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Characterization of Discrete-time $\mathcal{H}_2$ Control Performance Limitation Based on Poles and Zeros

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Abstract— **In this paper, we show that the best achievable performance in the $\mathcal{H}_2$ regulation/tracking problem for discrete-time SISO systems can be expressed in a remarkably simple form by the poles/zeros of the plants and the poles of the closed-loop systems. We also explain that the poles of the closed-loop systems can be related with system parameters by an algebraic approach. Furthermore, corresponding expressions in the δ-domain are given to reveal the relationship between the results in the z-domain and the s-domain. The derived expressions directly connect the system parameters and the resultant optimal $\mathcal{H}_2$ performance and, therefore, they are effectively utilized in system parameter tuning.**

I. INTRODUCTION

The original purpose in LQ or $\mathcal{H}_2$ optimal control, which is one of the important results in modern control theory that were solved in the 1960's and are still popular, was mainly on the pursuit of optimal controllers by employing then popularizing computers. In other words, the focus was on the search of algorithms to automatically derive controllers from given plants with theoretical justification. However, the relationship between the plant and the resulting optimized closed-loop system was unclear. This causes a problem in that we may have a plant and its optimal controller that yield a very poor performance, even if we can recover the performance by tuning some system parameters based on the knowledge of the relationship. Therefore, knowing the optimal performance as a function of the plant parameters is a relevant issue.

With such motivation in mind, a significant amount of effort has been expended in the name of performance limitations achievable by feedback control [1], [2]. Recent contributions include the search for expressions of performance limitations in certain $\mathcal{H}_2$ or $\mathcal{H}_\infty$ control problem frameworks [3]–[8]. The obtained expressions are provided in terms of plant characteristics such as unstable poles, non-minimum phase zeros, plant gain, and time-delay. Such results give a clear qualitative indication which characteristics and combinations thereof may deteriorate the best performance level achievable by feedback control [5], [6].

It is quite often the case also that the plant has only some real parameters that can be tuned. Once those parameters are fixed, one can compute the optimal controller, employing standard approaches such as the solution of algebraic Riccati equations or the LMI-based optimization. The task is therefore to find the best values for the parameters that lead to the very best performance, i.e., to achieve the 'best of the best'. The results on performance limitations mentioned above are not necessarily suited for this purpose. The reason is that plant characteristics such as unstable poles and non-minimum phase zeros may not always be expressed explicitly in terms of parameters and also that the analytic integral of the frequency gain of the plant cannot in general be evaluated in the presence of parameters. To overcome this difficulty and allow quantitative analyses, an algebraic approach is developed for the SISO continuous-time case in [9] that makes use of powerful algebraic tools of Gröbner bases [10] and Cylindrical Algebraic Decomposition (CAD) [11], which can be effective for the parametric case. Also derived in [9] are expressions of performance limitations for the continuous-time $\mathcal{H}_2$ regulation and tracking problems that can be exploited by an algebraic approach and an algebraic optimization approach.

The natural extension of our interests is the derivation of the discrete-time counterpart of the results reported in [9]. An important fact to note is that the results for the discrete-time case cannot be computed immediately from the continuous-time results through a simple transform such as the bilinear transform and, therefore, the discrete-time results are to be deduced by appropriate investigation. With the above background, the subject of this paper is an extension of the results in [9] to discrete-time systems. More specifically, we show that the achievable performance levels in the $\mathcal{H}_2$ regulation and tracking problems for discrete-time SISO systems can be expressed in terms of the poles and the zeros of the plants and the poles of the closed-loop systems. We then explain that the poles of the closed-loop systems can be related by algebraic calculus with plant parameters. Moreover, we give corresponding expressions in the δ-domain and discuss the relationship between the results in the z-domain and the s-domain derived in [9].

The paper is organized as follows. In Section II, we show an illustrative example to demonstrate the efficiency of the results of this paper. In Section III, we give several preparative lemmas and theorems and, in Section IV, we show one of our main results: an expression of the best achievable performance for the discrete-time $\mathcal{H}_2$ regulation problem. We then give another main result in Section V: an expression of the performance limitation for the discrete-time $\mathcal{H}_2$ tracking problem. Then, in Section VI, we give a further explanation on the numerical example investigated in

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Section II. Finally, we conclude in Section VII.

