots, N-1 \\ \hat{u}_k^{(N)*} &= \frac{1}{2} \left((H^{(N)})^T H^{(N)} + w^{(N)} I \right)^{-1} \\ &\qquad \cdot \left(\nu_k^{(N-1,2)} - 2(H^{(N)})^T F^{(N)} x_k^{(N)} \right). \end{aligned}$$

B. Subgradient Method

The objective function of the dual problem (14) is nondifferentiable, and therefore, we use a subgradient method to solve it. Also, because the dual variables, *i.e.*, the Lagrange multipliers must satisfy the linear equality constraints (10), we update a candidate of the optimal solution by using projection of the subgradient to the hyperplane associated with the equality constraint. This kind of solution method is called a projected subgradient method [8]. We develop a simple decentralized algorithm to solve the dual problem (14) by applying the projected subgradient method.

The subgradient of a concave function is defined as follows.

Definition 1: Suppose that $\phi : \Re^n \to \Re$ is concave, and $x \in \Re^n$. Then g is a *subgradient* of ϕ at x if there exists a $g \in \Re^n$ such that

$$\phi(\xi) \le \phi(x) + g^T(\xi - x)$$

holds for any $\xi \in \Re^n$.

The basic subgradient method for maximizing the objective functions (11)–(13) uses the iterations

$$\begin{aligned} \nu_{k,j+1}^{(1,1)} &= \nu_{k,j}^{(1,1)} + \alpha_j g_{k,j}^{(1,1)} \\ \begin{bmatrix} \nu_{k,j+1}^{(i-1,2)} \\ \nu_{k,j+1}^{(i,1)} \end{bmatrix} &= \begin{bmatrix} \nu_{k,j}^{(i-1,2)} \\ \nu_{k,j}^{(i,1)} \end{bmatrix} + \alpha_j \begin{bmatrix} g_{k,j}^{(i-1,2)} \\ g_{k,j}^{(i,1)} \end{bmatrix}, \\ & \quad i = 2, \dots, N-1 \\ \nu_{k,j+1}^{(N-1,2)} &= \nu_{k,j}^{(N-1,2)} + \alpha_j g_{k,j}^{(N-1,2)} \end{aligned}$$

where j is the number of iterations, $g_{k,j}^{(i,l)}$ are any subgradients of the objective functions $q^{(i)}$, and $\alpha_j > 0$ is the jth step size. At each iteration of the subgradient method, we take a step in the direction of positive subgradient.

Subgradients of the dual objective functions (11)–(13) are given as follows.

Theorem 1: For the dual objective functions (11)–(13), let $\hat{u}_k^{(i)*}, i = 1, \dots, N$ denote the optimal solutions providing their function values. Then

$$g_k^{(1,1)} = -\hat{u}_k^{(1)*} - r_k^{(1)} \tag{15}$$

$$\begin{cases} g_k^{(i-1,2)} = -\hat{u}_k^{(i)*} - r_k^{(i)} \\ g_k^{(i,1)} = -\hat{u}_k^{(i)*} - r_k^{(i)} \end{cases}, \ i = 2, \dots, N-1 \tag{16}$$

$$g_k^{(N-1,2)} = -\hat{u}_k^{(N)*} - r_k^{(N)} \tag{17}$$

are subgradients of $q^{(1)}(\nu_k^{(1,1)})$, $q^{(i)}(\nu_k^{(i-1,2)}, \nu_k^{(i,1)}), i = 2, \dots, N-1$, and $q^{(N)}(\nu_k^{(N-1,2)})$, respectively.

