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Input/State/Output Modeling and Control of Dynamical Systems with Active Singularities: Single- and Multi-Impact Sequences

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Abstract— **A general controller synthesis setting for systems with active, or controlled, singularities under incomplete information is extended to the cases of the single-impact and the multi-impact sequences - the main cases of interest in applications.**

I. Introduction

Dynamical systems with active, or controlled, singularities is a new class of systems introduced in [2] and [3] for singularities parametrized by the elasticity coefficient μ. The defining feature of this system class is the presence of *active, or controlled, constraints* capable of radically changing the attainability set of the post-impact system state. The engagement phase of the system with such constraint is termed *active singularity*, and the system motion in the domain of constraint violation - the *singular* motion phase. The development in [3] was, however, confined only to a single multi-impact, whereas the single impacts and the single-impact sequences in [2] were considered only under the full state accessibility. Thus, the most important case for a variety of applications, including power systems under faults [1], microgrids [8], [9], mobile sensor networks [7], impact actuators [6], [12], and robotic manipulators [5], [4], [10] - that of the output feedback optimal control realized in terms of the single- and the multi-impact optimal sequences has not been considered. This gap is partially filled in the present work. Due to space limitations, the optimal control law synthesis example is not presented and most of the proofs are omitted. Subsection V-A reviews the results of [3] on a single multi-impact to enhance readability.

II. Problem Statement

The conceptual framework described above leads to the following three distinct objectives:

i) *the controller synthesis setting objective* - development of an analytical setting that a) permits computation of the optimal control sequences by standard controller synthesis techniques in auxiliary time, and b) once these sequences are obtained, admits their reparametrization by μ for application to the original system,

ii) *the control law calculation objective* - calculation of an impulsive optimal control sequences that satisfy the performance criterion, and, finally,

iii) *the limit modeling objective* - derivation of the limit system corresponding to the original one.

The present work addresses only objectives i) and iii) - the controller synthesis setting development and the limit model derivation. Objective ii) - optimal control law computation - will be considered elsewhere.

Let the controlled dynamical system be described by the state vector $x(t) = (x_p(t), x_v(t))$, $x_p(t) \in R^n, x_v(t) \in R^n$, where vectors x_p and x_v are referred to as the sets of *generalized positions* and *generalized velocities*, respectively, and $t \in [0, T]$, where T is sufficiently large.

Suppose that system motion includes interaction with some elastic constraint. Let the elastic deformation of the constraint be parametrized by some coefficient $\mu > 0$, so that for finite μ the constraint would admit a system motion, although inhibited, within the domain occupied by it. Let the constraint-free domain be given by

$$\{(x_p, t) : G(x_p, t) > 0\} \tag{1}$$

where $G : R^n \times [0, T] \to R$ is a sufficiently smooth function. Following Section II, the system motions in the domain occupied by the constraint and in the constraint-free domain will be referred to as the *singular* and the *constraint-free* motion phases, respectively.

A. Motion in the Constraint-Free and the Singular Phases

Generalizing representation of [2] in the context of the example of Section II, let the system motion be described by

$$\begin{aligned} &\dot{x}_p(t) = F_p^r(x_p(t), x_v(t), t), \\ &\dot{x}_v(t) = F_v^r(x_p(t), x_v(t), u(t), t)+ \\ &+\mu F_v^s(x_p(t), x_v(t), w_1^\mu(\xi, t), t, \mu) \\ &+\mu F_v^{rs}(x_p(t), x_v(t), w_2^\mu(\xi, t), t, \mu), \end{aligned} \tag{2}$$

where $u(t) \in U \subset R^r$ is a control variable (a measurable function) in the constraint-free phase, U is a compact set, $F_p^r(x_p, x_v, u, t)$ and $F_v^r(x_p, x_v, u, t)$ are the generalized forces in the constraint-free phase, $w_1^\mu(\xi, t)$ is a control

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signal in the singular phase, $\mu F_v^s(x_p, x_v, w_1^\mu(\xi,t), t, \mu)$ is a generalized controlled force arising from a contact with the constraint in the inhibited area, $\mu F_v^{rs}(x_p, x_v, w_2^\mu(\xi,t), t, \mu)$ is an additional generalized controlled force in the constraint-free phase governed by a control signal $w_2^\mu(\xi,t)$ (a measurable function), and ξ is the sensor output signal.

The first of the latter two forces, $\mu F_v^s(x_p, x_v, w_1^\mu(\xi,t), t, \mu)$, is characterized by

$F_v^s(x_p, x_v, w_1^\mu, t, \mu) = 0$, if

1) $G(x_p, t) > 0$ or

2) $G(x_p, t) = 0$ and

$$\left.\frac{d}{dt}\right|_{F_p^r} G(x_p,t) = G'_{x_p}(x_p,t)F_p^r(x_p,x_v,t) + G'_t(x_p,t) = 0 \tag{3}$$

where G'_{x_p} and G'_t denote partial derivatives with respect to x_p and t, respectively, and $\left.\frac{d}{dt}\right|_{F_p^r} G(x_p,t)$ denotes the time derivative of $G(x_p,t)$ along the trajectories of $\dot{x}_p(t) = F_p^r(x_p(t), x_v(t), t)$. Noting that $G(x_p,t)$ does not depend on x_v, the last expression in (3) is seen to represent the time derivative of $G(x_p,t)$ along the trajectories of the entire system (2).