Notation: $\mathbb{N}$: the set of natural numbers, $\mathbb{Q}$: the set of rational numbers, $\mathbb{R}$: the set of real numbers, $\mathbb{C}$: the set of complex numbers, $\mathbb{R}[\cdot]$: the polynomial ring (possibly multivariate), $\mathbb{R}(\cdot)$: the rational function field (also possibly multivariate), $\mathbb{C}_- := \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) < 0\}$, $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) > 0\}$, $\mathbb{C}_0 := \{s \in \mathbb{C} \mid \mathrm{Re}\,(s) = 0\}$, $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$, $\overline{\mathbb{D}}^{\mathrm{c}} := \{z \in \mathbb{C} \mid |z| > 1\}$, $\partial\mathbb{D} := \{z \in \mathbb{C} \mid |z| = 1\}$, $\mathbb{D}_T := \{z \in \mathbb{C} \mid |T\delta + 1| < 1\}$, $\partial\mathbb{D}_T := \{z \in \mathbb{C} \mid |T\delta+1| = 1\}$. When scalar functions $f(s)$, $f(z)$, and $f(\delta)$ are defined in the s-domain, z-domain, and δ-domain, respectively, $f^{\sim}(s) := f(-s)$, $f^{\sim}(z) := f(\frac{1}{z})$, and $f^{\sim}(\delta) := f(\frac{-\delta}{T\delta+1})$, respectively, where $T (> 0)$ is the sampling time. (See Subsection III-B for a brief exposition of the δ transform.)

II. An Illustrative Example

In this section, a numerical example is presented to give readers the idea of what we are developing in this paper. The $\mathcal{H}_2$ regulation problem, which will be formally formulated in Section IV, is considered for a discrete-time plant with tuning parameters. The problem is tackled under the assumption that the optimal controller can always be computed for a fixed plant, and we are interested in finding the best values for the tuning parameters so that the optimal performance may be optimized over plant parameters.

In [5], an expression of the best $\mathcal{H}_2$ regulation performance achievable by an optimal controller is derived for a non-minimum phase discrete-time SISO plant $P(z)$ in terms of the unstable poles and the frequency gain of plants as

$$E_z^\star(P) = \exp\left\{\frac{1}{\pi}\int_0^\pi \log(1+|P(e^{j\theta})|^2)d\theta\right\}\prod_{i=1}^{n_\lambda}|\lambda_i^a|^2 - 1 \ , \quad (1)$$

where λ_i^a $(i = 1, \ldots, n_\lambda)$ represent the unstable poles of P. (See (16) and (18) for the details on the definition of $E_z^\star(P)$.) While such an expression is useful for qualitative analysis, it is not necessarily suited to quantitatively analyse the effect of tuning parameters on achievable performance levels. We thus aim at discovering another expression that can be utilized in the presence of parameters.

Now consider the plant

$$P(z) = \frac{z + 0.1 - q_1^2}{z^2 + (1 + 0.01q_2)z + 0.25 + q_2^2} \quad (2)$$

with tuning parameters $\mathbf{q} = (q_1, q_2)$. The values of these parameters are to be chosen from

$$\mathcal{Q} = \big\{\mathbf{q} = (q_1, q_2) \,\big|\, q_1 \in [-0.25, 0.25], q_2 \in [-0.5, 0.5]\big\} \ ,$$

so that the best achievable performance $E_z^\star$ may be minimized. To this end, we get an expression for $E_z^\star$ containing parameters, and then optimize it over $\mathcal{Q}$. Although it is in general impossible to get a closed-form expression for $E_z^\star$, the result in Section IV states that $E_z^\star$ can be expressed as

$$E_z^\star = m_2^2 - 1 \ , \quad (3)$$

where the quantity corresponding to m_2 (or, m_2^2) can be given as a root of an algebraic equation. For this example, m_2^2 is the largest real root of the following polynomial in x:

$$\begin{aligned} &x^4 + (-q_1^4 + \tfrac{1}{5}q_1^2 - q_2^4 + \tfrac{14999}{10000}q_2^2 - \tfrac{1}{50}q_2 - \tfrac{1029}{400})x^3 \\ &+(\tfrac{1}{2}q_1^4 - \tfrac{13}{5}q_1^2 - 2q_1^4q_2^2 - \tfrac{1}{50}q_1^2q_2^3 - \tfrac{8}{5}q_1^2q_2^2 - \tfrac{1}{40}q_1^2q_2 - \tfrac{19999}{10000}q_2^6 \\ &+ \tfrac{1}{50}q_2^5 + \tfrac{30001}{20000}q_2^4 + \tfrac{3}{250}q_2^3 - \tfrac{431183}{160000}q_2^2 + \tfrac{19}{800}q_2 + \tfrac{329}{800})x^2 \\ &+(-\tfrac{1}{16}q_1^4 + \tfrac{1}{80}q_1^2 - q_1^4q_2^4 - \tfrac{1}{2}q_1^4q_2^2 + \tfrac{1}{5}q_1^2q_2^4 + \tfrac{1}{10}q_1^2q_2^2 - q_2^8 \\ &+ \tfrac{9999}{10000}q_2^6 - \tfrac{1}{50}q_2^5 - \tfrac{37701}{20000}q_2^4 - \tfrac{1}{100}q_2^3 - \tfrac{190801}{160000}q_2^2 - \tfrac{1}{800}q_2 \\ &- \tfrac{1029}{6400})x + q_2^8 + q_2^6 + \tfrac{3}{8}q_2^4 + \tfrac{1}{16}q_2^2 + \tfrac{1}{256} \ . \end{aligned} \quad (4)$$