Proof: Let $J_k^{(1)*}$ denote the optimal value of $J_k^{(1)}$ corresponding to the optimal solution $\hat{u}_k^{(1)*}$. Then, the following relationship for (15) holds for any $\xi \in \Re^m$:

$$\begin{aligned} & q^{(1)}(\nu_k^{(1,1)}) + (g_k^{(1,1)})^T(\xi - \nu_k^{(1,1)}) \\ &= \min_{\hat{u}_k^{(1)} \in \mathcal{U}^{(1)}} \{J_k^{(1)} - (\nu_k^{(1,1)})^T(\hat{u}_k^{(1)} + r_k^{(1)})\} \\ &\quad + (g_k^{(1,1)})^T(\xi - \nu_k^{(1,1)}) \\ &= J_k^{(1)*} - (\nu_k^{(1,1)})^T(\hat{u}_k^{(1)*} + r_k^{(1)}) \\ &\quad + (g_k^{(1,1)})^T(\xi - \nu_k^{(1,1)}) \\ &= J_k^{(1)*} - \xi^T(\hat{u}_k^{(1)*} + r_k^{(1)}) \\ &\ge \min_{\hat{u}_k^{(1)} \in \mathcal{U}^{(1)}} \{J_k^{(1)} - \xi^T(\hat{u}_k^{(1)} + r_k^{(1)})\} \\ &= q^{(1)}(\xi). \end{aligned}$$

From the definition of a subgradient, we see that $g_k^{(1,1)} = -\hat{u}_k^{(1)*} - r_k^{(1)}$ is a subgradient of $q^{(1)}(\nu_k^{(1,1)})$ at $\nu_k^{(1,1)}$. In the same manner, we can prove that (16) and (17) are subgradients of the other dual functions $q^{(i)}(\nu_k^{(i-1,2)}, \nu_k^{(i,1)}), i = 2, \dots, N-1$, $q^{(N)}(\nu_k^{(N,2)})$. ■

The dual variables ν_k must be updated by using the above subgradients so that the equality constraints (10) are satisfied. To this end, denoting

$$E^{(i)} := \begin{bmatrix} (H^{(i)})^{-1} \\ (H^{(i+1)})^{-1} \end{bmatrix},$$

we project the subgradients onto the hyperplanes $\{\zeta \in \Re^{2m} | (E^{(i)})^T \zeta = 0\}$, and then update as usual. From the assumption $C^{(i)}B^{(i)} \neq 0$, $(E^{(i)})^T$ is fat and full rank, and therefore, the projection operator is given by

$$P(v) = v - E^{(i)}((E^{(i)})^T E^{(i)})^{-1}(E^{(i)})^T v.$$

Thus, the projected subgradient method is given by

$$\begin{aligned} \begin{bmatrix} \nu_{k,j+1}^{(i,1)} \\ \nu_{k,j+1}^{(i,2)} \end{bmatrix} &= \begin{bmatrix} \nu_{k,j}^{(i,1)} \\ \nu_{k,j}^{(i,2)} \end{bmatrix} + \alpha_j^{(i)}(I \\ & -E^{(i)}((E^{(i)})^T E^{(i)})^{-1}(E^{(i)})^T) \begin{bmatrix} g_{k,j}^{(i,1)} \\ g_{k,j}^{(i,2)} \end{bmatrix}, \\ & i = 1, \dots, N-1. \end{aligned} \tag{18}$$

Notice that this update rule is performed in each group for information exchange.

We can summarize a decentralized MPC algorithm to solve (2) by using dual decomposition as follows:

Decentralized MPC algorithm via dual decomposition

For the current state $x_k^{(i)}$, the following steps are performed and then implement $u_k^{(i)*}$.

Step 1. Given initial vectors $\nu_{k,0}^{(i)} = [(\nu_{k,0}^{(i,1)})^T \ (\nu_{k,0}^{(i,2)})^T]^T, i = 1, \dots, N-1$ satisfying

$$(E^{(i)})^T \begin{bmatrix} \nu_{k,0}^{(i,1)} \\ \nu_{k,0}^{(i,2)} \end{bmatrix} = 0 \tag{19}$$

(*e.g.*, zero vectors). Set $j = 1$.