The force $\mu F_v^{rs}(x_p, x_v, w_2^\mu(\xi,t), t, \mu)$, the last of the forces in (2), characterizes an external impulsive action on the system in the constrained-free domain during the so-called inter-singular motion introduced in Section II and formally defined further in Section IV, and satisfies the condition

$$F_v^{rs}(x_p, x_v, w_2^\mu, t, \mu) = 0, \quad \text{if} \quad G(x_p,t) < 0. \tag{4}$$

The introduction of this force lays the groundwork for addressing optimal control problems with complex multi-impact structure, such as that encountered in Section II. Once this structure is in place, whether or not the multi-impact will appear depends on the specific features of the problem at hand.

Let in the singular phase, when $G(x_p(t),t) \le 0$, components of the state vector $(x_p(t),\, x_v(t))$ be unobservable directly, and it be possible to observe only signal $\xi(t) \in R^k$. Then, the control variables in the singular phase can be taken to be continuous functionals of the sensor output signal $\xi(t)$ and measurable in time.

B. Sensor Equations and Admissible Control in the Singular Phase

To admit control of the sensing environment, let the sensor output signal $\xi(t)$ satisfy the equation

$$\dot{\xi}(t) = \mu H(x_p(t), x_v(t), \alpha^\mu(\xi,t), t, \mu), \tag{5}$$

where $\alpha^\mu(\xi,t)$ is a control signal and

$$H(x_p, x_v, \alpha^\mu, t, \mu) = 0 \quad \text{if} \quad G(x_p,t) > 0.$$

Let the motion in the singular phase begin at τ, where τ is the first instant when

$$G(x_p(\tau),\tau) = 0 \quad \text{and} \quad \left.\frac{d}{dt}\right|_{F_p^r} G(x_p(\tau),\tau) < 0. \tag{6}$$

Denoting by γ any of the control signals w_1, w_2, α, define its dependence on t and μ in the singular (interlaced singular) phase as

$$\gamma^\mu(\xi,t) = \begin{cases} \gamma(\xi, \sqrt{\mu}(t-\tau)), & t \ge \tau, \\ 0, & \text{otherwise}. \end{cases} \tag{7}$$

Let the following Lipschitz condition take place

$$|\gamma(\xi',t) - \gamma(\xi'',t)| \le L\|\xi' - \xi''\|_t, \quad L = \text{const}, \tag{8}$$

where

$$\begin{aligned} \|\xi\|_t &= \operatorname*{ess\,sup}_{\tau \le s \le t} |\xi(s)| = \\ \min\{\lambda &: |\xi(s)| \le \lambda, \ a.s.\ s \in [\tau,\ t]\}, \\ \text{or} \quad \|\xi\|_t &= \left(\int_\tau^t |\xi(s)|^2\, ds\right)^{1/2}. \end{aligned}$$

Definition 1. Admissible control $w_1^\mu(\xi,t)$ in a singular phase is a restricted measurable by t functional, where dependence on τ, t, μ, ξ is given by (7), (8) and a restriction has the form $w_1^\mu(\xi,t) \in W_1 \subset R^{r_1}$. Here W_1 is a compact set including zero element. Admissible controls $w_2^\mu(\xi,t)$ and $\alpha^\mu(\xi,t)$ are defined analogously.

It is assumed that the right hand sides of (2)-(5) are sufficiently smooth to guarantee unique solution of (2)-(5) for any admissible controls.

C. Controller Synthesis Setting and Limit Modeling Objectives

As indicated in Sections 1 and 2, equations (2) and (5) are not directly suitable for the controller synthesis and modeling due to their unbounded right hand sides (rhs). This problem is addressed by the following specific objectives:

Controller synthesis setting objective: provide an analytical setting that permits reduction of an ill-posed problem of synthesis of the singular phase control signals $w_1^\mu(\xi,t)$, $w_2^\mu(\xi,t)$, and $\alpha^\mu(\xi,t)$ in (2)-(5) to a well-posed two-step approximation procedure: a) synthesis of bounded singular phase control signals $w_1(\eta,s)$, $w_2(\eta,s)$, and $\alpha(\eta,s)$ in the auxiliary fictitious time s, and b) calculation of $w_1^\mu(\xi,t)$, $w_2^\mu(\xi,t)$, and $\alpha^\mu(\xi,t)$ implementable in the original system (2)-(5) using signals synthesized in a). Tasks a) and b), that can be viewed as the direct and the converse ones, respectively, are addressed in the next section by Theorem 1 and Remark 2 for single impacts and by Theorem 4 for single impact sequences, and in Section V by Theorem 5 and Remark 3 for single multi-impacts and Theorem 6 for multi-impacts sequences.