It is emphasized that there is a constructive method of obtaining the algebraic equation for m_2 (or, m_2^2) [12].

Given (3) and (4), one can take the derivative of $E_z^\star$ with respect to the parameters and then employ an orthodox optimization approach such as the multistart gradient method. However we can further employ an algebraic method based on CAD [11] to carry out the second optimization. The algebraic optimization approach reported in [13] can compute the minimum value of $E_z^\star$ when parameters vary inside the permissible region:

$$\min_{\mathbf{q}\in\mathcal{Q}} E_z^\star \simeq 1.41476 \ .$$

We claim that this minimum value is guaranteed to be the global optimum, and also that the value is in fact obtained as an algebraic number and can be computed with arbitrary accuracy.

III. Preparation

A. Discrete-time Polynomial Spectral Factorization

Spectral factorization is one of important mathematical tools for the analysis and synthesis in modern and post-modern control for finite-dimensional linear time-invariant systems. In the discrete-time case, given a *self-reciprocal* polynomial[1] of degree $2n$ in $\mathbb{R}[z]$

$$\begin{aligned} f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \\ + \frac{a_1}{z} + \cdots + \frac{a_{n-1}}{z^{n-1}} + \frac{a_n}{z^n} \ , \quad a_n \neq 0 \ , \end{aligned} \quad (5)$$

without roots of unit modulus, the task is to find its decomposition of the following form:

Definition 1: The *spectral factorization* of $f(z)$ in (5) is a decomposition of $f(z)$ of the following form:

$$f(z) = g(z)\,g\Big(\frac{1}{z}\Big) \ , \quad (6)$$

$$g(z) = b_nz^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0 \in \mathbb{R}[z] \ , \quad (7)$$

where $b_n > 0$, and all the roots of $g(z)$ belong to $\mathbb{D}$ only. The polynomial $g(z)$ is called the *spectral factor* of $f(z)$.

[1]The polynomial (5) is obviously *not* a polynomial, but it can be easily converted to a polynomial: $z^nf(z)$. As the convention in signal processing and control and for the brevity of the notation, we regard (5) as a polynomial.

An algebraic approach to polynomial spectral factorization is developed in [12], which is briefly reviewed here. Write as $\bar{g}_k$ the coefficient of the k-th order term of $g(z)g\left(\frac{1}{z}\right) - f(z)$:

$$g(z)g\left(\frac{1}{z}\right) - f(z) = \sum_{k=-n}^{n} \bar{g}_{|k|} z^k \ . \tag{8}$$

Then, for each k, we have

$$\bar{g}_k = \sum_{i=0}^{n-k} b_i b_{i+k} - a_k \ .$$

The polynomial spectral factorization problem thus reduces to finding a particular solution to the set of equations $\bar{g}_k = 0$ ($k = 0, 1, \ldots, n$). An algebraic approach may then be applicable to solve this set of equations. Here we employ a different parametrization of $g(z)$ to facilitate the solution of the set of equations by way of an algebraic approach called Gröbner bases [10]. The following parametrization allows us to appreciate an effective structural property that can be exploited:

$$g(z) = \beta_n(z+1)^n + \beta_{n-1}(z+1)^{n-1} + \cdots + \beta_0 \ . \tag{9}$$

Notice that b_i and β_j are related as $b_i = \sum_{j=i}^{n} \binom{j}{i} \beta_j$ and $\beta_j = \sum_{i=j}^{n} \binom{i}{j} (-1)^{i-j} b_i$ ($i, j = 0, 1, 2, \ldots, n$), where $\binom{j}{i}$ is the binomial coefficient for $i, j \in \mathbb{N}$. In spite of the simplicity of this conversion, its benefit is enormous.