Step 2. Solve the minimization problems in the functions (11)–(13) to obtain the optimal solutions $\hat{u}_{k,j}^{(i)*}, i = 1, \dots, N$.

Step 3. Calculate the subgradients $g_{k,j}^{(i,1)}, g_{k,j}^{(i,2)}, i = 1, \dots, N$ according to (15)–(17).

Step 4. Communicate the subgradients among each group, and update the dual variables by (18).

Step 5. If a stopping criterion is satisfied, stop. Otherwise, set $j = j + 1$, and go to Step 2.

As a stopping criterion of the above algorithm, we can use the number of iterations, the convergence tolerance of the dual variables, the norm of the subgradients, and so on.

We expect that the optimal dual variables are not so different between the current and the next sampling time in the MPC scheme. Therefore, we can use the dual variables computed at the current time as initial variables at the next time.

Notice that when a new subsystem is added to the overall system, the algorithm at the group with the new subsystem is only modified. This is one of the advantages of the proposed decentralized MPC algorithm.

IV. REMARKS ON SOME SPECIAL AND GENERAL CASES

In this section, we first consider a special case where all subsystems have the same dynamics, and present that the computation becomes simple. Next we discuss a case where a subsystem belongs to more than two groups while a subsystem belongs to at most two groups in the previous sections.

A. Case of subsystems with a unique dynamics

When all subsystems have the same dynamics, we can represent the coefficient matrices in the model (1) by $A = A^{(i)}, B = B^{(i)}, C = C^{(i)}, i = 1, \dots, N$. Since the following matrices are all the same as denoted by $H = H^{(i)}$ and

$$E := E^{(i)} = \begin{bmatrix} H^{-1} \\ H^{-1} \end{bmatrix},$$

we obtain

$$E(E^T E)^{-1} E^T = \frac{1}{2} \begin{bmatrix} I & I \\ I & I \end{bmatrix}.$$

This implies

$$E(E^T E)^{-1} E^T \begin{bmatrix} g_k^{(i,1)} \\ g_k^{(i,2)} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} g_k^{(i,1)} + g_k^{(i,2)} \\ g_k^{(i,1)} + g_k^{(i,2)} \end{bmatrix}$$

at the update rule of Step. 4 in the proposed algorithm. Therefore, it is not necessary to use E at the update rule (18). We compute nothing but the average of the subgradients $g_k^{(i,1)}$ and $g_k^{(i,2)}$ at each group. As shown above, the computation at the update rule becomes simple when the dynamics of the subsystems are the same.

B. Case of a subsystem belonging to more than two groups

We now consider the case where a subsystem belongs to three groups as shown in Fig. 3, and present the dual problem briefly.

Taking into account infomation exchange, we regard the equality constraint corresponding to (6) as

$$\begin{aligned} \hat{y}_k^{(1)} + \hat{p}_k^{(1)} &= \hat{y}_k^{(2)} + \hat{p}_k^{(2)} \\ \hat{y}_k^{(2)} + \hat{p}_k^{(2)} &= \hat{y}_k^{(3)} + \hat{p}_k^{(3)} \\ \hat{y}_k^{(2)} + \hat{p}_k^{(2)} &= \hat{y}_k^{(4)} + \hat{p}_k^{(4)}, \end{aligned}$$

and derive the dual problem as in the previous section. The dual funcion for subsystem $S^{(2)}$ is shown by

$$\begin{aligned} & q^{(2)}(\nu_k^{(1,2)}, \nu_k^{(2,1)}, \nu_k^{(3,1)}) \\ &= \min_{\hat{u}_k^{(2)} \in \hat{\mathcal{U}}_k^{(2)}} \{ J_k^{(2)} - (\nu_k^{(1,2)} + \nu_k^{(2,1)} + \nu_k^{(3,1)})^T \\ & \qquad \cdot (\hat{u}_k^{(2)} + r_k^{(2)}) \}, \end{aligned}$$