Limit Modeling Objective: obtain a model that generates a discontinuous motion controlled by $w_1(\eta,\cdot)$, $w_2(\eta,\cdot)$, and $\alpha(\eta,\cdot)$ representing a consistent approximation of motion of (1)-(2) controlled by $w_1^\mu(\xi,t)$, $w_2^\mu(\xi,t)$, and $\alpha^\mu(\xi,t)$. This objective is addressed in the next section by Corollary 2 and Theorem 3 for single impacts and Eq. (22) for single impact sequences, and in Section V by Remark 3 and system (22), (31) for single multi-impacts and multi-impacts sequences, respectively.

III. Singular Motion Phase under Single Impact

As indicated in Subsection II-A, in this case $F_v^{rs}(x_p, x_v, w_2^\mu(\xi,t), t, \mu) \equiv 0$.

A. Infinitesimal Dynamics Equation under Single Impact

According to (6), the singular motion phase begins at the first time τ that the system engages the constraint. Therefore, for a finite value of μ there exists a non-zero time interval of the constraint violation. Then, applying the space-time transformation

$$\begin{aligned} & s = \sqrt{\mu}\,(t-\tau), \quad t \geq \tau, \\ & y_p^\mu(s) = x_p(\tau) + \sqrt{\mu}\left[x_p(\tau+\mu^{-1/2}s) - x_p(\tau)\right], \\ & y_v^\mu(s) = x_v(\tau+\mu^{-1/2}s), \\ & \eta^\mu(s) = \xi(\tau+\mu^{-1/2}s) \end{aligned} \tag{9}$$

where s represents the auxiliary time variable (this transformation extends the one given in [2] to accommodate sensor dynamics) to system (2), (5), the new variables $\{y_p^\mu(s),\, y_v^\mu(s),\, \eta^\mu(s)\}$ are straightforwardly shown to satisfy the multiscale equation

$$\begin{aligned} \dot{y}_p^\mu(s) &= F_p^r\Big(\frac{y_p^\mu(s)-x_p(\tau)}{\mu^{1/2}} + x_p(\tau), y_v^\mu(s), \\ &\quad \tau+\mu^{-1/2}s\Big), \\ \dot{y}_v^\mu(s) &= \sqrt{\mu}\,F_v^s\Big(\frac{y_p^\mu(s)-x_p(\tau)}{\mu^{1/2}} + x_p(\tau), \\ &\quad y_v^\mu(s), w_1(\eta^\mu, s), \tau+\mu^{-1/2}s, \mu\Big) + \\ &+ \sqrt{\mu}\,F_v^{rs}\Big(\frac{y_p^\mu(s)-x_p(\tau)}{\mu^{1/2}} + x_p(\tau), \\ &\quad y_v^\mu(s), w_2(\eta^\mu, s), \tau+\mu^{-1/2}s, \mu\Big) + \\ &+ \mu^{-1/2}\,F_v^r\Big(\frac{y_p^\mu(s)-x_p(\tau)}{\mu^{1/2}} + x_p(\tau), \\ &\quad y_v^\mu(s), u(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s\Big), \\ \dot{\eta}^\mu(s) &= \sqrt{\mu}\,H\Big(\frac{y_p^\mu(s)-x_p(\tau)}{\mu^{1/2}} + x_p(\tau), y_v^\mu(s), \\ &\quad \alpha(\eta^\mu, s), \tau+\mu^{-1/2}s, \mu\Big), \end{aligned} \tag{10}$$

with the initial conditions given by
$y_p^\mu(0) = x_p(\tau),\;\; y_v^\mu(0) = x_v(\tau-),\;\; \eta^\mu(0) = \xi(\tau)$.

The next theorem describes the limit behavior of the new variables introduced through (9) as $\mu \to \infty$.

Assumption 1. Suppose that F_v^s (analogously F_v^{rs} and H) satisfies the Lipschitz condition in the following form: there exist $L > 0$, $\mu_0 > 0$ such that for any (x_p, x_p', x_v, x_v'), $t \in [0,T]$, $w_1 \in W_1$, and $\mu \geq \mu_0$

$$\begin{aligned} & \|F_v^s(x_p, x_v, w_1, t, \mu) - F_v^s(x_p', x_v', w_1, t, \mu)\| \leq \\ & \leq L\{\|x_p - x_p'\| + \mu^{-1/2}\|x_v - x_v'\|\}. \end{aligned} \tag{11}$$

Theorem 1: Along with Assumption 1 assume that:

1) for any admissible controls w_1, α and for any (x_p, τ) such that $G(x_p,\tau) = 0$ and $\frac{d}{dt}\Big|_{F_p^r} G(x_p(\tau),\tau) < 0$ there exists

$$\begin{aligned} & \lim_{\mu\uparrow\infty} \mu^{1/2} F_v^s\Big(\frac{y_p - x_p}{\mu^{1/2}} + x_p, y_v, w_1(\eta^\mu, s), \\ & \tau+\mu^{-1/2}s, \mu\Big) = \bar{F}_v^s(y_p, y_v, w_1(\eta, s), x_p, \tau), \\ & \lim_{\mu\uparrow\infty} \mu^{1/2} H\Big(\frac{y_p - x_p}{\mu^{1/2}} + x_p, y_v, \alpha(\eta^\mu, s), \\ & \tau+\mu^{-1/2}s, \mu\Big) = \bar{H}(y_v, \alpha(\eta, s), x_p, \tau), \end{aligned} \tag{12}$$

where convergence is uniform in any bounded vicinity of (y_p, y_v, η, s);

2) the system of differential equations

$$\begin{aligned} & \dot{y}_p(s) = F_p^r(x_p(\tau), y_v(s), \tau), \\ & \dot{y}_v(s) = \bar{F}_v^s(y_p(s), y_v(s), w_1(\eta, s), x_p(\tau), \tau), \\ & \dot{\eta}(s) = \bar{H}(y_v, \alpha(\eta, s), x_p(\tau), \tau) \end{aligned} \tag{13}$$

with $y_p(0) = x_p(\tau),\;\; y_v(0) = x_v(\tau-),\;\; \eta(0) = \xi(\tau)$ has the unique solution on some interval $[0, s^*(\tau)+\varepsilon]$, where $\varepsilon > 0$ and

$$s^*(\tau) = \inf_{s>0}\left\{\begin{array}{l} G_t'\Big|_{(x_p(\tau),\tau)} s \;+ G_{x_p}'\Big|_{(x_p(\tau),\tau)} \times \\ \times\; (y_p(s) - x_p(\tau)) = 0, \\ G_t'\Big|_{(x_p(\tau),\tau)} \;+\; G_{x_p}'\Big|_{(x_p(\tau),\tau)} \times \\ \times\; F_p^r(x_p(\tau), y_v(s), \tau) > 0 \end{array}\right\}. \tag{14}$$

Then, if $\mu \to \infty$,

$$(y_p^\mu(s), y_v^\mu(s), \eta^\mu(s)) \to (y_p(s), y_v(s), \eta(s)) \tag{15}$$

uniformly on $[0, s^*(\tau)+\varepsilon]$, and for all sufficiently large μ there exists

$$s_\mu^*(\tau) = \inf_{s>0}\left\{\begin{array}{l} G(x_p(\tau+\mu^{-1/2}s), \tau+\mu^{-1/2}s) = 0, \\ G_t'\Big|_{(x_p(\tau+\mu^{-1/2}s),\tau+\mu^{-1/2}s)} \;+ \\ +\; G_{x_p}'\Big|_{(x_p(\tau+\mu^{-1/2}s),\tau+\mu^{-1/2}s)} \times \\ \times F_p^r(x_p(\tau+\mu^{-1/2}s), x_v(\tau+\mu^{-1/2}s), \\ \tau+\mu^{-1/2}s) > 0, \end{array}\right\} \tag{16}$$

such that

$$s_\mu^*(\tau) \to s^*(\tau). \tag{17}$$

Remark 1. Conditions (14) and (16) mean that the "forces" $\bar{F}_v^s$ and F_v^s have the property to repulse the system from the inhibited domain under any admissible control signals.

Remark 2. Eq. (13), referred to as *the controlled infinitesimal dynamics equation*, represents dynamics of the singular phase in the limit as $\mu \uparrow \infty$ in the extended time s. Unlike (2), where $w_1^\mu(\xi,t)$ appears in the unbounded rhs term, (13) has bounded rhs and is, therefore, amenable to synthesis of the singular phase control signal $w_1(\eta,s)$ using standard optimal control methods.

B. Limit System Representation under Single Impact

The single jump limit representation of the original system is given by Corollary 2.

Corollary 2: For sufficiently small $\varepsilon > 0$ on the interval $[0, \tau + \varepsilon)$, solution of the original system (2) converges to some discontinuous functions $(\bar{x}_p(t), \bar{x}_v(t))$, such that

$$\bar{x}_p(t) = x_p(t), \quad \bar{x}_v(t) = x_v(t), \quad t < \tau, \quad \text{and}$$
$$\bar{x}_p(\tau) = \lim_{\mu\uparrow\infty} x_p(\tau + \mu^{-1/2} s^*_\mu(\tau)) = x_p(\tau),$$
$$\bar{x}_v(\tau) = \lim_{\mu\uparrow\infty} x_v(\tau + \mu^{-1/2} s^*_\mu(\tau)) = y_v(s^*(\tau)).$$

The functions $(\bar{x}_p(t), \bar{x}_v(t))$ can be interpreted as *the generalized solution* of the original system (2). Let us now use Corollary 2 to formulate a theorem describing evolution of the variables $(\bar{x}_p(t), \bar{x}_v(t))$.