Lemma 1 ([12]): Let $c_{k,\ell}$ ($k = 0, 1, 2, \ldots, n$, $k \le \ell \le n$) be

$$\begin{cases} c_{k,k} = 1 & k = 0, 1, 2, \ldots, n \ , \\ c_{0,\ell} = (-1)^{\ell} \cdot 2 & \ell = 1, 2, \ldots, n \ , \\ c_{k,\ell} = (-1)^{k+\ell} \cdot \frac{2}{(2k)!} \cdot \frac{(k+\ell-1)!}{(\ell-k)!} \cdot \ell & \begin{cases} k = 1, 2, \ldots, n \ , \\ k < \ell \le n \ . \end{cases} \end{cases}$$

Furthermore, let

$$\bar{\bar{g}}_k := \sum_{\ell=k}^{n} c_{k,\ell}\, \bar{g}_\ell \ \left(= \bar{g}_k + \sum_{\ell=k+1}^{n} c_{k,\ell}\, \bar{g}_\ell \right), \ k = 0, 1, \ldots, n \ .$$

Then it holds that the set of polynomials

$$\mathcal{G} := \{\bar{\bar{g}}_0, \bar{\bar{g}}_1, \ldots, \bar{\bar{g}}_n\} \tag{10}$$

forms the reduced Gröbner basis of the ideal generated by $\{\bar{g}_0, \bar{g}_1, \ldots, \bar{g}_n\}$ in $\mathbb{R}[\beta_0, \beta_1, \ldots, \beta_n]$ with respect to the graded reverse lexicographic order $\beta_n \succ \beta_{n-1} \succ \cdots \succ \beta_0$, with β_k^2 being the leading monomial of $\bar{\bar{g}}_k$. (The leading coefficients are 1.)

For Gröbner bases and associated ideas such as the graded reverse lexicographic order, readers are referred to, e.g., [10].

Algebraic geometry theory further confirms that the set of equations has a finite number of solutions and that the number of roots (multiplicities counted) is 2^{n+1}. The virtue of the parametrization (9) is condensed in the following theorem.

Theorem 1 ([12]): The ideal of spectral factorization has a Gröbner basis so-called *shape basis* with respect to any elimination ordering $\{\beta_0, \beta_1, \ldots, \beta_{n-1}\} \succ\succ \beta_n$:

$$\mathcal{F} := \left\{ \tilde{S}_f(\beta_n),\ \beta_{n-1} - \tilde{h}_{n-1}(\beta_n), \ldots,\ \beta_0 - \tilde{h}_0(\beta_n) \right\} \ ,$$

where $\tilde{S}_f$ is a polynomial of degree exactly 2^{n+1} and $\tilde{h}_i$'s are polynomials of degree strictly less than 2^{n+1}.

The theorem guarantees that all the coefficients of the spectral factor can be expressed as polynomials in β_n and therefore that the problem of polynomial spectral factorization can in essence be solved by finding a particular root of $\tilde{S}_f(\tilde{y})$. Indeed it can be proven that what we have to compute is the largest real root of $\tilde{S}_f(\tilde{y})$. Lemma 1 says that $\mathcal{G}$ is already a Gröbner basis, therefore, we can effectively compute a shape basis $\mathcal{F}$ from $\mathcal{G}$ by way of the basis conversion (change-of-order) technique [14].

The above development is stated for $f(z) \in \mathbb{R}[z]$, but the approach can immediately expanded to the case of parametric $f(z) \in \mathbb{Q}(\mathbf{q})[z]$. Readers are referred to [12] for details.

B. δ Transform

In addition to the ordinary z transform, the δ transform is proposed as an operator on discrete-time domain signals [15]. The δ operator is defined as the following forward difference:

$$\delta = T^{-1}(z-1) \ , \tag{11}$$

where $T\,(>0)$ is the sampling time. For a sequence $x(k)$, its δ transform is defined as

$$x_T(\delta) = \mathcal{D}\{x(k)\} := T \sum_{k=0}^{\infty} x(k)(T\delta + 1)^{-k} \ .$$

Due to (11), $\mathbb{D}_T$ is the stability region in the δ-domain. Also, as T tends to 0 in (11), the δ operator behaves like the derivative, i.e., s in the Laplace domain.