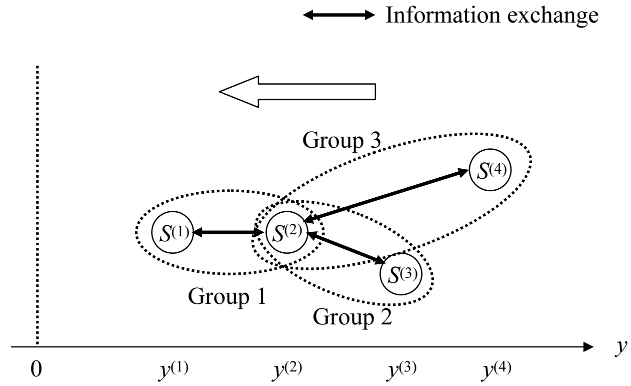


Fig. 3. Case of a subsystem belonging to three groups.

and the equality constraints on the groups are shown by

$$[(\nu_k^{(1,1)})^T \ (\nu_k^{(1,2)})^T] \begin{bmatrix} (H^{(1)})^{-1} \\ (H^{(2)})^{-1} \end{bmatrix} = 0$$

$$[(\nu_k^{(2,1)})^T \ (\nu_k^{(2,2)})^T] \begin{bmatrix} (H^{(2)})^{-1} \\ (H^{(3)})^{-1} \end{bmatrix} = 0$$

$$[(\nu_k^{(3,1)})^T \ (\nu_k^{(3,2)})^T] \begin{bmatrix} (H^{(2)})^{-1} \\ (H^{(4)})^{-1} \end{bmatrix} = 0.$$

We can solve the optimization problem (5) by performing the proposed algorithm for the above dual function and constraints.

As shown above, we can extend the decentralized MPC algorithm to the case where the group of information exchange does not have a chain structure.

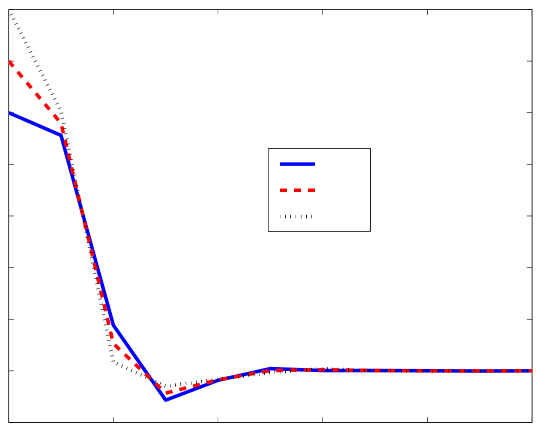
V. NUMERICAL EXAMPLES

In this section, we show two examples of the proposed method. We first consider a simple example for a system with the chain structure shown in Fig. 2. Let the number of subsystems be $N = 3$, and we deal with the discrete-time system of a double integrator $1/s^2$ discretized with sampling time 0.5 (s). As a simple output constraint, we impose $y_k^{(1)} = y_k^{(2)} = y_k^{(3)}$, $k = 1, 2, \ldots$, while there is no constraint on the control input. We set the time interval for performance evaluation and the control horizon by $m = 5$, and the weight in the performance index by $w^{(i)} = 0.01, i = 1, 2, 3$. Let the initial outputs be $y_0^{(1)} = 5, y_0^{(2)} = 6, y_0^{(3)} = 7$.