Theorem 3: Let $(x^\mu_p(t), x^\mu_v(t))$ denote the ordinary solution of the original system (2) where a superscript μ is used to indicate dependence of this solution on parameter μ. Then, the generalized solution $(\bar{x}_p(t), \bar{x}_v(t))$ of the original system (2) is a pointwise limit of its ordinary solution as $\mu \to \infty$, and satisfies on an interval $[0, \tau+\varepsilon]$ the system of generalized differential equations

$$\begin{aligned}
&\dot{\bar{x}}_p(t) = F^r_p(\bar{x}_p(t), \bar{x}_v(t), t),\\
&\dot{\bar{x}}_v(t) = F^r_v(\bar{x}_p(t), \bar{x}_v(t), u(t), t) +\\
&+\Psi_v\big(\bar{x}_p(\tau), \bar{x}_v(\tau-), w_{1\tau}(\cdot), \tau\big)\delta(t-\tau),
\end{aligned} \tag{18}$$

with $\bar{x}_p(0) = x_p(0)$, $\bar{x}_v(0) = x_v(0)$, $\bar{x}_p(\tau) = x_p(\tau)$, $\bar{x}_v(\tau-) = x_v(\tau-)$.

Here $\Psi_v(\cdot)$ is a v-component of the shift operator along the paths of (13) so that

$$y_v(s^*(\tau)) = y_v(0) + \Psi_v\big(y_p(0), y_v(0), w_{1\tau}(\cdot), \tau\big),$$

where $w_{1\tau}(\cdot) = \{w_1(\eta, s) : \ 0 \le s \le s^*(\tau)\}$ (analogous notation will be used for control signal α). Since (18) encompasses limit motions corresponding to both regular and singular original system motion phases, it will be further referred to as *the full limit system.*

IV. System Motion under Single Impact Sequences

Limit System Representation under Single Impact Sequence

The following derivation is a natural extension of Theorem 3 for the case of sequences of single collisions with the constraint. Let us fix an arbitrary admissible control $u(t)$, $t \in [0, T]$ and define recursively a finite sequence of times

$$0 < \tau_1 < \cdots < \tau_i < \cdots \le T, \quad i \le N < \infty$$

as follows.

First step. Set $\tau_1 = \tau$ from (6), i.e.

$$\tau_1 = \begin{cases} \inf\limits_{0<t\le T} \Big\{ t : G(\bar{x}_p(t), t) = 0, \ \dfrac{d}{dt}\Big|_{F^r_p} G(\bar{x}_p(t), t) < 0 \Big\}, \\ T, \quad \text{if the set is empty.} \end{cases} \tag{19}$$

Here we substitute $\bar{x}_p(t)$ instead of $x_p(t)$ due to their coincidence for $t \in [0, \tau_1]$.

Second step. For some admissible controls $w_{1\tau_1}(\cdot)$, $\alpha_{\tau_1}(\cdot)$ we find a solution of the limit system (13). This gives $y_v(s^*(\tau_1))$ and hence a shift operator

$$\Psi_v\big(y_p(0), y_v(0), w_{1\tau_1}(\cdot), \tau_1\big) = y_v(s^*(\tau_1)) - y_v(0).$$

Third step. Taking the values $\bar{x}_p(\tau_1)$ and $\bar{x}_v(\tau_1)$=$y_v(s^*(\tau_1))$ = $\bar{x}_v(\tau_1-) + \Psi_v\big(\bar{x}_p(\tau_1), \bar{x}_v(\tau_1-), w_{1\tau_1}(\cdot), \tau_1\big)$ as initial ones, define time τ_2 analogously to τ_1 (as in the first step)

$$\tau_2 = \begin{cases} \inf\limits_{\tau_1<t\le T} \Big\{ t : G(\bar{x}_p(t), t) = 0, \ \dfrac{d}{dt}\Big|_{F^r_p} G(\bar{x}_p(t), t) < 0 \Big\}, \\ T, \quad \text{if the set is empty,} \end{cases} \tag{20}$$

and so on. This yields

$$\bar{x}_v(\tau_i) = \bar{x}_v(\tau_i-) + \Psi_v\big(\bar{x}_p(\tau_i), \bar{x}_v(\tau_i-), w_{1\tau_i}(\cdot), \tau_i\big). \tag{21}$$

The variables $(\bar{x}_p(t), \bar{x}_v(t))$ generated by this procedure satisfy on $[0, T]$ the system of generalized differential equations

$$\begin{aligned}
&\dot{\bar{x}}_p(t) = F^r_p(\bar{x}_p(t), \bar{x}_v(t), t),\\
&\dot{\bar{x}}_v(t) = F^r_v(\bar{x}_p(t), \bar{x}_v(t), u(t), t) +\\
&+\sum_{\tau_i\le T} \Psi_v\big(\bar{x}_p(\tau_i), \bar{x}_v(\tau_i-), w_{1\tau_i}(\cdot), \tau_i\big)\delta(t-\tau_i)
\end{aligned} \tag{22}$$

with $\bar{x}_p(0) = x_p(0)$, $\bar{x}_v(0) = x_v(0)$.