In the δ-domain, the $\mathcal{H}_2$-norm of $y_T(k) := y(kT)$ is defined as $\|y_T(k)\|_2 := \left(T \sum_{k=0}^{\infty} |y_T(k)|^2\right)^{\frac{1}{2}}$ [5]. Note that this $\mathcal{H}_2$-norm is the square root of the sampling time T times the $\mathcal{H}_2$-norm of $y_T(k)$ in the z-domain. This in fact corresponds to the $\mathcal{H}_2$-norm of the continuous-time signal $y(t)$ passed through the 0-th order holder with sampling time T. Furthermore, as $T \to 0$, $\|y(t)\|_2$ is recovered, which is a useful feature to relate the continuous-time results and the discrete-time results as we will see in the rest of the paper. The $\mathcal{H}_2$-norm of a function $y_T(\delta)$ in the δ-domain is defined as

$$\|y_T(\delta)\|_2 = \left\{ \frac{1}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left\| y_T\left(\frac{e^{j\omega T} - 1}{T}\right) \right\|^2 d\omega \right\}^{\frac{1}{2}} \ .$$

Parseval's relation also holds in the δ-domain:

$$\|y_T(\delta)\|_2 = \|y_T(k)\|_2 \ , \tag{12}$$

where $y_T(\delta) = \mathcal{D}\{y_T(k)\}$.

The transfer function in the δ-domain is defined in a manner similar to the s- and z-domains, i.e., as the ratio of the input and the output of the system. For a z-domain transfer function $F(z)$, let $G_T(\delta) = F(T\delta + 1)$. Due to (12) and the fact that the unit pulse signal in the δ-domain is

$$d(k) = \begin{cases} \frac{1}{T} & k = 0 \ , \\ 0 & k \ne 0 \ , \end{cases}$$

we can immediately deduce that [5]

$$\|G_T(\delta)\|_2^2 = T^{-1} \|F(z)\|_2^2 \ . \tag{13}$$

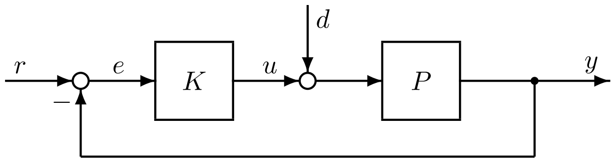

Fig. 1. Unity feedback system configuration.

C. Normalized Coprime Factorization

For a (SISO) system P, its normalized coprime factorization is the representation of P as a ratio of two stable system under the normalization condition:

$$P = \frac{N}{M} \ ,$$

where N and M are both stable and satisfy $N^{\sim}N + M^{\sim}M = 1$. The plants in the s-, z-, and δ-domains can be dealt with in a unified manner. Let λ represent the transform variable of the domain of the system, i.e., λ is read as s, z, or δ according to the domain. Write the system $P(\lambda)$ as

$$P(\lambda) = \frac{P_N(\lambda)}{P_D(\lambda)} \ , \tag{14}$$

where $P_N(\lambda)$ and $P_D(\lambda)$ are coprime polynomials. Without loss of generality, $P_D(\lambda)$ is assumed to be monic. Let $M_D(\lambda)$ be the spectral factor of

$$P_N^{\sim}(\lambda)P_N(\lambda) + P_D^{\sim}(\lambda)P_D(\lambda) = M_D^{\sim}(\lambda)M_D(\lambda) \ . \tag{15}$$

Namely, $M_D(\lambda)$ is the polynomial that satisfies (15) and also is stable in the definition of the each domain. The discrete-time case was discussed in detail in Subsection III-A. Then, normalized coprime factorization can be given as

$$N(\lambda) = \frac{P_N(\lambda)}{M_D(\lambda)} \ , \quad N(\lambda) = \frac{P_D(\lambda)}{M_D(\lambda)} \ .$$

IV. Discrete-time $\mathcal{H}_2$ Regulation Problem

In this section, we formulate the discrete-time $\mathcal{H}_2$ regulation problem and give its performance limitations with respect to the poles of the plants and the closed-loop systems in the z-domain and the δ-domain.

A. Problem Formulation

We deal with the closed-loop system depicted in Fig. 1, where P and K represent a plant and a controller of discrete-time SISO linear time-invariant systems, and $r \in \mathbb{R}$, $d \in \mathbb{R}$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$ represent the reference signal, the disturbance, the control input, and the output of the plant, respectively. In this section, we assume the reference r to be zero and the disturbance d to be an unit pulse signal. Then, the cost functions to be minimized are

$$E_z(P,\ K) := \sum_{k=0}^{\infty} \left(|y(k)|^2 + |u(k)|^2 \right) \ , \tag{16}$$

$$E_\delta(P,\ K) := T\sum_{k=0}^{\infty} \left(|y(k)|^2 + |u(k)|^2 \right) \ , \tag{17}$$

in the z-domain and the δ-domain, respectively, and their lower bounds, i.e., the performance limitations of the $\mathcal{H}_2$ regulation problem, are defined as

$$E_z^{\star}(P) := \inf_{K \text{ stabilizing}} E_z(P,\ K) \ , \tag{18}$$

$$E_\delta^{\star}(P) := \inf_{K \text{ stabilizing}} E_\delta(P,\ K) \ , \tag{19}$$

respectively.