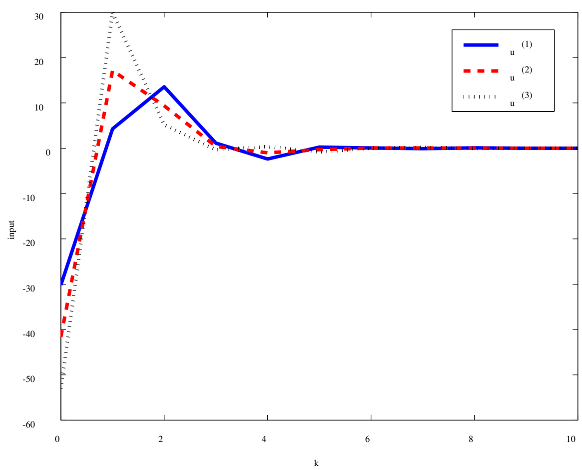
We solve the problem using the decentralized MPC algorithm. We use a diminishing step size $\alpha_j^{(i)} = 0.02/\sqrt{j}, i = 1, 2, 3$ in (18). Let the stopping criterion of the algorithm be the maximum number of iterations ($M = 5$). Fig. 4 shows the control outputs and inputs of the subsystems. It is seen from the figure that the output constraint is almost satisfied.

Fig. 5 shows the duality gap for the problem at $k = 0$ The duality gap converges to zero and is relatively small after 5 iterations.

Next, we show an example of 15 subsystems. The settings are almost the same as the above example. Let the output constraint be $y_k^{(1)} = \cdots = y_k^{(15)}$, and initial outputs be $y_0^{(1)} = 5, y_0^{(2)} = 6, \ldots, y_0^{(15)} = 19$ ($y_0^{(i)} = i + 4, i = 1, \ldots, 15$).



(a) Control output.



(b) Control input.

Fig. 4. A simple example of 3 subsystems.

Under these settings, the control outputs of the subsystems are shown in Fig. 6. As seen from the figure, the output constraint is satisfied after 10 iterations.

VI. CONCLUSION

In this paper, we have proposed a decentralized MPC method based on the dual decomposition method. When the proposed algorithm is iterated sufficiently, the original MPC problem is solved exactly in a decentralized manner. The numerical examples have shown that the proposed MPC scheme converges even for the case where the number of iteration in the algorithm is not so large.

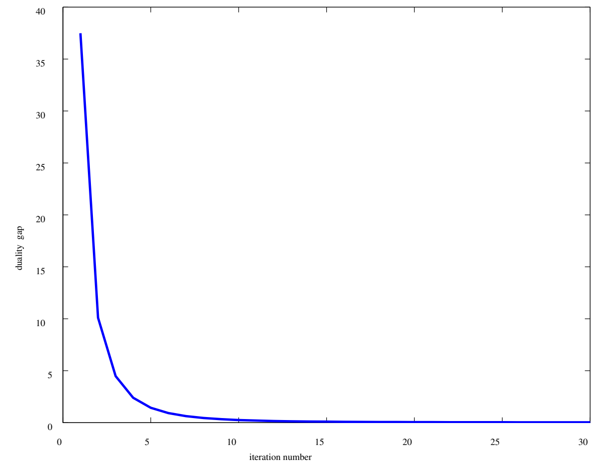


Fig. 5. Duality gap versus iteration number.

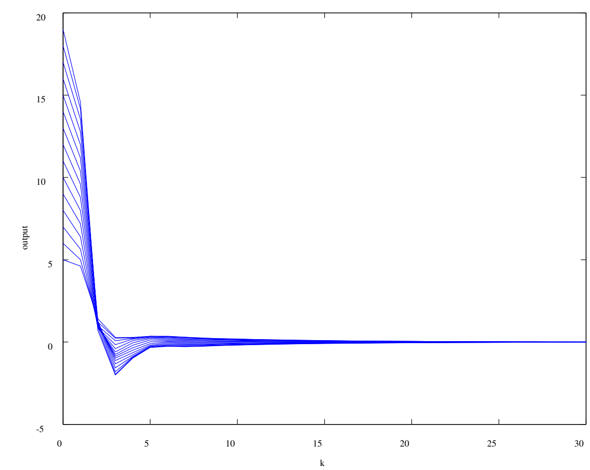


Fig. 6. Control output in an example of 15 subsystems.

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