In this description, the state of the limit system (22) changes continuously on half-intervals $[0, \tau_1), \ldots, [\tau_{i-1}, \tau_i), \ldots$ and undergoes a discontinuous change at every instant τ_i. Due to equation (21), the values of these changes depend on the state immediately preceding the jump and the impulsive control signal $w_{\tau_i}(\cdot)$ applied during the singularity phase corresponding to the instant τ_i.

The shift operator representation of jumps implies the Lipschitzian character of function $\Psi_v(\cdot)$, thereby guaranteeing the existence and uniqueness of the solution of (22).

A. Control Law Implementation under Single Impact Sequence

Consider the system (2) with some fixed $\mu > \mu_0$ under the control signals equal to those used in (22), where $w_{1\tau_i}(\cdot), \alpha_{\tau_i}(\cdot)$, $i = 1, \ldots, N$, can be extended, if necessary, beyond the point $s^*(\tau_i)$ in an arbitrary admissible manner. Denoting a solution of this system by $(x^\mu_p(t), x^\mu_v(t))$, define recursively a finite sequence of instants of system trajectory intersections with the constraint boundary

$$0 < \tau^\mu_1 < \tau^{\mu*}_1 < \cdots < \tau^\mu_i < \tau^{\mu*}_i < \cdots \le T, \quad i \le N < \infty$$

as follows.

First step. Set $\tau^\mu_1 = \tau_1 = \tau$, i.e.

$$\tau^\mu_1 = \begin{cases} \inf\limits_{0<t\le T} \Big\{ t : G(x^\mu_p(t), t) = 0, \ \dfrac{d}{dt}\Big|_{F^r_p} G(x^\mu_p(t), t) < 0 \Big\}, \\ T, \quad \text{if the set is empty.} \end{cases} \tag{23}$$

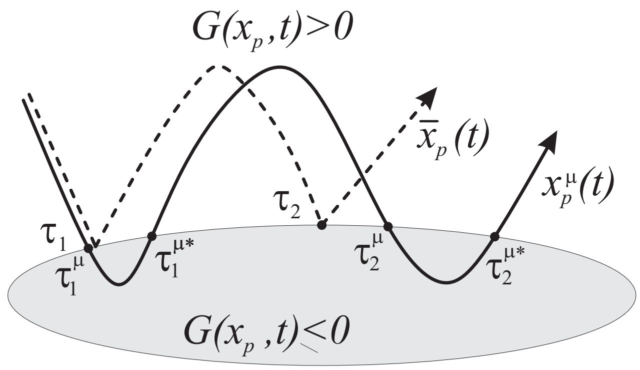

Fig. 1. Multiple sequential collisions: prelimit and limit trajectories

Second step. Denote, as in (7), by γ any of the controls w_1, α mentioned above and define under controls $\gamma(\xi, \sqrt{\mu}(t-\tau_1^\mu)),\ t \geq \tau_1^\mu$, an exit instant

$$\tau_1^{\mu*} = \begin{cases} \inf\limits_{\tau_1^\mu < t \leq T} \Big\{ t : G(x_p^\mu(t), t) = 0,\ \dfrac{d}{dt}\Big|_{F_p^r} G(x_p^\mu(t), t) > 0 \Big\}, \\ T, \quad \text{if the set is empty.} \end{cases} \tag{24}$$

Third step. Define an entering instant τ_2^μ analogously to (20)

$$\tau_2^{\mu} = \begin{cases} \inf\limits_{\tau_1^\mu < t \leq T} \Big\{ t : G(x_p^\mu(t), t) = 0,\ \dfrac{d}{dt}\Big|_{F_p^r} G(x_p^\mu(t), t) < 0 \Big\}, \\ T, \quad \text{if the set is empty.} \end{cases} \tag{25}$$

Forth step. Define the second exit instant $\tau_2^{\mu*}$ analogously to $\tau_1^{\mu*}$, as shown in Fig. 1, and so on.

Theorem 4: If $\mu \to \infty$, the corresponding sequence of ordinary solutions $(x_p^\mu(t), x_v^\mu(t))$ of the system (2) with the above-indicated admissible control signals $u(t),\ t \in [0,T]$, and $\gamma^\mu(\xi,t) = \gamma(\xi, \sqrt{\mu}(t-\tau_i^\mu)),\ t \in [\tau_i^\mu, \tau_i^{\mu*}]$, converges everywhere on $[0,T]$, except, possibly, at the points $\{\tau_i\}$, to the general solution $(\bar{x}_p(t), \bar{x}_v(t))$ of the system (22).

V. System Motion under Single Multi-Impacts and Their Sequences

A. Infinitesimal Dynamics Equation under Single Multi-Impact

In order to consider the case $F_v^{rs}(x_p, x_v, w_2^\mu(\xi,t), t, \mu) \not\equiv 0$ introduce the following definitions.

Definition 2. Constrained-free system motion between two sequentially occurring singular phases will be called *inter-singular motion* if its duration goes to zero as $\mu \to \infty$.