B. Expression in the z-domain

In this subsection, we give an expression of $E_z^{\star}(P)$ with respect to the poles of the plant P and the closed-loop system composed of P and the optimal K.

We make the following assumptions:

Assumption 1: P is strictly proper.

Assumption 2: P is minimum phase.

Then, an expression of $E_z^{\star}$ is given as in (1). With (1) and the roots of the polynomial spectral factor, we can derive the following theorem:

Theorem 2: Suppose that $P(z)$ given as in (14) satisfies Assumptions 1 and 2. Let n denote the degree of $P_D(z)$, λ_k $(k = 1, \ldots, n)$ the poles of $P(z)$, α_i^z $(i = 1, \ldots, n)$ the roots of the spectral factor $M_D(z)$ in (15), and m_n be the leading coefficient of $M_D(z)$. Then, the performance limitation $E_z^{\star}$ of the $\mathcal{H}_2$ regulation problem is given by

$$E_z^{\star}(P) = \frac{\prod_{k=1}^{n} \lambda_k}{\prod_{i=1}^{n} \alpha_i^z} - 1 = m_n^2 - 1 \ . \tag{20}$$

The proof is omitted due to space limitation.

We emphasize that the expression (20) involves all the poles of $P(z)$, unlike (1), where only the unstable poles appear. This fact indeed facilitates the algebraic approach. It can be shown that $M_D(z)$ is the characteristic polynomial of the optimal closed-loop system attaining $E_z^{\star}$, and, therefore, (20) is given by the ratio of the poles of the plant and those of the resulting closed-loop system.

C. Expression in the δ-domain

In this subsection, we give an expression of $E_\delta^{\star}(P)$ in terms of the poles of the plant and of the closed-loop system.

We also suppose the that plant $P(\delta)$ satisfies Assumptions 1 and 2 in the δ-domain. Then, it is also known that $E_\delta^{\star}(P)$ can be expressed as [5]

$$E_\delta^{\star}(P) = \frac{1}{T}\left[|\Lambda_o(\infty)|^2 \prod_{i=1}^{n_\rho} |T\rho_i^a + 1|^2 - 1 \right] \ ,$$

where

$$|\Lambda_o(\infty)|^2 = \exp\left\{ \frac{T}{\pi} \int_0^{\frac{\pi}{T}} \log\left(1 + \left| P\left(\frac{e^{j\omega T} - 1}{T} \right) \right|^2 \right) \right\} \ ,$$

and ρ_i^a $(i = 1, \ldots, n_\rho)$ are the unstable poles of P. With the equivalence of the δ-domain and the z-domain by linear transformation as mentioned in Subsection III-B, Theorem 2 and (13), we can derive the following theorem:

Theorem 3: Suppose that $P(\delta)$ satisfies Assumptions 1 and 2. Let T denote the sampling time, n the degree of $P_D(\delta)$, ρ_k $(k = 1, \ldots, n)$ the poles of $P(\delta)$, and α_i^δ $(i =$

$1,\ldots,n$) the roots of the spectral factor $M_D(\delta)$ of (15). Then, the performance limitation $E_\delta^\star$ of the $\mathcal{H}_2$ regulation problem is given by

$$E_\delta^\star(P) = \frac{1}{T}\left[\frac{\prod_{k=1}^{n}(T\rho_k+1)}{\prod_{i=1}^{n}(T\alpha_i^\delta+1)} - 1\right] . \tag{21}$$

The proof is omitted due to space limitation.

Here, $M_D(\delta)$ is also the characteristic polynomial of the optimal closed-loop system attaining $E_\delta^\star$, and (21) is given by the poles of the plant and the resulting closed-loop system.

When the plant $P(\delta)$ in the δ-domain is derived from $P(s)$ in the s-domain by the 0-th order holder, we then recover

$$\lim_{T\to 0} E_\delta^\star = \sum_{k=1}^{n} p_k - \sum_{i=1}^{n} \alpha_i^s , \tag{22}$$

where p_k denote the poles of $P(s)$, α_i^s the roots of the spectral factor $M_D(s)$ of (15). The right hand side of (22) is equal to the result for the continuous-time case [9]. This fact shows the continuity of the performance limitation in the δ-domain and the s-domain as $T \to 0$.

V. Discrete-time $\mathcal{H}_2$ Tracking Problem

In this section, we formulate the discrete-time $\mathcal{H}_2$ tracking problem and give expressions of the performance limitations in the z-domain and the δ-domain.