Definition 3. An arbitrary finite connected sequence of alternating singular and inter-singular motions, which starts and ends with singular phase will be referred to as the *interlaced singular phase* of the system motion.

Definition 4. A pair of admissible controls $w_1^\mu(\xi,t)$ and $w_2^\mu(\xi,t)$ in (2) such that $w_1^\mu(\xi,t) = 0,\ G(x_p(t),t) > 0$ and $w_2^\mu(\xi,t) = 0, G(x_p(t),t) < 0$ is said to be *a temporal multi-impulse control* if they exist only on the disjoint finite subset of the time subintervals within the time interval of an isolated interlaced singular phase.

Definitions 2-4 characterize the *temporal multi-impact mode* of system interaction with the constraint. As it is seen from these definitions, this mode is comprised by the time subintervals partitioning the time interval of the interlaced singular phase such that in each subinterval, alternating, either $G(x_p(t),t) > 0$ and $w_2^\mu(\xi,t) \geq 0$ or $G(x_p(t),t) < 0$ and $w_1^\mu(\xi,t) \geq 0$. In the rest of the paper, the qualifier "temporal" will be mostly omitted with no loss of clarity, since the paper does not consider spatially distributed simultaneous impacts.

Let us now extend the theorems formulated above to the case of interlaced singular phase of the system motion. We begin with the theorem that describes the limit behavior of variables (9), satisfying (10), developing first the necessary background.

Assumption 2. Assume that in condition 1) of Theorem 1 the expression (12) is supplemented by the relation

$$\begin{aligned} &\lim_{\mu\uparrow\infty} \mu^{1/2} F_v^{rs}\Big(\frac{y_p - x_p}{\mu^{1/2}} + x_p, y_v, w_2(\eta^\mu, s), \\ &\tau + \mu^{-1/2}s, \mu\Big) = \bar{F}_v^{rs}(y_p, y_v, w_2(\eta, s), x_p, \tau), \end{aligned} \tag{26}$$

for any admissible control w_2.

Now, fix some admissible controls u, w_1, w_2, α on $[0,T]$ and define recursively a finite sequence of instants

$$0 = s_1(\tau) < s_1^*(\tau) < \cdots < s_j(\tau) < s_j^*(\tau) < \cdots,$$

$j = 1, \ldots, N_1$, determining $s_1^*(\tau)$ from (14), as given in equation (33) of [3].

Along with the controlled infinitesimal dynamics equation (13) consider its counterpart for $F_v^{rs} \not\equiv 0$ given by

$$\begin{aligned} \dot{y}_p(s) &= F_p^r(x_p(\tau), y_v(s), \tau), \\ \dot{y}_v(s) &= \bar{F}_v^{rs}(y_p(s), y_v(s), w_2(\eta, s), x_p(\tau), \tau), \\ \dot{\eta}(s) &= 0, \end{aligned} \tag{27}$$

which is supposed to have the unique solution on an interval $[s_1^*(\tau), s_2(\tau) + \varepsilon_1]$, where $\varepsilon_1 > 0$ and $y_p(s_1^*(\tau))$, $y_v(s_1^*(\tau))$, $\eta(s_1^*(\tau))$ coincide with the terminal values of solutions of (13) at instant $s^*(\tau)$, i.e. a solution of the system (27) is a continuous extension of the solution of the system (13). Define $s_2(\tau)$ as in equation (42) of [3]. Next, taking the terminal values $y_p(s_2(\tau))$, $y_v(s_2(\tau))$, $\eta(s_2(\tau))$ of solutions of (27) as initial conditions for the system (13), assume that it has a unique solution on an interval $[s_2(\tau), s_2^*(\tau) + \varepsilon_2]$, where $\varepsilon_2 > 0$ and $s_2^*(\tau)$ is given by equation (43) of [3], and so on. Set $\varepsilon_0 = \min\limits_j\{\varepsilon, \varepsilon_j\}$ and denote by $(y_p(s), y_v(s), \eta(s))$ defined above the solution of the systems (13) and (27) for $s \in [0,\ s_{N_1}^*(\tau) + \varepsilon_0]$. Next, for sufficiently large $\mu > \mu_0$ consider the systems (2) and (10) under fixed control signals defined above. Denoting solutions of those systems by $(x_p^\mu(t),\ x_v^\mu(t))$ and $(y_p^\mu(s),\ y_v^\mu(s),\ \eta^\mu(s))$, respectively, define recursively a finite sequence of instants of system trajectory intersections with the constraint boundary

$$0 = s_{\mu 1}(\tau) < s^*_{\mu 1}(\tau) < \cdots < s_{\mu j}(\tau) < s^*_{\mu j}(\tau) < \cdots,$$

$j = 1, \dots, N_1$, determining $s^*_{\mu 1}(\tau)$, $s_{\mu 2}(\tau)$, and $s^*_{\mu 2}(\tau)$ as given in Boxes I, II, and III of [3] and so on.
Here the intervals $[s_{\mu j}(\tau), s^*_{\mu j}(\tau)]$, $j = 1, \dots, N_1$, correspond to the singular phases of motion and the intervals $(s^*_{\mu j}(\tau), s_{\mu, j+1}(\tau))$, $j = 1, \dots, N_1 - 1$, correspond to the inter-singular ones.

Theorem 5: Let under conditions of Theorem 1 and Assumption 2 $\mu \to \infty$. Then

$$(y^\mu_p(s), y^\mu_v(s), \eta^\mu(s)) \to (y_p(s), y_v(s), \eta(s)) \tag{28}$$

uniformly on $[0, s^*_{N_1}(\tau) + \varepsilon_0]$, and for $j = 1, \dots, N_1$

$$s_{\mu j}(\tau) \quad \to \quad s_j(\tau), \tag{29}$$

$$s^*_{\mu j}(\tau) \quad \to \quad s^*_j(\tau). \tag{30}$$

Remark 3. It is obvious that Corollary 2 and Theorem 3, with slight changes, are valid in the case of multi-impulse control. Indeed, it is sufficient to assign $s^*(\tau) = s^*_{N_1}(\tau)$ and $s^*_\mu(\tau) = s^*_{\mu N_1}(\tau)$, and, by integrating the system (13), (27) for $s \in [0, s^*(\tau)]$, to calculate v-component of the shift operator $\Psi_v(\cdot)$ used in (18) so that

$$y_v(s^*(\tau)) = y_v(0) + \Psi_v\big(y_p(0), y_v(0), w_{1\tau}(\cdot), w_{2\tau}(\cdot), \tau\big). \tag{31}$$

B. Control Law Implementation under Multi-Impact Sequence

Theorem 4 admits natural generalization for the case of multi-impulse control as well. The sequence $\{\tau_i\}$, $i = 1, \dots, N$, is defined similarly. But the double sequence $\{\tau^\mu_i, \tau^{\mu *}_i\}$, $i = 1, \dots, N$, splits into N finite series

$$\tau^\mu_i = \tau^\mu_{i1} < \tau^{\mu *}_{i1} < \tau^\mu_{i2} < \tau^{\mu *}_{i2} < \cdots < \tau^\mu_{iN_i} < \tau^{\mu *}_{iN_i} = \tau^{\mu *}_i,$$

where

$$\tau^\mu_{ij} = \tau^\mu_i + \mu^{-1/2} s_{\mu j}(\tau^\mu_i), \quad \tau^{\mu *}_{ij} = \tau^\mu_i + \mu^{-1/2} s^*_{\mu j}(\tau^\mu_i),$$
$$j = 1, \dots, N_i.$$

Here each series corresponds to one multi-impulse, as shown in Fig. 2.

Theorem 6: Let $(\bar{x}_p(t), \bar{x}_v(t))$, $t \in [0, T]$, be a solution of the system (22), with shift operators replaced by those defined by (31), further referred to as system (22), (31), under some admissible controls u, w_1, w_2 and α. Then, if $\mu \to \infty$, the corresponding sequence of ordinary solutions $(x^\mu_p(t), x^\mu_v(t))$ of the system (2) with the same control signals $u(t)$, $t \in [0, T]$, and $\gamma^\mu(\xi, t) = \gamma(\xi, \sqrt{\mu}(t - \tau^\mu_i))$, $t \in [\tau^\mu_i, \tau^{\mu *}_i]$, converges everywhere on $[0, T]$, except, possibly, at the points $\{\tau_i\}$, to the general solution $(\bar{x}_p(t), \bar{x}_v(t))$ of the system (22), (31). Here γ is any of the controls w_1, w_2 and α, which, generally, admit an extension in the neighborhood of the points $\tau^{\mu *}_i$ as in Theorem 3 of [4]).

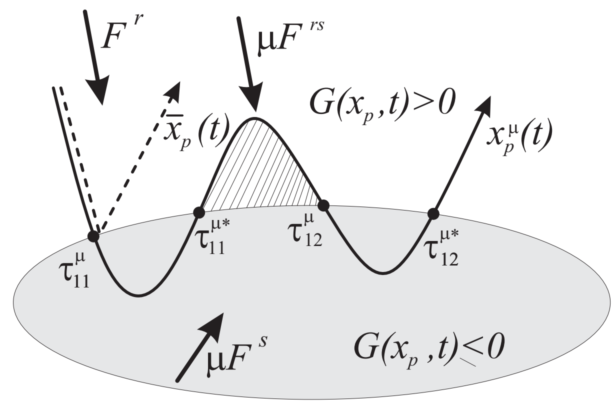

Fig. 2. The first multi-impulse in the multi-impact sequence. Prelimit and limit trajectories. Crosshatched region corresponds to intersingular motion

VI. Conclusions

The framework of [2] and [3] is extended to permit the design of the observations-based optimal control laws requiring the single-impulse and/or the multi-impulse finite control sequences to attain the desired control objective. Future research will focus on general techniques for computation of the optimal control laws for this class of systems as well as extending the results of the present work to the case of the infinite sequences and the finite time accumulation points under the incomplete observation.

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