A. Problem Formulation

We also consider the closed-loop system in Fig. 1, where $d(k) \equiv 0$ and r is a unit step function given as

$$r(k) = \begin{cases} 1 & k \geq 0 , \\ 0 & k < 0 . \end{cases}$$

Let $e(k)$ denote the tracking error, i.e., $e(k) := r(k) - y(k)$. Then the cost functions of the tracking problem with a penalty on control input are given by

$$J_z(P,\ K) := \sum_{k=0}^{\infty}\left(|e(k)|^2 + |u(k)|^2\right) ,$$
$$J_\delta(P,\ K) := T\sum_{k=0}^{\infty}\left(|e(k)|^2 + |u(k)|^2\right) ,$$

in the z-domain and the δ-domain, respectively. The performance limitations of the $\mathcal{H}_2$ tracking problem are defined by

$$J_z^\star(P) := \inf_{K\ \text{stabilizing}} J_z(P,\ K) ,$$
$$J_\delta^\star(P) := \inf_{K\ \text{stabilizing}} J_\delta(P,\ K) ,$$

respectively.

B. Expression in the z-domain

In this subsection, we give an expression of $J_z^\star(P)$ by the zeros of the plant and the poles of the closed-loop system.

First, suppose the following assumptions:

Assumption 3: The plant $P(z)$ can be described by $P(z) = \frac{P_0(z)}{z-1}$ and $P_0(z)$ is stable and proper.

Assumption 4: The plant P does not have zeros at $z = 1$.

Assumption 3 implies that P has at least one integrator and P_0 is stable. Under these assumptions, an expression of $J_z^\star$ is known as [5]

$$J_z^\star(P) = \sum_{i=1}^{n_\eta}\frac{\eta_i^a+1}{\eta_i^a-1} + \frac{1}{2\pi}\int_0^\pi \log\left(1+\frac{1}{|P(e^{j\theta})|^2}\right)\frac{d\theta}{1-\cos\theta} , \tag{23}$$

where η_i^a ($i = 1,\ldots,n_\eta$) are non-minimum phase zeros of P. For continuous-time systems, an expression of the performance limitation of the $\mathcal{H}_2$ tracking problem can be given from the result on the $\mathcal{H}_2$ regulation problem via the reciprocal transform [16]. However, For discrete-time systems, such a method is not applicable. In this paper, we thus give an expression by a direct method with (23) and the roots of the spectral factor just as the case in the $\mathcal{H}_2$ regulation problem.

Theorem 4: Suppose that the plant $P(z)$ given as in (14) satisfies Assumptions 3 and 4. Let n and m denote the degrees of $P_D(z)$ and $P_N(z)$, respectively. Denote by η_k ($k = 1,\ldots,m$) the zeros of $P(z)$, and by α_i^z ($i = 1,\ldots,n$) the roots of the spectral factor $M_D(z)$ in (15). Then, the performance limitation $J_z^\star$ in the $\mathcal{H}_2$ tracking problem is given by

$$J_z^\star(P) = \sum_{k=1}^{m}\frac{\eta_k}{\eta_k-1} - \sum_{i=1}^{n}\frac{\alpha_i^z}{\alpha_i^z-1} . \tag{24}$$

The proof is omitted due to space limitation.

In the $\mathcal{H}_2$ tracking problem, we can also show that $M_D(z)$ is the characteristic polynomial of the optimal closed-loop system attaining $J_z^\star$, and, therefore, the theorem implies that the performance limitation can be expressed by the zeros of the plant and the poles of the resulting optimal closed-loop system.

C. Expression in the δ-domain

In this subsection, we give an expression of $J_\delta^\star(P)$ in terms of the zeros of the plant and the closed-loop system poles.

We also suppose that the plant $P(\delta)$ satisfies the following assumptions:

Assumption 5: The plant $P(\delta)$ can be described by $P(\delta) = \frac{P_0(\delta)}{\delta}$ and $P(\delta)$ is stable and proper.

Assumption 6: The plant P does not have zeros at $\delta = 0$.

Similar to the case of the z-domain, Assumption 5 implies that P has at least one integrator and P_0 is stable. Under these assumptions, an expression of $J_\delta^\star$ is also known as [5]

$$J_\delta^\star(P) = \sum_{i=1}^{n_\zeta}\left(\frac{2}{\zeta_i^a}+T\right) + \frac{T^2}{2\pi}\int_0^{\pi/T}\log\left(1+\frac{1}{|P(\frac{e^{j\omega T}-1}{T})|^2}\right)\frac{d\omega}{1-\cos\omega T} ,$$

where ζ_i^a ($i = 1,\ldots,n_\zeta$) are the non-minimum phase zeros.

With the equivalence of the δ-domain and the z-domain by linear transformation as in the $\mathcal{H}_2$ regulation problem, Theorem 4 and (13), we can derive the following theorem:

Theorem 5: Suppose that $P(\delta)$ given as in (14) satisfies Assumptions 5 and 6. Let T denote the sampling time, n the degree of $P_D(\delta)$, m the degree of $P_N(\delta)$, ζ_k $(k = 1, \ldots, m)$ the zeros of $P(\delta)$, and α_i^δ $(i = 1, \ldots, n)$ the roots of the spectral factor $M_D(\delta)$ in (15). Then, the performance limitation $J_\delta^\star$ of the $\mathcal{H}_2$ tracking problem is given by

$$J_\delta^\star(P) = \sum_{k=1}^{m} \left(\frac{1}{\zeta_k} + T \right) - \sum_{i=1}^{n} \left(\frac{1}{\alpha_i^\delta} + T \right). \qquad (25)$$

The proof is omitted due to space limitation.

Here, $M_D(\delta)$ is also the characteristic polynomial of the optimal closed-loop system attaining $J_\delta^\star$, and, therefore, (25) is given by the zeros of the plant and the poles of the resulting closed-loop system.

When the plant $P(\delta)$ in the δ-domain is derived from $P(s)$ in the s-domain by the 0-th order holder, we then get

$$\lim_{T\to 0} J_\delta^\star = \sum_{k=1}^{m} \frac{1}{z_k} - \sum_{i=1}^{n} \frac{1}{\alpha_i^s} \ , \qquad (26)$$

where z_k $(k = 1, \ldots, m)$ denote the zeros of $P(s)$, and α_i^s $(i = 1, \ldots, n)$ the roots of $M_D(s)$. The right hand side of (26) is equal to the result for the continuous-time case [9]. This fact shows the continuity of the performance limitation in the δ-domain and the s-domain as $T \to 0$.

VI. Numerical Example Revisited

In this section, we revisit the numerical example considered in Section II, giving a further detail of the solution based on the development of this paper. Let $P_N(z)$ and $P_D(z)$ be the numerator and the denominator of $P(z)$ given in (2), respectively, and construct a polynomial in the left hand side of (15). Write its spectral factor as in (9), i.e.,

$$M_D(z) = m_2(z+1)^2 + m_1(z+1) + m_0 \ .$$

Comparing the coefficients of the both side of (15), a set of polynomial equations is obtained. Lemma 1 then helps us to obtain the reduced Gröbner basis of the ideal generated by the polynomial parts of the equations with respect to the graded reverse lexicographic order $m_2 \succ m_1 \succ m_0$:

$$\begin{gathered}\big\{ m_0^2 - q_1^4 - \tfrac{9}{5} q_1^2 - q_2^4 + \tfrac{1}{50} q_2^3 - \tfrac{5001}{10000} q_2^2 + \tfrac{1}{200} q_2 - \tfrac{349}{400}, \\ m_1^2 + m_0 m_1 - 2 m_0 m_2 + q_1^2 - \tfrac{1}{100} q_2^3 + 3 q_2^2 - \tfrac{1}{80} q_2 - \tfrac{7}{20}, \\ m_2^2 + m_0 m_2 + m_1 m_2 - q_2^2 - \tfrac{1}{4} \big\} \ .\end{gathered}$$

By means of the basis conversion (change-of-order) technique [10], [17], the shape basis can be obtained, and the first polynomial contains m_2 only (and not m_0 and m_1), which is in fact a polynomial in m_2^2. Substituting x for m_2^2, we get (4). It is noted that the true m_2^2 is always the largest real root of (4) [12].

In order to compute the minimum value of $E_z^\star$ as $\mathbf{q}$ varies inside $\mathcal{Q}$, the relationship (4) is useful. An algebraic optimization method proposed in [13] can be employed and the result stated in Section II is obtained. It is repeated that, thanks to (4), we can find the true (global) optimal value.

VII. Concluding Remarks

In this paper, we gave the expressions of the best achievable performance in the $\mathcal{H}_2$ regulation/tracking problem for discrete-time SISO systems by the poles/zeros of the plants and the poles of the closed-loop systems. We also extended the results to δ-domain for connecting the results in the z-domain and the s-domain. The derived expressions are also usable for tuning system parameters.

For the future works, the extension of the results to non-minimum phase plants, or general case of relative degree remains.

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